Open Mathematics

Research Article

Qinghua Zhang*

Abstract-valued Orlicz spaces of range-varying type

https://doi.org/10.1515/math-2018-0080
Received December 16, 2017; accepted May 8, 2018.

Abstract: This paper mainly deals with the abstract-valued Orlicz spaces of range-varying type. Using notions of Banach space net and continuous modular net etc., we give definitions of $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ and $L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, and discuss their geometrical properties as well as the representation of $L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})^*$. We also investigate some functionals and operators on $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, giving expression for the subdifferential of the convex functional generated by another continuous modular net. After making some investigations on the Bochner-Sobolev spaces $W^{1,\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ and $W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)})$, and the intersection space $W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)}) \cap L^{\varphi_{\vartheta(\cdot)}}(I, V_{\vartheta(\cdot)})$, a second order differential inclusion together with an anisotropic nonlinear elliptic equation with nonstandard growth are also taken into account.

Keywords: Continuous modular net, $X_{\theta(\cdot)}$–valued function space, Geometrical property, Doubly nonlinear differential inclusion

MSC: 46B10, 46E30, 46E40

1 Introduction

In this paper we study a new type of Orlicz space, whose members are abstract-valued functions taking values in a varying space. Orlicz space, which was introduced firstly by Orlicz [1] in 1931, is a type of semimodular space commonly generated by a Φ or generalized Φ function. Typical examples of this type are Lebesgue and Sobolev spaces with variable exponents, i.e. $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ (see [2] for references). Due to the wide applications in many fields of applied mathematics, Orlicz space received a growing interest of scholars in the latest decades. Using the anisotropic function spaces, Antontsev-Shmarev in [3–5] studied the parabolic equations of variable nonlinearity, including a model porus medium problem. By means of time discretization and subdifferential calculus, Akagi etc in [6, 7] dealt with the doubly nonlinear parabolic equations involving variable exponents. The work [8] considered the application of Orlicz space in Navier-Stokes equation, and [9] investigated an obstacle problem with variable growth and low regularity of the data.

To deal with the evolution equations with variable exponents, a new type of functions, called $X_{\theta(\cdot)}$–valued functions, are needed. As the valued space varies upon the time, it is difficult to give a suitable definition of "measurability" for these functions. By introducing the concepts of bounded topological lattice $\mathcal{A}$, regular Banach space net $\{X_\alpha : \alpha \in \mathcal{A}\}$ and order-continuous exponent $\theta : I \to \mathcal{A}$, Zhang- Li in [10] firstly gave definition of the space $L^0(I, X_{\theta(\cdot)})$, which contains all the $X_{\theta(\cdot)}$–valued functions measurable in a special manner. Like the measurable functions of range-fixed type, members of $L^0(I, X_{\theta(\cdot)})$ are all norm-measurable. Based on the useful character, from $L^0(I, X_{\theta(\cdot)})$ the authors extracted two types of function spaces: continuous type $C^-(I, X_{\theta(\cdot)})$ and integral type $L^{p(\cdot)}(I, X_{\theta(\cdot)})$. After showing their completeness and

***Corresponding Author: Qinghua Zhang:** Department of Mathematics, Nantong University, 9 Seyuan Road, Nantong City 226019, Jiangsu Province, China, E-mail: zhangqh1971@126.com

connections between them together with some concrete examples, the authors paid attention to a semilinear evolution equation with the nonlinearity having a time-dependent domain to illustrate the application of the $X_{\theta(\cdot)}$–valued functions.

It is worth remarking that some Banach space net can be produced by a continuous modular net $\{\varrho_\alpha : \alpha \in \mathcal{A}\}$. According to whether or not being built on the continuous modular nets, Zhang-Li in [11] divided the $X_{\theta(\cdot)}$–valued function spaces of integral type into two subclasses: norm-modular ones and modular-modular ones. A norm-modular space, like $L^{p(\cdot)}(I, X_{\theta(\cdot)})$, is commonly produced by the semimodular

$$\varrho_\phi(f) = \int_A \phi(t, \|f(t)\|_{\theta(t)})dt$$

with a generalized Φ function ϕ, while a modular-modular space is derived from a continuous modular net $\{\varrho_\alpha : \alpha \in \mathcal{A}\}$ with the semimodular

$$\Phi_\varrho(f) = \int_I \varrho(Mf(t))dt.$$

Here, M is a continuous operator from a topological linear space X to a closed cone V of another topological linear space W, called a V–modular (refer to [11]).

Here we will drop the extra map M, and use merely $\{\varrho_\alpha : \alpha \in \mathcal{A}\}$ and θ to reconstruct the semimodular, namely

$$\Phi_{\varrho_{\theta(\cdot)}}(f) = \int_I \varrho_{\theta(t)}(f(t))dt.$$

This change brings much convenience to us to study the duality and reflexivity of the abstract-valued Orlicz spaces of modular-modular type.

The main part of this paper is organized as follows: As preparations, in Section 2, we study the abstract-valued Orlicz space generated by a single modular. Section 3 is devoted to the abstract-valued Orlicz space generated by a series of modular. Using different measurability of the $X_{\theta(\cdot)}$–valued functions, we introduce two different spaces: $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ and $L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, both of them are complete according to the same norm. We show that, under some suitable situations, $L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ is separable, and its dual space can be represented by

$$L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})^* = L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)}).$$

We also make some evaluations on the proper conditions for the equality

$$L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) = L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$$

as well as the reflexivity of $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$.

For the sake of applications, in Section 4, we make some discussions on the functionals and operators on the modular-modular space $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, including functional $\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}$ defined by another continuous modular system $\{\varphi_\beta : \beta \in \mathcal{B}\}$ and another order-continuous map $\vartheta : I \to \mathcal{B}$, and operators $Z_{\theta(\cdot)}$ and $\partial\varphi_{\vartheta(\cdot)}$, which are subdifferentials of $\Phi_{\varrho_{\theta(\cdot)}}$ and $\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}$ respectively. Here $Z_{\theta(\cdot)}$ plays the role of an extended dual map, and $\partial\varphi_{\vartheta(\cdot)}$ usually arises in a differential equation as the driving operator. Under some extra assumptions, such as the weak lower-continuity of $\{\varrho_\alpha\}$ and $\{\varrho^*_\alpha\}$, and the strong coercivity of $\{\varrho_\alpha\}$ and $\{\varphi_\beta\}$, demicontinuity and coercivity of $Z_{\theta(\cdot)}$ and the representation $\partial\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(u)(t) = \partial\varphi_{\vartheta(t)}(u(t))$ are obtained. This is an attempt to extend the convex functional and its subdifferential generated by a single function to that generated by a series of functions (compare to [12, Ch. 2], and [7]).

After making some investigations on the Bochner-Sobolev spaces $W^{1,\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ and $W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)})$, and the intersection

$$\mathcal{W} = W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)}) \cap L^{\varphi_{\vartheta(\cdot)}}(I, V_{\vartheta(\cdot)}),$$

including the continuous and compact embedding of $W^{1,\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ into the space $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ along with the estimate

$$\|u\|_{W^{1,\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})} \le C(\|u'\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})} + \|u\|_{L^{\varphi_{\vartheta(\cdot)}}(I,V_{\vartheta(\cdot)})}),$$

in Section 5, we study a type of second order nonlinear differential inclusion

$$-\frac{d}{dt}\partial\varrho_{\theta(t)}(u'(t)) + \partial\varphi_{\vartheta(t)}(u(t)) \ni f(t,u(t)) \text{ for a.e. } t \in I, \tag{1}$$

with the periodic boundary condition, where the operator $f : I \times X \to X$ owns a nonstandard growth

$$\varrho^*_{\theta(t)}(f(t,u)) \le \mu\varrho_{\theta(t)}(u) + h(t),\ u \in X_{\theta(t)}$$

for a small number $\mu > 0$ and a nonnegative function $h \in L^1(I)$. By introducing the Nemytskij operator $F(u) = f(\cdot, u)$, and the second order differential operator $D^2_{\theta(\cdot)}$ defined by

$$\langle\langle D^2_{\theta(\cdot)}u, v\rangle\rangle_{\theta(\cdot)} = \int_I \langle\partial\varrho_{\theta(\cdot)}(u'(t)), v'(t)\rangle_{\theta(t)}dt,\ u, v \in W^{1,\varrho_{\theta(\cdot)}}_{\text{per}}(I, X_{\theta(\cdot)}),$$

we obtain a continuous and compact operator

$$(D^2_{\theta(\cdot)} + \partial\Phi_{\varphi_{\vartheta(\cdot)}})^{-1} \circ F : L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) \to L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}),$$

which by Leray-Schauder's alternative theorem contains a fixed point solving the differential inclusion (1) in the weak sense. To illustrate these results, at the end of the paper, an anisotropic elliptic equation defined on a cylinder $I \times \Omega$ of $\mathbb{R}^{N+1}_+$ with a Caratheodory type nonlinearity $\mu g(t,x,u)$ are investigated. Because of the nonstandard growth

$$|g(t,x,u)| \le C(1 + |u|^{p(t,x)-1})$$

for a.e. $(t,x) \in I \times \Omega$ and all $u \in \mathbb{R}$ fulfilled by the nonlinearity, and the periodic boundary condition $u(0,x) = u(T,x)$, study of the anisotropic elliptic equation seems somewhat meaningful.

Framework of our study can be incorporated in the theory of convex analysis and function spaces with variable exponents. Results obtained here have their meaning in the study of nonlinear evolution equations with nonstandard growth.

As preliminaries, let us firstly make a brief review on the Orlicz space of scalar type. For the detailed discussions please refer to [2, Ch. 2] with the references therein.

Let $\mathbb{K}$ be a scalar (real or complex) field, and X be a $\mathbb{K}$–linear space. A convex function $\varrho : X \to [0,\infty]$ is called a semimodular on X, if the following hypotheses are all satisfied:

- $\varrho(0) = 0$,
- for every $u \in X$, the function $\lambda \mapsto \varrho(\lambda u)$ is left continuous on $[0,\infty)$, and
- $\varrho(\lambda u) = \varrho(u)$ provided $|\lambda| = 1$,
- $\varrho(\lambda u) = 0$ for all $\lambda > 0$ implies that $u = 0$.

If in addition, $\varrho(u) = 0$ means $u = 0$, then ϱ is called a modular. Given a semimodular ϱ on X, the corresponding subspace

$$X_\varrho = \{u \in X : \varrho(\lambda u) < \infty \text{ for some } \lambda > 0\}$$

endowed with the norm

$$\|u\|_\varrho = \inf\{\lambda > 0 : \varrho\Big(\frac{u}{\lambda}\Big) \le 1\}$$

becomes a normed linear space. X_ϱ is called the semimodular space, while $\|\cdot\|_\varrho$ is called the Luxemburg norm. Both of them are generated by ϱ. Recall that in X_ϱ the unit ball property is holding, that is $\|u\|_\varrho \le 1$ if and only if $\varrho(u) \le 1$.

Scalar Orlicz space is a common semimodular space produced by the integral semimodular. Suppose that $\phi : [0,\infty) \to [0,\infty]$ is a Φ function, i.e., ϕ is convex, left continuous, $\phi(0) = 0$ and $\lim_{t\to 0}\phi(t) = 0$, $\lim_{t\to\infty}\phi(t) = \infty$. Suppose also (A,μ) is a σ–finite and complete measure space, and $L^0(A,\mu)$ is the linear space containing all the measurable scalar function defined on A. Then integration

$$\varrho_\phi(f) = \int_\Omega \phi(|f(x)|)d\mu$$

defines a semimodular on $L^0(A,\mu)$. The corresponding semimodular space, denoted by $L^\phi(A,\mu)$, is called an Orlicz space. According to the Luxemburg norm $L^\phi(A,\mu)$ is a Banach space. Moreover, ϱ_ϕ is a modular in case that ϕ is positive, i.e., $\phi(t)>0$ whenever $\lambda>0$. Suppose further $\phi: A\times[0,\infty)\to[0,\infty]$ is a generalized Φ function, that is, for a.e. $x\in A$, $\phi(x,\cdot)$ is a Φ functions, and for all $t\in[0,\infty)$, the function $x\mapsto\phi(x,t)$ is measurable on A, then for all $f\in L^0(A,\mu)$, integration $\int_\Omega\phi(x,|f(x)|)d\mu$ makes sense. This defines another semimodular and induces another semimodular space, which is the generalization of Orlicz space, called a Musielak-Orlicz space.

Taking a measurable subset $\Omega\subseteq\mathbb{R}^N$, and a measurable exponent $p:\Omega\to[1,\infty)$, define $A=\Omega$ with Lebesgue measure, and $\phi(x,t)=t^{p(x)}$. Then we obtain a generalized Φ function, from which we can construct an integral modular $\varrho_{p(\cdot)}$ through

$$\varrho_{p(\cdot)}(f)=\int_\Omega|f(x)|^{p(x)}dx,\ f\in L^0(\Omega),$$

and induce an important Musielak-Orlicz space, denoted by $L^{p(\cdot)}(\Omega)$, and called the Lebesgue space with variable exponent. One knows that if $p^+=\operatorname{esssup}_{x\in\Omega}p(x)<\infty$, then $L^{p(\cdot)}(\Omega)$ is separable, and the unit ball property turns to be

$$\min\{\|f\|_{p(\cdot)}^{p^-},\|f\|_{p(\cdot)}^{p^+}\}\le\varrho_{p(\cdot)}(f)\le\max\{\|f\|_{p(\cdot)}^{p^-},\|f\|_{p(\cdot)}^{p^+}\},$$

where $\|\cdot\|_{p(\cdot)}:=\|\cdot\|_{\varrho_{p(\cdot)}}$ is the Luxemburg norm. Furthermore, if additionally $p^-=\operatorname{essinf}_{x\in\Omega}p(x)>1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex (of course reflexive) with the dual space $L^{p'(\cdot)}(\Omega)$, where $1/p'(x)+1/p(x)=1$ for a.e. $x\in\Omega$.

2 Orlicz space generated by a single modular

Let X be a linear space and $\varrho:X\to[0,\infty]$ be a semimodular, which induces a semimodular space X_ϱ with the Luxemburg norm $\|\cdot\|_\varrho$. Let I be a finite or infinite interval, namely $I=[0,T]$ for some $0<T<\infty$ or $I=[0,\infty)$. A function $f:I\to X_\varrho$ is said to be measurable, if for every open set $G\subseteq X$, the preimage $\{t\in I:f(t)\in G\}$ is a measurable subset of I. Moreover, f is called strongly measurable, if there is a sequence of X_ϱ–valued simple functions convergent to f almost everywhere. Of course, a strongly measurable function is measurable definitely, and vice versa provided X is separable (cf. [13, §1.2]). Denote by $L^0(I,X_\varrho)$ the set of all strongly measurable X_ϱ–valued functions defined on I. Recall that a semimodular ϱ is lower-continuous on the induced space X_ϱ, thus for all $a>0$, the set $\{u\in X_\varrho:\varrho(u)>a\}$ is open in X_ϱ. Consequently, for each $f\in L^0(I,X_\varrho)$, the multifunction $t\mapsto\varrho(f(t))$ is also measurable. Hence integration

$$\Phi_\varrho(f)=\int_I\varrho(f(t))dt$$

makes sense. One can easily verify that Φ_ϱ is also a semimodular on $L^0(I,X_\varrho)$ with the semimodular space

$$L^\varrho(I,X_\varrho)=\big\{f\in L^0(I,X_\varrho):\Phi_\varrho(\lambda f)<\infty\text{ for some }\lambda>0\big\}$$

and the Luxemburg norm denoted by $\|\cdot\|_{L^\varrho(I,X_\varrho)}$.

Theorem 2.1. *$L^\varrho(I,X_\varrho)$ is a Banach space in case that X_ϱ is complete.*

This theorem is a special case of Theorem 3.7, which is given in §3 with a proof.

Remark 2.2. *Suppose that $f\in L^\infty(I,X_\varrho)$, and the one-dimension Lebesgue measure of the set $E_0=\{t\in I:f(t)\neq0\}$ is finite. Then we have*

$$\Phi_\varrho\Big(\frac{f}{M+1}\Big)\le|E_0|<\infty,$$

where $M \geq 0$ is the essential supremum of $\|f(t)\|_\varrho$. Thus $f \in L^\varrho(I, X_\varrho)$ and $\|f\|_{L^\varrho(I,X_\varrho)} \leq (M+1)\max\{1, |E_0|\}$. Furthermore, by the estimate

$$\|u\|_\varrho \leq \varrho(u) + 1, \ \forall\ u \in X_\varrho,$$

we also have

$$L^\infty(I, X_\varrho) \hookrightarrow L^\varrho(I, X_\varrho) \hookrightarrow L^1(I, X_\varrho)$$

in case that $I = [0, T]$ is bounded.

A semimodular ϱ is said to be satisfying the Δ_2–condition, if there exists a constant $d_2 \geq 2$ such that

$$\varrho(2u) \leq d_2\varrho(u) \text{ for all } u \in X.$$

Recall that, under the Δ_2–condition, ϱ turns to be a continuous modular satisfying
- $u \in X_\varrho$ if and only if $\varrho(\lambda u) < \infty$ for all $\lambda > 0$, and
- $u_n \to u$ in X_ϱ if and only if $\varrho(u_n - u) \to 0$.

Moreover, Φ_ϱ also satisfies the Δ_2–condition with the same constant d_2.

Proposition 2.3. *If ϱ satisfies the Δ_2–condition, then the set of all simple X_ϱ–valued functions, say $\mathcal{S}(I, X_\varrho)$, is dense in $L^\varrho(I, X_\varrho)$.*

Proof. For each $f \in L^\varrho(I, X_\varrho)$, there is correspondingly a sequence of simple X_ϱ–valued functions $\{s_k\}$ such that $s_k(t) \to f(t)$ and consequently $s_k\chi_{E_0}(t) \to f(t)$ in X_ϱ for a.e. $t \in I$ for the set E_0 defined in Remark 2.2. Let

$$E_k = \big\{t \in I : \varrho(s_k\chi_{E_0}(t) - f(t)) \leq \varrho(f(t))\big\},$$

and let $\varphi_k = s_k\chi_{E_0}\chi_{E_k}$, which is also a simple X_ϱ–valued function. Notice that

$$\begin{aligned}
&\big\{t \in I : \varrho(\varphi_k(t) - f(t)) \to 0\big\}\\
&= \bigcap_{\varepsilon>0}\bigcup_{K\geq 1}\bigcap_{k\geq K}\big\{t \in I : \varrho(\varphi_k(t) - f(t)) \leq \varepsilon\big\}\\
&\supseteq \bigcap_{\varepsilon>0}\bigcup_{K\geq 1}\bigcap_{k\geq K}\big\{t \in I : \varrho(s_k\chi_{E_0}(t) - f(t)) \leq \min\{\varepsilon, \varrho(f(t))\}\big\}\\
&= \big\{t \in I : \varrho(s_k\chi_{E_0}(t) - f(t)) \to 0\big\},
\end{aligned}$$

we have $\varrho(\varphi_k(t) - f(t)) \to 0$ a.e. on I as $k \to \infty$, which combined with $\varrho(\varphi_k(t) - f(t)) \leq \varrho(f(t))$ for a.e. $t \in I$, and Lebesgue's convergence theorem, yields $\Phi_\varrho(\varphi_k - f) \to 0$ or equivalently $\varphi_k \to f$ in $L^\varrho(I, X_\varrho)$ as $k \to \infty$. Thus density of $\mathcal{S}(I, X_\varrho)$ in $L^\varrho(I, X_\varrho)$ has been proved. □

Remark 2.4. *Since $I = \bigcup_{n=1}^\infty I_n$, where $I_n = I \cap [0, n]$, $n = 1, 2, \cdots$, we can also prove that $\mathcal{S}_c(I, X_\varrho)$ and $L_c^\infty(I, X_\varrho)$, subsets of $\mathcal{S}(I, X_\varrho)$ and $L^\infty(I, X_\varrho)$ respectively, containing the functions with compact supports, are both dense in $L^\varrho(I, X_\varrho)$.*

Corollary 2.5. *Under the Δ_2–condition of ϱ and the separability assumption of X_ϱ, $L^\varrho(I, X_\varrho)$ is a separable space.*

Remark 2.6. *Given a semimodular ϱ, recall that the dual functional ϱ^* is also a semimodular on X_ϱ^*, and the double dual ϱ^{**} is equal to ϱ on the space X_ϱ (cf. [2, §2.2] or [14, §3.2]). Moreover, for all $u \in X_\varrho$ and $\xi \in X_\varrho^*$, Young's inequality*

$$\langle \xi, u\rangle \leq \varrho(u) + \varrho^*(\xi)$$

holds. The equality also holds if and only if $\xi \in \partial\varrho(u)$ or equivalently $u \in \partial\varrho^*(\xi)$ if we regard X_ϱ as a closed subspace of X_ϱ^{**}. Here $\partial\varrho$ is the the subdifferential operator of ϱ and $\partial\varrho^*$ is that of ϱ^*. Recall that as the subdifferential operators of lower-semicontinuous and convex proper functionals, $\partial\varrho$ and $\partial\varrho^*$ can be viewed

as two maximal monotone and semiclosed subsets of the product spaces $X_\varrho \times X_\varrho^*$ and $X_\varrho^* \times X_\varrho^{**}$ respectively. As for the multivalued inverse map $(\partial\varrho)^{-1}$, we know that $(\partial\varrho)^{-1} \subseteq X_\varrho^* \times X_\varrho$ has the same properties as $\partial\varrho$ has, together with the inclusion $(\partial\varrho)^{-1} \subseteq \partial\varrho^*$ holding. Furthermore, if in addition $\mathcal{R}(\partial\varrho) = X_\varrho^*$, then for all $\xi \in X_\varrho^*$, the image $(\partial\varrho)^{-1}(\xi)$ is a nonempty convex and closed subset of X_ϱ. Therefore for all $\xi \in L^0(I, X_\varrho^*)$, the multifunction $t \mapsto (\partial\varrho)^{-1}(\xi(t))$ is graph measurable with nonempty closed image everywhere, consequently under the additional separability assumption of X_ϱ, we can assert that $(\partial\varrho)^{-1}(\xi(\cdot))$ has a measurable selection, or in other words, there is a function $u \in L^0(I, X_\varrho)$ such that $u(t) \in (\partial\varrho)^{-1}(\xi(t))$ for a.e. $t \in I$ (see[15, §8.3] for references).

Theorem 2.7. *Suppose that ϱ satisfies the Δ_2–condition, X_ϱ is a separable Banach space, and $\mathcal{R}(\partial\varrho) = X_\varrho^*$. Suppose also ϱ^* is a modular, and X_ϱ^* has the Radon-Nikodym's property w.r.t. every finite subinterval of I. Then the dual space $L^\varrho(I, X_\varrho)^*$ is isomorphic to $L^{\varrho^*}(I, X_\varrho^*)$.*

Proof. For each $\xi \in L^{\varrho^*}(I, X_\varrho^*)$, define a functional Λ_ξ through

$$\langle\langle \Lambda_\xi, f\rangle\rangle = \int_I \langle \xi(t), f(t)\rangle dt, \quad \forall\, f \in L^\varrho(I, X_\varrho). \tag{2}$$

By Hölder's inequality

$$\Big| \int_I \langle \xi(t), f(t)\rangle dt\Big| \le 2\|\xi\|_{L^{\varrho^*}(I,X_\varrho^*)}\|f\|_{L^\varrho(I,X_\varrho)},$$

it is easy to check that $\Lambda_\xi \in L^\varrho(I, X_\varrho)^*$, and $\|\Lambda_\xi\| \le 2\|\xi\|_{L^{\varrho^*}(I,X_\varrho^*)}$.

Conversely, for each $\Lambda \in L^\varrho(I, X_\varrho)^*$, we will prove the existence of a unique $\xi \in L^{\varrho^*}(I, X_\varrho^*)$ such that $\Lambda = \Lambda_\xi$ as in (2). If $\Lambda = 0$, then take $\xi = 0$ and there is nothing to do. If $\Lambda \ne 0$, then $|||\Lambda|||^* > 0$, where $|||\Lambda|||^*$ denotes the $L^\varrho(I, X_\varrho)^*$–norm of Λ. Without loss of generality, in the following discussions we may assume that $|||\Lambda|||^* = 1$. For each positive integer n, define an X_ϱ^*–valued function τ_n on the set of all measurable subsets of $I_n = I \cap [0, n]$ as follows:

$$\langle \tau_n(E), u\rangle = \langle\langle \Lambda, \chi_E u\rangle\rangle, \quad \forall\, u \in X_\varrho.$$

Here χ_E is the characteristic function of E.

Suppose that $\{E_k\}$ is a sequence of mutually disjoint measurable subsets of I_n and $u \in X_\varrho$. Since for all $\lambda > 0$,

$$\Phi_\varrho\Big(\lambda \sum_{k=1}^\infty \chi_{E_k} u\Big) = \int_I \sum_{k=1}^\infty \chi_{E_k}\varrho(\lambda u)dt = \varrho(\lambda u)\sum_{k=1}^\infty |E_k| \le n\varrho(\lambda u) < \infty,$$

we can conclude that

$$\sum_{k=1}^\infty \chi_{E_k} u = \chi_{\cup_{k=1}^\infty E_k} u \text{ in } L^\varrho(I, X_\varrho), \text{ and } \Big\|\sum_{k=1}^\infty \chi_{E_k} u\Big\|_{L^\varrho(I,X_\varrho)} \le n\|u\|_\varrho.$$

Consequently,

$$\Big\langle \tau_n\Big(\bigcup_{k=1}^\infty E_k\Big), u\Big\rangle = \sum_{k=1}^\infty \langle\langle \Lambda, \chi_{E_k} u\rangle\rangle = \sum_{k=1}^\infty \langle \tau_n(E_k), u\rangle,$$

and

$$\Big|\Big\langle \tau_n\Big(\bigcup_{k=1}^\infty E_k\Big), u\Big\rangle\Big| \le n|||\Lambda|||^*\|u\|_\varrho.$$

Therefore τ_n is an X_ϱ^*–valued measure on I_n with a bounded total variation no more than $n|||\Lambda|||^*$. Now since X_ϱ^* has the Radon-Nikodym's property w.r.t. every finite subinterval of I, there is a unique $\xi_n \in L^1(I_n, X_\varrho^*)$ such that

$$\langle\langle \Lambda, \chi_E u\rangle\rangle = \langle \tau_n(E), u\rangle = \int_{I_n} \langle \xi_n(t), \chi_E u\rangle dt$$

for every measurable subset E of I_n. By the uniqueness of ξ_n in the above representation, we have that $\xi_{n+1}(t) = \xi_n(t)$ a.e. on I_n. Let $\xi(t) = \xi_n(t)$ for $t \in I_n$, then we obtain a globally defined and strongly measurable X_ϱ^*–valued function satisfying

$$\langle\langle \Lambda, f\rangle\rangle = \int_I \langle \xi(t), f\rangle dt \tag{3}$$

for the function $f = u\chi_E$ with a bounded measurable subset E of I and a point $u \in X_\varrho$ and consequently for all $f \in \mathcal{S}(I, X_\varrho)$ with compact supports.

Given a function $f \in L_c^\infty(I, X_\varrho)$, from Corollary 2.3, we can find a sequence of X_ϱ–valued simple functions with compact supports, say $\{s_k\}$, such that $s_k \to f$ in both $L^\infty(I, X_\varrho)$ and $L^\varrho(I, X_\varrho)$, and (3) is satisfied by s_k for all $k \in \mathbb{N}$. Taking limits as $k \to \infty$ in both sides of (3), we can deduce that (3) is also satisfied by $f \in L_c^\infty(I, X_\varrho)$.

Remark 2.6 shows that the multivalued function $t \mapsto (\partial\varrho)^{-1}(\xi(t))$ is measurable, and it has a strongly measurable selection since X_ϱ is separable. Denote the selection by u, then we have $u(t) \in (\partial\varrho)^{-1}(\xi(t)) \subseteq \partial\varrho^*(\xi(t))$ a.e. on I. For each $n \in \mathbb{N}^+$, let

$$J_n = \{t \in I_n : \|u(t)\|_{X_\varrho} \le n\}, \text{ and } u_n(t) = u(t)\chi_{J_n}.$$

Then $u_n \in L_c^\infty(I, X_\varrho)$ and

$$\int_{J_n} \varrho^*(\xi(t))dt + \int_I \varrho(u_n(t))dt = \int_I \langle \xi(t), u_n(t)\rangle dt = \langle\langle \Lambda, u_n\rangle\rangle \le \|u_n\|_{L^\varrho(I,X_\varrho)}. \tag{4}$$

As $\Lambda \neq 0$ and ϱ^* is a modular, neither $\xi(t)$ nor $\varrho^*(\xi(t))$ is equal to 0 a.e. on I. Thus $\int_{J_n} \varrho^*(\xi(t))dt > 0$, consequently from (4) we get

$$\Phi_\varrho(u_n) < \|u_n\|_{L^\varrho(I,X_\varrho)}$$

for n large enough. Hence by the unit ball property, we assert that $\|u_n\|_{L^\varrho(I,X_\varrho)} \le 1$, which in turn yields $\int_{J_n} \varrho^*(\xi(t))dt \le 1$. Let $n \to \infty$, and take the fact $|I \setminus \bigcup_{n=1}^\infty J_n| = 0$ into account, we have $\int_I \varrho^*(\xi(t))dt \le 1$, which leads to the conclusion $\xi \in L^{\varrho^*}(I, X_\varrho^*)$ with the estimate

$$\|\xi\|_{L^{\varrho^*}(I,X_\varrho^*)}dt \le 1.$$

Finally, using the density of $L_c^\infty(I, X_\varrho)$ in $L^\varrho(I, X_\varrho)$, we can conclude that (3) holds for all $f \in L^\varrho(I, X_\varrho)$. Therefore $\Lambda = \Lambda_\xi$ as in (2) and $\|\xi\|_{L^{\varrho^*}(I,X_\varrho^*)} = |||\Lambda_\xi|||^* = 1$. Thus the proof has been completed in the case $|||\Lambda_\xi|||^* = 1$, and the general case can be dealt with by the scaling arguments. □

Remark 2.8. *Here the separability assumption of X_ϱ can be replaced by the strict convexity assumption of ϱ. As a matter of fact, if ϱ is strictly convex, then $\partial\varrho$ is injective, or equivalently $(\partial\varrho)^{-1}$ is single-valued.*

Recall that every reflexive space satisfies the Radon-Nikodym's property with respect to every complete and finite measure space. Furthermore if X_ϱ is reflexive, then $\partial\varrho^* = (\partial\varrho)^{-1}$ and $\partial\varrho = (\partial\varrho^*)^{-1}$. Putting these facts into Theorem 2.7, we have

Corollary 2.9. *Suppose that both ϱ and its dual ϱ^* satisfy the Δ_2–condition, the semimodular space X_ϱ is reflexive and separable. Then*

$$L^\varrho(I, X_\varrho)^* \cong L^{\varrho^*}(I, X_\varrho^*),\ L^{\varrho^*}(I, X_\varrho^*)^* \cong L^\varrho(I, X_\varrho),$$

and the function space $L^\varrho(I, X_\varrho)$ is also reflexive.

Given a semimodular $\varrho : X \to [0, \infty]$, we say ϱ is uniformly convex, i.e. we mean that for every $\varepsilon \in (0, 1)$, there is a $\delta \in (0, 1)$, for which either

$$\varrho\left(\frac{u-v}{2}\right) \le \varepsilon\frac{\varrho(u)+\varrho(v)}{2} \quad \text{or} \quad \varrho\left(\frac{u-v}{2}\right) \le (1-\delta)\frac{\varrho(u)+\varrho(v)}{2} \tag{5}$$

holds. According to [2, §2.4], we know that every uniformly convex semimodular satisfying the Δ_2-condition generates a uniformly convex space. Similarly, for a semimodular ϱ, its uniform convexity can be inherited by the Nemytskij functional Φ_ϱ. Summing up, we have

Theorem 2.10. *Under the uniform convexity assumption and the Δ_2-condition of ϱ, $L^\varrho(I, X_\varrho)$ is a uniformly convex space.*

3 Orlicz space generalized by a series of semimodular

Suppose that $\mathcal{A}$ is a topological lattice, i.e. $\mathcal{A}$ is an ordered topological space, and for every order-bounded subset of $\mathcal{A}$ its order supremum and order infimum exist in $\mathcal{A}$ simultaneously. In this paper, $\mathcal{A}$ is always assumed to be a totally order-bounded topological lattice, or $\mathcal{BTL}$ in abbreviation. Its order supremum and infimum are denoted by α^+ and α^- respectively. In a $\mathcal{BTL}$ $\mathcal{A}$, a sequence $\{\alpha_k\}$ is said to be approaching a point β, if the two conditions $\alpha_k \prec \beta$, $\forall\ k \in \mathbb{N}$ and $\lim_{k\to\infty} \alpha_k = \beta$ are both fulfilled.

Definition 3.1. *Given a family of Banach spaces $\{X_\alpha : \alpha \in \mathcal{A}\}$, we say it is a Banach space net, or $\mathcal{BSN}$ for short, provided*

- *$\alpha \prec \beta$ implies $X_\beta \hookrightarrow X_\alpha$.*

We say $\{X_\alpha\}$ is norm-continuous, if

- *for every sequence $\{\alpha_k\}$ approaching β, the limit of norms $\lim_{k\to\infty} \|x\|_{\alpha_k} = \|x\|_\beta$ holds at all $x \in X_\beta$.*

$\{X_\alpha\}$ is called uniformly bounded, whenever

- *there is a constant $C \geq 1$ such that for all $\alpha, \beta \in \mathcal{A}$ with $\alpha \prec \beta$ and all $x \in X_\beta$, inequality $\|x\|_\alpha \leq C\|x\|_\beta$ always holds.*

And $\{X_\alpha\}$ is said to be successive, if

- *for any sequence $\{\alpha_k\}$ approaching β and any point $x \in X_{\alpha^-}$ with the constraints: $x \in X_{\alpha_k}$ for all $k \in \mathbb{N}$ and $C = \sup_{k\to\infty} \|x\|_{\alpha_k} < \infty$, we have $x \in X_\beta$ and $\|x\|_\beta \leq C$.*

Finally, a $\mathcal{BSN}$ $\{X_\alpha\}$ is called regular provided it is norm-continuous, uniformly bounded and successive at the same time.

Remark 3.2. *Given a $\mathcal{BSN}$ $\{X_\alpha : \alpha \in \mathcal{A}\}$, the family of dual spaces $\{X_\alpha^* : \alpha \in \mathcal{A}\}$, where $\mathcal{A}$ takes the inverse order $\succ$ instead of $\prec$, is also a $\mathcal{BSN}$, called the dual space net or $\mathcal{DSN}$ in symbol. Here we use the convention: $\langle \xi, x\rangle_\alpha = \langle \xi, x\rangle_\beta$ provided $\xi \in X_\alpha^*$, $x \in X_\beta$ and $\alpha \prec \beta$. It is easy to see that, if $\{X_\alpha\}$ is uniformly bounded, then $\{X_\alpha^*\}$ is also uniformly bounded with the same bounds. However, whether or not $\{X_\alpha^*\}$ inherits the norm-continuity and successive property from $\{X_\alpha\}$ is not clear.*

Definition 3.3. *Suppose that X is a linear space, and $\{\varrho_\alpha : \alpha \in \mathcal{A}\}$ is a family of semimodulars defined on X. We say $\{\varrho_\alpha\}$ is a continuous modular net, or $\mathcal{CMN}$ in abbreviation, i.e. we mean that the following hypotheses are satisfied:*

1. *every ϱ_α generates a Banach space $X_{\varrho_\alpha} =: X_\alpha$,*
2. *there exist two positive constants C_i, $i = 1, 2$ for which inequality*

$$\varrho_{\alpha_1}(u) \leq C_1 \varrho_{\alpha_2}(u) + C_2, \tag{6}$$

   holds for all $u \in X$ and all $\alpha_i \in \mathcal{A}$, $i = 1, 2$ with $\alpha_1 \prec \alpha_2$, and
3. *if $\{\alpha_k\}$ approaches α in $\mathcal{A}$, then*

$$\lim_{k\to\infty} \varrho_{\alpha_k}(u) = \varrho_\alpha(u).$$

The following proposition reveals the relationship between $\mathcal{CMN}$ and $\mathcal{BSN}$. For its proof, please refer to [10].

Proposition 3.4. *Given a $\mathcal{CMN}$ $\{\varrho_\alpha : \alpha \in \mathcal{A}\}$, the family of semimodular spaces $\{X_\alpha : \alpha \in \mathcal{A}\}$ is a regular $\mathcal{BSN}$.*

Remark 3.5. *Similar to the scalar ones, for two indexes $\alpha_i \in \mathcal{A}$, $i = 1, 2$ with $\alpha_1 \prec \alpha_2$, we have $L^{\varrho_{\alpha_2}}(I, X_{\alpha_2}) \hookrightarrow L^{\varrho_{\alpha_1}}(I, X_{\alpha_1})$ with the imbedding constant $C = \max\{1, C_1 + C_2 T\}$ in the case $I = [0, T]$.*

Let I be an interval as in Section 2, and $\Pi(I)$ be the collection of all bounded subintervals of I. Consider the map $\theta : I \to \mathcal{A}$. When we say θ is order-continuous, we mean that for any nest of intervals $\{J_k \in \Pi(I) : k = 1, 2, \cdots\}$ shrinking to t, the limit

$$\lim_{k\to\infty} \theta^-_{J_k} = \lim_{k\to\infty} \theta^+_{J_k} = \theta(t)$$

always holds, where θ^-_J and θ^+_J denote the order infimum and supremum of θ on J respectively.

Remark 3.6. *Here we give up the extra assumption that θ is continuous according to the topology of $\mathcal{A}$, which was stated but not used in [11].*

Define

$$L^0_-(I, X_{\theta(\cdot)}) = \left\{f \in L^0(I, X) : f|_J \in L^0(J, X_{\theta^-_J}) \text{ for all } J \in \Pi(I)\right\},$$

and

$$L^0(I, X_{\theta(\cdot)}) = \left\{f \in L^0_-(I, X_{\theta(\cdot)}) : f(t) \in X_{\theta(t)} \text{ for a.e. } t \in I\right\}.$$

Obviously, both of them are linear spaces according to the sum and scalar multiplication of abstract-valued functions, and $L^0(I, X_{\theta(\cdot)}) \subseteq L^0_-(I, X_{\theta(\cdot)})$.

For each positive integer n, let $t_{n,k} = kT/2^n$ or $t_{n,k} = k/2^n$, $J_{n,1} = [0, t_{n,1}]$, $J_{n,k+1} = (t_{n,k}, t_{n,k+1}]$ and $\theta^\pm_{n,k} = \theta^\pm_{J_{n,k}}$, for $k = 1, 2, \cdots, 2^n$ if $I = [0, T]$ or $k = 1, 2, \cdots$ if $I = [0, \infty)$. Define a step function θ_n through $\theta^\pm_n(t) = \theta^\pm_{n,k}$ for $t \in J_{n,k}$. Obviously, $\{\theta^\pm_n\}$ is decreasing (increasing) in n and converging to $\theta(t)$ as $n \to \infty$ for all $t \in I$. Similar to the constant ones, for every $n \in \mathbb{N}$, function space $L^0(I, X_{\theta^\pm_n(\cdot)})$ is well defined, on which $\Phi_{\theta^\pm_n}(f) = \int_I \varrho_{\theta^\pm_n(t)}(f(t))dt$ is a semimodular. It induces a Banach space, denoted by $L^{\varrho_{\theta^\pm_n(\cdot)}}(I, X_{\theta^\pm_n(\cdot)})$.

There is a natural relation among the three types of function spaces mentioned above, that is

$$L^0(I, X_{\theta(\cdot)}) \subseteq L^0_-(I, X_{\theta(\cdot)}) \subseteq L^0(I, X_{\theta^-_n(\cdot)}).$$

Thus for each $f \in L^0_-(I, X_{\theta(\cdot)})$, the function $t \mapsto \varrho_{\theta_n(t)}(f(t))$ is measurable, $n = 1, 2, \cdots$. Note that

$$\varrho_{\theta(t)}(f(t)) = \lim_{n\to\infty} \varrho_{\theta_n(t)}(f(t))$$

by the continuity of $\{\varrho_\alpha\}$, so the composite function $t \mapsto \varrho_{\theta(t)}(f(t))$ is also measurable. Let

$$\Phi_{\varrho_{\theta(\cdot)}}(f) = \int_I \varrho_{\theta(t)}(f(t))dt,\ f \in L^0_-(I, X_{\theta(\cdot)}).$$

we then obtain a semimodular, whose semimodular space is denoted by $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$. Obviously, every member of $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ lies in $L^0(I, X_{\theta(\cdot)})$, hence $\Phi_{\varrho_{\theta(\cdot)}}$ can also be considered as a semimodular on $L^0(I, X_{\theta(\cdot)})$, correspondingly $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ can be regarded as the semimodular space generated from $L^0(I, X_{\theta(\cdot)})$.

Theorem 3.7. *For every $\mathcal{CMN}$ $\{\varrho_\alpha : \alpha \in \mathcal{A}\}$ and every order-continuous map $\theta : I \to \mathcal{A}$, $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ is a Banach space.*

Proof. Suppose that $\{f_k\}$ is a cauchy sequence in $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$. Then for every $\lambda > 0$, we have

$$\lim_{k,l\to\infty} \Phi_{\varrho_{\theta(\cdot)}}(\lambda(f_k - f_l)) = \lim_{k,l\to\infty} \int_I \varrho_{\theta(t)}(\lambda(f_k(t) - f_l(t))dt = 0.$$

Thus there is sequence of positive integers, say $\{k_i\}$, satisfying $k_i < k_{i+1}$ for all $i \in \mathbb{N}$, $\lim_{i\to\infty} k_i = \infty$, and

$$\int_I \varrho_{\theta(t)}(2^i(f_k(t) - f_j(t)))dt < \frac{1}{2^i} \text{ whenever } k, j \geq k_i, \tag{7}$$

Especially we have

$$\int_I \varrho_{\theta(t)}(2^i(f_{k_{i+1}}(t)-f_{k_i}(t)))dt < \frac{1}{2^i},$$

which in turn yields

$$|E_i| \le |\{t \in I : \varrho_{\theta(t)}(2^i(f_{k_{i+1}}(t)-f_{k_i}(t))) > 1\}| < \frac{1}{2^i},$$

where $E_i = \{t \in I : \|f_{k_{i+1}}(t)-f_{k_i}(t)\|_{\theta(t)} > 1/2^i\}$, $i = 1, 2, \cdots$.

Let $E = \bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty} E_i$, then we have $m(E) = 0$, and for each $t \in I \setminus E$, there exists $j \in \mathbb{N}$ such that $t \in I \setminus \bigcup_{i=j}^{\infty} E_i$, or equivalently $\|f_{k_{i+1}}(t)-f_{k_i}(t)\|_{\theta(t)} \le 1/2^i$ for all $i \ge j$. Consequently, for the integer $l \ge j$, we have

$$\sum_{i=l}^{\infty} \|f_{k_{i+1}}(t)-f_{k_i}(t)\|_{\theta(t)} \le \frac{1}{2^{l-1}}.$$

This infers that series $f_{k_1}(t) + \sum_{i=l}^{\infty}(f_{k_{i+1}}(t)-f_{k_i}(t))$ is absolutely continuous in $X_{\theta(t)}$ on the set $I \setminus E$. Then by the completeness of $X_{\theta(t)}$, we conclude that $\{f_{k_i}(t)\}$ is convergent in $X_{\theta(t)}$ a.e. on I, and the limit function f belongs to $L^0(I, X_{\theta(\cdot)})$.

Taking any $\lambda > 0$ and $\varepsilon > 0$, there exists $i \in \mathbb{N}$ such that $2^i > \lambda$ and $1/2^i < \varepsilon$, thus using inequality (7), Fatou's lemma together with the lower semicontinuity of $\varrho_{\theta(t)}$, we obtain

$$\int_I \varrho_{\theta(t)}(\lambda(u_k(t)-u(t)))dt \le \liminf_{j\to\infty} \int_I \varrho_{\theta(t)}(2^i(u_k(t)-u_{k_j}(t)))dt < \varepsilon, \quad \forall\ k > k_i,$$

which means that $u \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ and $u_k \to u$ in $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ as $k \to \infty$. This shows the completeness of $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$. □

The following propositions can be proved by inequality (6), continuity of $\{\varrho_\alpha\}$, Fatou's lemma, together with the unit ball property.

Proposition 3.8. *Suppose that I is a bounded interval, then*

$$L^{\varrho_{\theta_n^+(\cdot)}}(I, X_{\theta_n^+(\cdot)}) \hookrightarrow L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) \hookrightarrow L^{\varrho_{\theta_n^-(\cdot)}}(I, X_{\theta_n^-(\cdot)}),$$

and there exists a constant $C > 0$ such that

$$C^{-1}\|f\|_{L^{\varrho_{\theta_n^-(\cdot)}}(I, X_{\theta_n^-(\cdot)})} \le \|f\|_{L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})} \le C\|f\|_{L^{\varrho_{\theta_n^+(\cdot)}}(I, X_{\theta_n^+(\cdot)})}.$$

Proposition 3.9. *A function $f \in L^0_-(I, X_{\theta(\cdot)})$ with the property*

$$K = \limsup_{n\to\infty} \|f\|_{L^{\varrho_{\theta_n^-(\cdot)}}(I, X_{\theta_n^-(\cdot)})} < \infty$$

lies in $f \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ definitely with the estimate $\|f\|_{L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})} \le K$.

Corollary 3.10. *Under the bounded assumption of I, function space $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ is equivalent to*

$$\Big\{f \in L^0_-(I, X_{\theta(\cdot)}) : \sup_{J\in\Pi(I)} \|f\|_{L^{\varrho_{\theta_J^-}}(I, X_{\theta_J^-})} < \infty\Big\}$$

and

$$\|f\|_{L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})} \le \sup_{J\in\Pi(I)} \|f\|_{L^{\varrho_{\theta_J^-}}(I, X_{\theta_J^-})} \le C\|f\|_{L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})}$$

for some constant $C > 0$.

Assume that $\{X_\alpha\}$ generated by $\{\varrho_\alpha\}$ is a dense $\mathcal{BSN}$, i.e. X_{α_2} is a dense subspace of X_{α_1} whenever $\alpha_1 \prec \alpha_2$. It is easy to see that, under this situation $\mathcal{S}(I, X_{\alpha^+})$ is contained in $L^0(I, X_{\theta(\cdot)})$, consequently for every

$f \in \mathcal{S}(I, X_{\alpha^+})$, the multifunction $t \mapsto \varrho_{\theta(t)}(f(t))$ is measurable. Moreover, by invoking Proposition 2.3 we can prove that, if every modular ϱ_α satisfies the Δ_2–condition, then for each $J \in \Pi(I)$, $\mathcal{S}(J, X_{\alpha^+})$ hence $L^{\varrho_{\alpha^+}}(J, X_{\alpha^+})$ is dense in $L^{\varrho_{\theta_J^+}}(J, X_{\theta_J^+})$.

Analogous to the range-invariant ones, we can define the space of strongly measurable functions with varying ranges, that is

$$\begin{aligned} L^0_+(I, X_{\theta(\cdot)}) = \big\{ & f \in L^0(I, X) : f(t) \in X_{\theta(t)} \text{ for a.e. } t \in I, \\ & \text{and there exists a sequence } \{s_n\} \text{ of } \mathcal{S}(I, X_{\alpha^+}) \text{ s.t.} \\ & \|s_n(t) - f(t)\|_{\theta(t)} \to 0 \text{ as } n \to \infty \text{ for a.e. } t \in I \big\}. \end{aligned}$$

Remark 3.11. *In this definition, the set $\mathcal{S}(I, X_{\alpha^+})$ can be replaced by $L^0(I, X_{\alpha^+})$, both of which are contained in $L^0(I, X_{\theta(\cdot)})$. As a result, one can easily check that $L^0_+(I, X_{\theta(\cdot)})$ is a subspace of $L^0(I, X_{\theta(\cdot)})$.*

Suppose that $\{\varrho_\alpha : \alpha \in \mathcal{A}\}$ is a $\mathcal{CMN}$ generating a dense $\mathcal{BSN}$ $\{X_\alpha\}$. Similar to $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, we can define $L^{\varrho_{\theta(\cdot)}}_+(I, X_{\theta(\cdot)})$ through

$$L^{\varrho_{\theta(\cdot)}}_+(I, X_{\theta(\cdot)}) = \big\{ f \in L^0_+(I, X_{\theta(\cdot)}) : \Phi_{\varrho_{\theta(\cdot)}}(\lambda f) < \infty \text{ for some } \lambda > 0 \big\}$$

with the same Luxemburg norm. Note that in such situation, $\mathcal{S}(I, X_{\alpha^+})$ is a linear subspace of $L^{\varrho_{\theta(\cdot)}}_+(I, X_{\theta(\cdot)})$.

We say that the $\mathcal{CMN}$ $\{\varrho_\alpha\}$ satisfies the Δ_2–condition, if every ϱ_α satisfies the Δ_2–condition, and the constant C_2 or the related function σ is independent of α. It is easy to see that under this condition, $\Phi_{\varrho_{\theta(\cdot)}}$ is also a Δ_2–type modular. Hence following the same process as in Proposition 2.3, we can prove that

Proposition 3.12. *Under the Δ_2-condition of $\{\varrho_\alpha\}$ and the density assumption of $\{X_\alpha\}$, each function of $L^{\varrho_{\theta(\cdot)}}_+(I, X_{\theta(\cdot)})$ can be approximated by a sequence of $\mathcal{S}(I, X_{\alpha^+})$ according to the norm $\|\cdot\|_{L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})}$.*

Theorem 3.13. *Under the same assumptions as above, $L^{\varrho_{\theta(\cdot)}}_+(I, X_{\theta(\cdot)})$ is a Banach space.*

Proof. Taken any Cauchy sequence $\{f_n\}$ in $L^{p(\cdot)}_+(I, X_{\theta(\cdot)})$, by virtue of Theorem 3.7, there is a function $f \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ for which $\|f_n - f\|_{L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})} \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, on account of Proposition 3.12, there is a $\varphi_n \in \mathcal{S}(I, X_{\alpha^+})$ such that $\|\varphi_n - f_n\|_{L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})} < 1/n$, which in turn yields $\|\varphi_n - f\|_{L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})} \to 0$ as $n \to \infty$. Hence there is a subsequence, say $\{\varphi_n\}$ itself, satisfying $\varphi_n(t) \to f(t)$ in $X_{\theta(t)}$ as $n \to \infty$ for a.e. $t \in I$. Therefore $f \in L^{\varrho_{\theta(\cdot)}}_+(I, X_{\theta(\cdot)})$ and the proof is completed. □

Remark 3.14. *By reviewing the above proof, one can easily find that, under present situations, condition $L^0(I, X_{\theta(\cdot)}) = L^0_+(I, X_{\theta(\cdot)})$ is sufficient for $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) = L^{\varrho_{\theta(\cdot)}}_+(I, X_{\theta(\cdot)})$.*

Theorem 3.15. *Besides the Δ_2–condition of $\{\varrho_\alpha\}$ and the density assumption of $\{X_\alpha\}$, assume that X_{α^+} is separable. Then the function space $L^{\varrho_{\theta(\cdot)}}_+(I, X_{\theta(\cdot)})$ is also separable.*

This theorem is a straight consequence of Proposition 3.12.

For each $\alpha \in \mathcal{A}$, denote by ϱ^*_α the Fenchel duality of ϱ_α, i.e.

$$\varrho^*_\alpha(\xi) = \sup_{u \in X_\alpha} \{\langle \xi, u \rangle_\alpha - \varrho_\alpha(u)\}, \ \xi \in X^*_\alpha.$$

Since ϱ_α is a semimodular, ϱ^*_α is also a semimodular on X^*_α, and the semimodular space derived by ϱ^*_α is exactly X^*_α itself (see [2, §2.2]). Define

$$\tilde{\varrho}^*_\alpha(\xi) = \begin{cases} \varrho^*_\alpha(\xi), & \text{if } \xi \in X^*_\alpha, \\ \infty, & \text{if } \xi \in X^*_{\alpha^+} \setminus X^*_\alpha, \end{cases}$$

then we obtain another family of semimodulars defined on $X^*_{\alpha^+}$, called the dual modular net or in symbol $\mathcal{DMN}$ of $\{\varrho_\alpha\}$. Since for each $\alpha \in \mathcal{A}$, the effective domains and the induced semimodular spaces are equal,

in the coming arguments, we will not distinguish $\tilde{\varrho}_\alpha^*$ and ϱ_α^*, and prefer to use $\{\varrho_\alpha^*\}$ instead of $\{\tilde{\varrho}_\alpha^*\}$ to denote the $\mathcal{DMN}$ of $\{\varrho_\alpha\}$.

Suppose $\alpha_1 \prec \alpha_2$, then $X_{\alpha_1}^* \hookrightarrow X_{\alpha_2}^*$, and for all $\xi \in X_{\alpha_2}^*$, by (6) we have

$$\begin{aligned}\varrho_{\alpha_2}^*(\xi) &= \sup_{u\in X_{\alpha_2}} \{\langle \xi, u\rangle_{\alpha_2} - \varrho_{\alpha_2}(u)\} \\ &\le \sup_{u\in X_{\alpha_2}} \{\langle \xi, u\rangle_{\alpha_2} - \frac{1}{C_1}\varrho_{\alpha_1}(u)\} + \frac{C_2}{C_1} \\ &\le \sup_{u\in X_{\alpha_1}} \{\langle \xi, u\rangle_{\alpha_1} - \frac{1}{C_1}\varrho_{\alpha_1}(u)\} + \frac{C_2}{C_1} = \frac{1}{C_1}\varrho_{\alpha_1}^*(C_1\xi) + \frac{C_2}{C_1}.\end{aligned} \tag{8}$$

Similar to Proposition 3.4, from this property we can show that the dual space family $\{X_\alpha^* : \alpha \in \mathcal{A}\}$, where $\mathcal{A}$ takes the inverse order, is a uniformly bounded net. Moreover, assume that the function $\alpha \mapsto \varrho_\alpha$ is sequently continuous, in other words, if $\{\alpha_k\}$ converges to α in $\mathcal{A}$, then for all $u \in X$, the limit

$$\lim_{k\to\infty} \varrho_{\alpha_k}(u) = \varrho_\alpha(u)$$

holds. Under this assumption, we can deduce that, for all sequences $\{\tilde{\alpha}_k\}$ satisfying $\tilde{\alpha}_k \succ \alpha$ and $\tilde{\alpha}_k \to \alpha$, inequality

$$\varrho_\alpha^*(\xi) \le \limsup_{k\to\infty} \varrho_{\tilde{\alpha}_k}^*(\xi)$$

holds for all $\xi \in X_{\alpha^+}^*$, which in turn leads to the successive property of $\{X_\alpha^*\}$. Unfortunately, the inverse inequality, hence continuity of $\{X_\alpha^*\}$ can not be guaranteed under present situations. For the sake of convenience, hereinafter, we always assume that $X_{\alpha^+}^* \subseteq X$, and the $\mathcal{DMN}$ $\{\varrho_\alpha^*\}$ is assumed to be a $\mathcal{CMN}$ defined on X. We also assume that the $\mathcal{BSN}$ $\{X_\alpha\}$ and its dual net $\{X_\alpha^*\}$ are compatible, i.e. $\langle \xi, u\rangle_{\alpha_2} = \langle \xi, u\rangle_{\alpha_1}$ provided $u \in X_{\alpha_2}$, $\xi \in X_{\alpha_1}^*$ and $\alpha_1 \prec \alpha_2$. This convention has been already used in (8). All the assumptions mentioned above will be used later without any other comments.

Theorem 3.16. *Suppose that the following hypotheses are all satisfied:*
- *$\{\varrho_\alpha\}$ satisfies the Δ_2–condition, and $\mathcal{R}(\partial\varrho_\alpha) = X_\alpha^*$ for all $\alpha \in \mathcal{A}$,*
- *$\{X_\alpha\}$ is a dense $\mathcal{BSN}$, and X_{α^+} is separable,*
- *for every $\alpha \in \mathcal{A}$, ϱ_α^* is a modular, and*
- *X_α^* has the Radon-Nikodym's property w.r.t. every $J \in \Pi(I)$.*

Then the dual space $L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})^*$ is equivalent to $L^{\varrho_{\theta(\cdot)}^*}(I, X_{\theta(\cdot)}^*)$ in the sense of isomorphism.

Proof. Firstly for each $\xi \in L^{\varrho_{\theta(\cdot)}^*}(I, X_{\theta(\cdot)}^*)$, define the linear functional Λ_ξ as follows:

$$\langle\langle \Lambda_\xi, f\rangle\rangle_{\theta(\cdot)} = \int_I \langle \xi(t), f(t)\rangle_{\theta(t)} dt, \quad \forall\, f \in L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}). \tag{9}$$

Suppose that $\|\xi\|_{L^{\varrho_{\theta(\cdot)}^*}(I, X_{\theta(\cdot)}^*)} = \|f\|_{L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})} = 1$, then by Young's inequality we have

$$\langle\langle \xi, f\rangle\rangle_{\theta(\cdot)} \le \int_I \varrho_{\theta(t)}^*(\xi(t))dt + \int_I \varrho_{\theta(t)}(f(t))dt \le 2.$$

Therefore $\Lambda_\xi \in L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})^*$, and $\|\Lambda_\xi\|_{L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})^*} \le 2\|\xi\|_{L^{\varrho_{\theta(\cdot)}^*}(I, X_{\theta(\cdot)}^*)}$. This claim also holds for arbitrary $\xi \in L^{\varrho_{\theta(\cdot)}^*}(I, X_{\theta(\cdot)}^*)$ by scaling arguments.

Conversely, given a functional $\Lambda \in L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})^*$, we will find a function $\xi \in L^{\varrho_{\theta(\cdot)}^*}(I, X_{\theta(\cdot)}^*)$ such that $\Lambda = \Lambda_\xi$ in the sense of (9) with the norm equivalent to that of ξ. If $\Lambda = 0$, then we take $\xi = 0$ and there is nothing to do. If $\Lambda \neq 0$, then without loss of generality, assume that $\|\Lambda\|_{L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})^*} = 1$. Taking any $J \in \Pi(I)$ and any $f \in L^{\varrho_{\theta_J^+}}(J, X_{\theta_J^+})$, consider the zero extension of f out of J and denote it by $\tilde{f}$. Obviously, $\tilde{f} \in L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ and

$$\|\tilde{f}\|_{L_+^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})} \le C\|f\|_{L^{\varrho_{\theta_J^+}}(J, X_{\theta_J^+})} \tag{10}$$

for some constant $C > 0$ depending on $|J|$ but independent of f, which means that the restriction of Λ to $L^{\theta_J^+}(J, X_{\theta_J^+})$, denoted by $\Lambda|_{J,\theta_J^+}$, lies in $L^{\theta_J^+}(J, X_{\theta_J^+})^*$. So by invoking Theorem 2.7, there is a unique function $\xi_J \in L^{\varrho^*_{\theta_J^+}}(I, X^*_{\theta_J^+})$ such that

$$\langle\langle \Lambda|_{J,\theta_J^+}, f\rangle\rangle_{\theta_J^+} = \int_J \langle \xi_J(t), f(t)\rangle_{\theta_J^+} dt$$

for all $f \in L^{\theta_J^+}(J, X_{\theta_J^+})$, and

$$\|\xi_J\|_{L^{\varrho^*_{\theta_J^+}}(I, X^*_{\theta_J^+})} \le C\|\Lambda|_{J,\theta_J^+}\|_{L^{\varrho_{\theta_J^+}}(J, X_{\theta_J^+})^*} \le C\|\Lambda\|_{L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})^*} \tag{11}$$

for some constant $C > 0$ depending only on the length of J.

Suppose that $J_1, J_2 \in \Pi(I)$ satisfy $J_2 \subseteq J_1$, then $L^{\varrho_{\theta_{J_2}^+}}(J, X_{\theta_{J_2}^+})$ is densely imbedded in $L^{\varrho_{\theta_{J_1}^+}}(J, X_{\theta_{J_1}^+})$, hence by the uniqueness of the representation ξ_{J_1}, we can assert that $\xi_{J_2}(t) = \xi_{J_1}(t)$ for a.e. $t \in J_1$. Define $\xi(t) = \xi_J(t)$ if $t \in J$ for arbitrary $J \in \Pi(I)$, then we obtain a well defined function $\xi \in L^0_-(I, X^*_{\theta(\cdot)})$.

In case that $I = [0, T]$ is bounded, all the constants C in (10) and (11) can be selected independent of $J \in \Pi(I)$, thus via Corollary 3.10, we can derive that $\xi \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$ and $\|\xi\|_{L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})} \le C\|\Lambda\|_{L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})^*}$ for some $C > 0$ independent of Λ.

For each $f \in L^{\varrho_{\theta(\cdot)}}_+(I, X_{\theta(\cdot)})$, select a sequence $\{\varphi_k\} \subseteq \mathcal{S}(I, X_{\alpha^+})$ converging to f according to the $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$–norm. Since for every $k \in \mathbb{N}$,

$$\langle\langle \Lambda, \varphi_k\rangle\rangle_{\theta(\cdot)} = \langle\langle \Lambda|_{I,\alpha^+}, \varphi_k\rangle\rangle_{\alpha^+} = \int_I \langle \xi(t), \varphi_k(t)\rangle_{\alpha^+} dt = \int_I \langle \xi(t), \varphi_k(t)\rangle_{\theta(t)} dt,$$

letting $k \to \infty$, we obtain

$$\langle\langle \Lambda, f\rangle\rangle_{\theta(\cdot)} = \int_I \langle \xi(t), f(t)\rangle_{\theta(t)} dt,$$

which shows that $\Lambda = \Lambda_\xi$.

It remains to prove (9) in the case $I = [0, \infty)$. Firstly the above discussions tell us that $\xi \in L^{\varrho^*_{\theta(\cdot)}}(J, X^*_{\theta(\cdot)})$ for all $J \in \Pi(I)$. Consequently $\xi \in L^0(I, X^*_{\theta(\cdot)})$, and the scalar function $t \mapsto \langle \xi(t), f(t)\rangle_{\theta(t)}$ is measurable on I whenever $f \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$.

Given a function $f \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, for each $n \in \mathbb{N}$, let $f_n = f\chi_{[0,n]}$, then we obtain an approximate sequence of f in $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ satisfying

$$\langle\langle \Lambda, f_n\rangle\rangle_{\theta(\cdot)} = \langle\langle \Lambda|_{[0,n]}, f_n\rangle\rangle_{\theta(\cdot)} = \int_{[0,n]} \langle \xi(t), f(t)\rangle_{\theta(t)} dt.$$

Let $n \to \infty$, using the fact $\lim_{n\to\infty}\langle\langle \Lambda, f_n\rangle\rangle_{\theta(\cdot)} = \langle\langle \Lambda, f\rangle\rangle_{\theta(\cdot)}$, we can deduce that $t \mapsto \langle \xi(t), f(t)\rangle_{\theta(t)}$ is integrable on I, and

$$\langle\langle \Lambda, f\rangle\rangle_{\theta(\cdot)} = \int_I \langle \xi(t), f(t)\rangle_{\theta(t)} dt.$$

Thus $\Lambda = \Lambda_\xi$ by the arbitrariness of f. The remaining task for us is to show $\xi \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$. For this purpose, notice that the effective domain $\mathcal{D}(\varrho_\alpha)$ is equal to X_α and the latter is separable, so the dual modular ϱ^*_α can be represented by

$$\varrho^*_\alpha(\eta) = \sup_{k\ge 1}\big\{\langle \eta, v_k\rangle_\alpha - \varrho_\alpha(v_k)\big\}, \ \forall\ \eta \in X^*_\alpha, \tag{12}$$

where $\{v_k\}$ is a countable dense subset of X_α. By the density of $\{X_\alpha\}$, if we take $\{v_k\}$ as the dense sequence of X_{α^+} with $v_1 = 0$, then (12) holds with $\alpha = \theta(t)$ and $\eta \in X^*_{\theta(t)}$ for all $t \in I$.

For each $n \in \mathbb{N}$, define

$$r_n(t) = \chi_{[0,n]}(t) \max_{1\le k\le n}\big\{\langle \xi(t), v_k\rangle_{\theta(t)} - \varrho_{\theta(t)}(v_k)\big\}.$$

Obviously, $\{r_n\}$ is a nondecreasing sequence of nonnegative ($v_1 = 0$) measurable functions converging to $\varrho^*_{\theta(t)}(\xi(t))$ almost everywhere. Moreover, there is a sequence of simple functions $\{s_n\} \subseteq \mathcal{S}(I, X_{\alpha^+})$ such that

$$r_n(t) = \langle \xi(t), s_n(t)\rangle_{\theta(t)} - \varrho_{\theta(t)}(s_n(t)).$$

Due to the facts $\mathcal{S}(I, X_{\alpha^+}) \subseteq L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ and $\|\Lambda\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})^*} = 1$, we have

$$\begin{aligned} 1 &\geq \Phi^*_{\varrho_{\theta(\cdot)}}(\Lambda) \geq \langle\langle \Lambda, s_n\rangle\rangle - \Phi_{\varrho_{\theta(\cdot)}}(s_n) \\ &= \int_I \left\{\langle \xi(t), s_n(t)\rangle_{\theta(t)} - \varrho_{\theta(t)}(s_n(t))\right\} dt, \end{aligned}$$

where $\Phi^*_{\varrho_{\theta(\cdot)}}$ is the dual modular of $\Phi_{\varrho_{\theta(\cdot)}}$. Taking limit of the second line as $n \to \infty$, we obtain

$$\Phi_{\varrho^*_{\theta(\cdot)}}(\xi) = \int_I \varrho^*_{\theta(t)}(\xi(t))dt \leq 1.$$

Therefore $\xi \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$ and $\|\xi\|_{L^{\varrho^*_{\theta(\cdot)}}(I,X^*_{\theta(\cdot)})} \leq 1$.

Finally by means of scaling transformation, we can obtain the desired estimate

$$\|\xi\|_{L^{\varrho^*_{\theta(\cdot)}}(I,X^*_{\theta(\cdot)})} \leq \|\Lambda\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})^*}.$$

Thus we have completed the proof. □

Remark 3.17. *There is a by-product produced from the above proof, that is under all the hypotheses of Theorem 3.16, we have $\Phi_{\varrho^*_{\theta(\cdot)}}(\xi) = \Phi^*_{\varrho_{\theta(\cdot)}}(\Lambda_\xi)$ for all $\xi \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$. This is a natural extension of that of the scalar case.*

Corollary 3.18. *In addition to the assumptions of the above theorem, assume that $L^0(I, X_{\theta(\cdot)}) = L^0_+(I, X_{\theta(\cdot)})$, then*

$$L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})^* \cong L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)}). \tag{13}$$

Theorem 3.19. *Suppose the following conditions are all satisfied.*
- *both $\{\varrho_\alpha\}$ and $\{\varrho^*_\alpha\}$ satisfy the Δ_2–condition,*
- *$\{X_\alpha\}$ and $\{X^*_\alpha\}$ are two dense $\mathcal{BSN}$s,*
- *X_{α^+} and $X^*_{\alpha^-}$ are both separable, and*
- *for every $\alpha \in \mathcal{A}$, X_α is reflexive,*
- *$L^0(I, X_{\theta(\cdot)}) = L^0_+(I, X_{\theta(\cdot)})$, and $L^0(I, X^*_{\theta(\cdot)}) = L^0_+(I, X^*_{\theta(\cdot)})$.*

Then $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ is a reflexive space.

Given a $\mathcal{CMN}$ $\{\varrho_\alpha\}$, assume that it is uniformly convex, in other words, every ϱ_α is uniformly convex, and for each $\varepsilon \in (0, 1)$, the corresponding number $\delta \in (0, 1)$ appearing in (5) is independent of α.

Theorem 3.20. *Under the uniform convexity assumption and the Δ_2–condition of $\{\varrho_\alpha\}$, the function space $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ is uniformly convex.*

Remark 3.21. *Putting all the hypotheses in Theorem 3.19, 3.20 together, we obtain not only the uniform convexity of $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, but the representation (13) as well.*

Example 3.22. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, and let $\mathcal{P}(\Omega)$ be the set of all measurable functions taking values in $[1, \infty]$, and*

$$\mathcal{P}_b(\Omega) = \{p \in \mathcal{P}(\Omega) : 1 \leq p^- \leq p^+ < \infty\},$$

where notations p^+ and p^- denote the essential supremum and infimum of p on Ω respectively. For any $p \in \mathcal{P}_b(\Omega)$, functional

$$\varrho_p(f) = \int_\Omega \frac{1}{p(x)}|f(x)|^{p(x)}dx$$

is a continuous modular on the linear space $X = L^0(\Omega)$, which induces a separable Banach space $X_p := L^{p(x)}(\Omega)$. Evidently ϱ_p satisfies the Δ_2-condition with the function $\omega(t) = t^{p^+}$. If in addition $p^- > 1$, then ϱ_p is uniformly convex, its dual modular ϱ_p^* equals $\varrho_{p'}$, and the dual space $L^{p(x)}(\Omega)^*$ is equivalent to $L^{p'(x)}(\Omega)$. Here $p'(x)$ is the conjugate exponent of $p(x)$, that is $1/p(x) + 1/p'(x) = 1$ for a.e. $x \in \Omega$. It is also easy to see that, $L^{p_2(x)}(\Omega)$ is a dense subspace of $L^{p_1(x)}(\Omega)$ provided $p_1(x) \le p_2(x)$ a.e. on Ω.

Fix two numbers $\underline{p}$ and $\bar{p}$ in $[1, \infty)$ with $\underline{p} \le \bar{p}$, let

$$\mathcal{A}_b = \{p \in \mathcal{P}_b(\Omega) : p(x) \in [\underline{p}, \bar{p}] \text{ for a.e. } x \in \Omega\}.$$

Then equipped with the order: $p \prec q$ by $p(x) \le q(x)$ a.e. on Ω, and the topology determined by: $p_n \to p$ in $\mathcal{A}_b$ if and only if $p_n(x) \to p(x)$ a.e. on Ω, $\mathcal{A}_b$ becomes a $\mathcal{BTL}$. Meanwhile, $\{\varrho_p : p \in \mathcal{A}_b\}$ is a $\mathcal{CMN}$ defined on X satisfying the Δ_2-condition with the common function $\omega(t) = t^{\bar{p}}$, and $\{X_p : p \in \mathcal{A}_b\}$ is a dense regular $\mathcal{BSN}$ generated by $\{\varrho_p\}$ (cf. [11]).

Assume that $I = [0, T]$, and $Q = I \times \Omega$ is a cylinder. Recall that, each $u \in L^0(Q)$ has an $X-$realization Pu in $L^0(I, X)$ satisfying $Pu(t)(x) = u(t, x)$ for a.e. $x \in \Omega$ and a.e. $t \in I$, and conversely, each $u \in L^0(I, X)$ has a scalar realization $\tilde{u}$ in $L^0(\Omega)$ satisfying $\tilde{u}(t, x) = u(t)(x)$ for a.e. $(t, x) \in Q$. Moreover, for all $q \in (1, \infty)$, the projection $P : L^q(Q) \to L^q(I, L^q(\Omega))$ is a linear isometrical isomorphism with the inverse $P^{-1}u = \tilde{u}$. If $q \in \mathcal{P}_b(\Omega)$ is a variable exponent, then $P : L^{q(\cdot)}(Q) \to L^{q^-}(I, L^{q(\cdot)}(\Omega))$ is also continuous (refer to [7]). In the following discussion we will omit the notation P and simply use a single letter u to represent a scalar function and its $X-$realization, or an $X-$valued function and its scalar realization without any other remarks, if there is no confusion arising.

Suppose that $p \in \mathcal{P}_b(Q)$ is a Caratheodory type function satisfying

1. *$p(t, \cdot)$ is measurable on Ω for every $t \in I$, and*
2. *$p(\cdot, x)$ is continuous on I for a.e. $x \in \Omega$.*

Let $\underline{p} = p_Q^-$, $\bar{p} = p_Q^+$, and define $\theta(t) = p(t, \cdot) =: p(t)$, then we obtain an order-continuous exponent $\theta : I \to \mathcal{A}_b$ with

$$\theta_J^+ = \sup\{p(t, x) : t \in J\},\ \theta_J^- = \inf\{p(t, x) : t \in J\}$$

for all $J \in \Pi(I)$, and $X_{\theta(t)} = L^{p(t,x)}(\Omega)$, $X^*_{\theta(t)} = L^{p'(t,x)}(\Omega)$ for all $t \in I$. [11] reveals that

$$L^0(I; L^{p(\cdot,x)}(\Omega)) = L^0_+(I; L^{p(\cdot,x)}(\Omega)).$$

Thus, $L^{\varrho_{p(\cdot)}}(I; L^{p(\cdot,x)}(\Omega)) = L^{\varrho_{p(\cdot)}}_+(I; L^{p(\cdot,x)}(\Omega))$ is a separable Banach space. Furthermore, if $\underline{p} > 1$, then

$$L^0(I; L^{p'(\cdot,x)}(\Omega)) = L^0_+(I; L^{p'(\cdot,x)}(\Omega)),$$

and all the assumptions arising in Theorem 3.19, 3.20 are fulfilled, thus by Remark 3.21, we get the uniform convexity, hence the reflexivity of $L^{\varrho_{p(\cdot)}}(I; L^{p(\cdot,x)}(\Omega))$, together with the expression

$$L^{\varrho_{p(\cdot)}}(I; L^{p(\cdot,x)}(\Omega))^* \cong L^{\varrho_{p'(\cdot)}}(I; L^{p'(\cdot,x)}(\Omega)).$$

It is worth mentioning that for the same exponent $p(\cdot, \cdot)$, projection P is also an isometrical isomorphism from $L^{p(t,x)}(Q)$ onto $L^{\varrho_{p(\cdot)}}(I; L^{p(\cdot,x)}(\Omega))$ (see [11] for references). This is a natural extension of the property of P from the case of constant exponents to the case of variable ones.

4 Functionals and Operators On $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$

In this section we will study some functionals and operators on the function space $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, including the subdifferential of $\Phi_{\varrho_{\theta(\cdot)}}$, whose representation will be taken into account. For this purpose, we need the coercive assumption on ϱ_α, ϱ_α^*, as well as $\Phi_{\varrho_{\theta(\cdot)}}$ and $\Phi_{\varrho^*_{\theta(\cdot)}}$. Coercivity, which says

$$\frac{\varrho(u)}{\|u\|_X} \to \infty \text{ as } \|u\|_X \to \infty,$$

is an important property of a lower semicontinuous (or lsc for short) and proper convex function ϱ defined on a Banach space X. Using the coercive property of ϱ, we can obtain the boundedness of a sequence in X under some situations. For example, if there is a sequence $\{u_n\} \subseteq X$ satisfying

$$\frac{\varrho(u_n)}{\|u_n\|_X} \le K \text{ for some } K > 0,$$

then there is a constant $C > 0$ depending only on K such that $\|u_n\|_X \le C$ for all $n \in \mathbb{N}$.

It is easy to check that if ϱ_α is a coercive modular, then its dual ϱ_α^* satisfies $\mathrm{Dom}(\varrho_\alpha^*) = X_\alpha^*$. As a matter of fact, taking any $\xi \in X_\alpha^*$, by the coercivity of ϱ_α, there is a constant $M > 0$ for which $\varrho_\alpha(u) \ge (\|\xi\|_\alpha^* + 1)\|u\|_\alpha$ provided $\|u\|_\alpha \ge M$. Consequently,

$$\varrho_\alpha^*(\xi) = \sup_{\|u\|_\alpha \le M} \{\langle \xi, u\rangle_\alpha - \varrho_\alpha(u)\} \le M\|\xi\|_\alpha^* < \infty.$$

In general, coercivity of $\Phi_{\varrho_{\theta(\cdot)}}$ could not be derived from the coercive assumption of all the ϱ_αs naturally. Under some special conditions, however, all of ϱ_α, $\alpha \in \mathcal{A}$ and $\Phi_{\varrho_{\theta(\cdot)}}$ are coercive simultaneously. The following assumption, which is called strong coercivity of $\{\varrho_\alpha\}$, is a desired one.

$$\varrho_\alpha\Big(\frac{u}{\gamma^{-1}(s)}\Big) \le \frac{\varrho_\alpha(u)}{s} \text{ for all } u \in X,\ s > 0 \text{ and } \alpha \in \mathcal{A}, \tag{14}$$

where $\gamma : [1, \infty) \to [1, \infty)$ is strictly increasing function satisfying

$$\lim_{s\to\infty} \frac{\gamma(s)}{s} = \infty. \tag{15}$$

By (15), there is a constant $K \ge 1$ such that $\gamma^{-1}(s) \le s$ whenever $s \ge K$. Now taking any $\alpha \in \mathcal{A}$ and $u \in X_\alpha$ with $\|u\|_\alpha \ge 2K$, and using (14), we can deduce that

$$\frac{\varrho_\alpha(u)}{\|u\|_\alpha} \ge \frac{1}{2}\varrho_\alpha\Big(\frac{u}{\gamma^{-1}(\|u\|_\alpha/2)}\Big) \ge \frac{1}{2}\frac{\|u\|_\alpha/2}{\gamma^{-1}(\|u\|_\alpha/2)}\varrho_\alpha\Big(\frac{u}{\|u\|_\alpha/2}\Big) \ge \frac{1}{2}\frac{\|u\|_\alpha/2}{\gamma^{-1}(\|u\|_\alpha/2)},$$

which combined with (15) yields the coercivity of ϱ_α.

Furthermore, for any $u \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ with $\|u\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})} \ge 1$, we have that

$$\int_I \varrho_{\theta(t)}\Big(\frac{u(t)}{\gamma^{-1}(\Phi_{\varrho_{\theta(\cdot)}}(u))}\Big)dt \le \frac{1}{\Phi_{\varrho_{\theta(\cdot)}}(u)}\int_I \varrho_{\theta(t)}(u(t))dt = 1,$$

which means $\|u\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})} \le \gamma^{-1}(\Phi_{\varrho_{\theta(\cdot)}}(u))$. Consequently,

$$\lim_{\|u\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})}\to\infty} \frac{\Phi_{\varrho_{\theta(\cdot)}}(u)}{\|u\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})}} \ge \lim_{\Phi_{\varrho_{\theta(\cdot)}}(u)\to\infty} \frac{\Phi_{\varrho_{\theta(\cdot)}}(u)}{\gamma^{-1}(\Phi_{\varrho_{\theta(\cdot)}}(u))} = \infty,$$

which shows the coercivity of $\Phi_{\varrho_{\theta(\cdot)}}$.

In the following arguments, we also need the strict convexity of ϱ_α^* for all $\alpha \in \mathcal{A}$, and the weak lower-semicontinuity of $\{\varrho_\alpha\}$ and $\{\varrho_\alpha^*\}$, which says

– For any sequence $\{\alpha_k\}$ approaching β in $\mathcal{A}$ and any sequence $\{u_k\} \subseteq X_\beta$ converging weakly to u in X_β, $\{\xi_k\}$ converging star-weakly to ξ in X_β^*, we have

$$\varrho_\beta(u) \le \liminf_{k\to\infty} \varrho_{\alpha_k}(u_k) \quad \text{and} \quad \varrho_\beta^*(\xi) \le \liminf_{k\to\infty} \varrho_{\alpha_k}^*(\xi_k). \tag{16}$$

For the sake of convenience, all the hypotheses listed in Theorem 3.19, 3.20 are denoted by $H(A)$ as a whole. Furthermore, assumptions of strong coercivity of ϱ_α and ϱ_α^*, and weak lower-semicontinuity of $\{\varrho_\alpha\}$ and $\{\varrho_\alpha^*\}$ are put together, denoted by $H(B)'$, which jointly with the strict convexity assumption of ϱ_α and ϱ_α^* for all $\alpha \in \mathcal{A}$ are denoted by $H(B)$. Without any other specific comments, in further discussion we always assume that $H(A)$ and $H(B)$ are both verified for the given $\mathcal{CMN}$ $\{\varrho_\alpha\}$ and its $\mathcal{DMN}$ $\{\varrho_\alpha^*\}$.

Consider the subdifferential operator $\partial\varrho_\alpha$. Similar to Remark (2.8), we can check that $\partial\varrho_\alpha$ is single-valued provided ϱ_α^* is strict convex. Furthermore, by the coercivity of ϱ_α, we can also find that $\partial\varrho_\alpha : X_\alpha \to X_\alpha^*$ is a demicontinuous, monotone and coercive operator, whose range is the whole space X_α^*. Since

$$\langle \partial\varrho_\alpha(u), u\rangle_\alpha = \varrho_\alpha(u) + \varrho_\alpha^*(\partial\varrho_\alpha(u))$$

for all $u \in X_\alpha$, $\varrho_\alpha(u)$ can be regarded as the extension of the traditional dual map where not modulars but norms of X_ϱ and X_α^* are involved.

Lemma 4.1. *For all $u \in L^\infty(I, X_\alpha)$, the compound operator $t \mapsto \partial\varrho_\alpha(u(t))$ lies in $L^\infty(I, X_\alpha^*)$. Moreover, if $u \in L^{\varrho_\alpha}(I, X_\alpha)$, then $\partial\varrho_\alpha(u(\cdot)) \in L^{\varrho_\alpha^*}(I, X_\alpha^*)$.*

Proof. By the demicontinuity of $\partial\varrho_\alpha$, one can easily see that the function $\xi(t) = \partial\varrho_\alpha(u(t))$ is strongly measurable. Furthermore by Remark 2.6, we have

$$\varrho_\alpha^*(\xi(t)) + \varrho_\alpha(u(t)) = \langle \xi(t), u(t)\rangle_\alpha \le \|\xi(t)\|_\alpha^* \|u(t)\|_\alpha,$$

which together with the coercivity of ϱ_α^*, yields the boundedness of $\varrho_\alpha^*(\xi(t))$ uniformly for a.e. $t \in I$, hence the inclusion $\xi \in L^\infty(I, X_{\varrho_\alpha}^*)$.

Suppose that $u \in L^{\varrho_\alpha}(I, X_{\varrho_\alpha})$, and $\{s_k\} \subseteq \mathcal{S}(I, X_{\varrho_\alpha})$ converges to u in $L^{\varrho_\alpha}(I, X_\alpha)$. From above arguments, we know that for every $k \in \mathbb{N}$, the X_α^*–valued simple function $\xi_k(t) = \partial\varrho_\alpha(s_k(t))$ lies in $L^\infty(I, X_\alpha^*)$, consequently it lies $L^{\varrho_\alpha^*}(I, X_{\varrho_\alpha}^*)$, and

$$\varrho_\alpha^*(\xi_k(t)) + \varrho_\alpha(s_k(t)) = \langle \xi_k(t), s_k(t)\rangle_\alpha \text{ a.e. on } I. \tag{17}$$

Taking integrations on both sides and using generalized Hölder's inequality, we have

$$\Phi_{\varrho_\alpha^*}(\xi_k) + \Phi_{\varrho_\alpha}(s_k) = \int_I \langle \xi_k), s_k(t)\rangle_\alpha dt \le 2\|\xi_k\|_{L^{\varrho_\alpha^*}(I, X_\alpha^*)} \|s_k\|_{L^{\varrho_\alpha}(I, X_\alpha)}.$$

Then by the coercivity of $\Phi_{\varrho_\alpha^*}$ and the boundedness of $\{\xi_k\}$ in $L^{\varrho_\alpha}(I, X_\alpha)$, we get the boundedness of $\{\xi_k\}$ in $L^{\varrho_\alpha^*}(I, X_\alpha^*)$. Therefore there is a subsequence, say $\{\xi_k\}$ itself, convergent to some $\tilde{\xi}$ weakly in $L^{\varrho_\alpha^*}(I, X_\alpha^*)$. Suppose that $\{v_i\}$ is a countable dense subset of X_α, then for every two positive integers i, n, we have

$$\lim_{k\to\infty} \int_{I_n} \langle \xi_k(t) - \tilde{\xi}(t), v_i\rangle_\alpha dt = 0.$$

It follows that the scalar function $h_k(t) = \langle \xi_k(t) - \tilde{\xi}(t), v_i\rangle_\alpha$ is convergent to 0 in measure on I_n. As a result, $\{h_k\}$ has a subsequence convergent to 0 a.e. on I_n. Then by means of the diagonalizing method, we can find another subsequence, denoted still by $\{h_k\}$ such that $\lim_{k\to\infty} h_k(t) = 0$ for a.e. $t \in I$, which combined with the boundedness of $\{\xi_k(t)\}$ derived from (17) and the density of $\{v_i\}$ in X_α, results in the weak convergence of $\xi_k(t)$ to $\tilde{\xi}(t)$ in X_α^* as $k \to \infty$ for a.e. $t \in I$. Now taking limits in (17), and using continuity of ϱ_α and weak lower-continuity of ϱ_α^*, we obtain

$$\begin{aligned}
\varrho_\alpha^*(\tilde{\xi}(t)) + \varrho_\alpha(u(t)) &\le \liminf_{k\to\infty} \varrho_\alpha^*(\xi_k(t)) + \lim_{k\to\infty} \varrho_\alpha(s_k(t)) \\
&\le \liminf_{n\to\infty} \left(\varrho_\alpha^*(\xi_k(t)) + \varrho_\alpha(u(t))\right) \\
&\le \lim_{k\to\infty} \langle \xi_k(t), s_k(t)\rangle_\alpha = \langle \tilde{\xi}(t), u(t)\rangle_\alpha,
\end{aligned}$$

which in turn tields $\tilde{\xi}(t) = \partial\varrho_\alpha(u(t))$ for a.e. $t \in I$. Thus the lemma has been proved since $\tilde{\xi} \in L^{\varrho_\alpha^*}(I, X_{\varrho_\alpha}^*)$. □

In studying the subdifferential of the Nemytzkij functional of a series of modulars, we always assume that $I = [0, T]$.

Proposition 4.2. *For each $u \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, the $X_{\theta(\cdot)}^*$–valued function $\partial\varrho_{\theta(t)}(u(t))$ belongs to $L^{\varrho_{\theta(\cdot)}^*}(I, X_{\theta(\cdot)}^*)$.*

Proof. For every 2^n-mean partition of I, define the step function $\theta_n^-(t)$ as in the preceding section, and let $\xi_n(t) = \partial\varrho_{\theta_n^-(t)}(u(t))$ for a.e. $t \in I$. From Lemma 4.1, we can derive that ξ_n lies in $L^{\varrho_{\theta_n^-(\cdot)}}(I, X^*_{\theta_n^-(\cdot)})$, hence it lies in $L^{\varrho_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$. Much similar to the proof of Lemma 4.1, and using the strong coercivity of $\{\varrho^*_\alpha\}$, density of $\{\varrho_\alpha\}$, separability of X_{α^+} together with the relations $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) = L^{\varrho_{\theta(\cdot)}}_+(I, X_{\theta(\cdot)})$ and $L^{\varrho^*_{\theta_n^-(\cdot)}}(I, X^*_{\theta_n^-(\cdot)}) \hookrightarrow L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$, we can deduce that

$$\varrho^*_{\theta_n^-(t)}(\xi_n(t)) + \varrho_{\theta_n^-(t)}(u(t)) = \langle \xi_n(t), u(t)\rangle_{\theta_n^-(t)} \text{ a.e. on } I, \tag{18}$$

$\{\xi_n\}$ is bounded in $L^{\varrho^*_{\theta_n^-(\cdot)}}(I, X^*_{\theta_n^-(\cdot)})$ and consequently in $L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$. Thus by passing to a subsequence, we can assume that $\{\xi_n\}$ converges weakly to some ξ in $L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$, and $\xi_n(t) \rightharpoonup \xi(t)$ in $X^*_{\theta(t)}$ as $n \to \infty$ a.e. on I. Now taking limits in (18), and using (16), we obtain

$$\begin{aligned}\varrho^*_{\theta(t)}(\xi(t)) + \varrho_{\theta(t)}(u(t)) &\le \liminf_{n\to\infty} \varrho^*_{\theta_n^-(t)}(\xi_n(t)) + \lim_{n\to\infty} \varrho_{\theta_n^-(t)}(u(t)) \\ &\le \lim_{n\to\infty} \langle \xi_n(t), u(t)\rangle_{\theta_n^-(t)} = \langle \xi(t), u(t)\rangle_{\theta(t)}\end{aligned}$$

for a.e. $t \in T$. Therefore $\xi(t) = \partial\varrho_{\theta(t)}(u(t))$ almost everywhere and the proposition has been proved. □

Remark 4.3. *In terms of Lemma 4.1 and Remark 3.17, we can claim that for every $u \in L^{\varrho_\alpha}(I, X_{\varrho_\alpha})$, the subdifferential $\partial\Phi_{\varrho_\alpha}(u)$ equals $\partial\varrho_\alpha(u(\cdot))$. Moreover, if we drop the strict convexity assumption of ϱ^*_α, then we have the classical representation:*

$$\partial\Phi_{\varrho_\alpha}(u) = \left\{\xi \in L^{\varrho^*_\alpha}(I, X^*_\alpha) : \xi(t) \in \partial\varrho_\alpha(u(t)) \text{ for a.e. } t \in I\right\}.$$

Similarly, under the conditions of Proposition 4.2, we have $\partial\Phi_{\varrho_{\theta(\cdot)}}(u) = \partial\varrho_{\theta(\cdot)}(u(\cdot))$ for all $u \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, and

$$\partial\Phi_{\varrho_{\theta(\cdot)}}(u) = \left\{\xi \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)}) : \xi(t) \in \partial\varrho_{\theta(t)}(u(t)) \text{ for a.e. } t \in I\right\},$$

if the strict convexity assumptions of ϱ^*_α for all $\alpha \in \mathcal{A}$ are moved.

The following theorem is a natural corollary of Proposition 4.2 and Remark 4.3.

Theorem 4.4. *Under hypotheses $H(A)$ and $H(B)$, the operator $Z_{\theta(\cdot)}$ defined through $Z_{\theta(\cdot)}(u) = \partial\varrho_{\theta(\cdot)}(u(\cdot))$ is demicontinuous, coercive and bounded, together with*

$$\langle\langle Z_{\theta(\cdot)}(u), u\rangle\rangle_{\theta(\cdot)} = \Phi_{\varrho_{\theta(\cdot)}}(u) + \Phi_{\varrho^*_{\theta(\cdot)}}(u)$$

for all $u \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$. In a word, $Z_{\theta(\cdot)} : L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) \to L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$ is a generalized dual map.

Remark 4.5. *Due to the facts that $\mathcal{D}(\Phi_{\theta(\cdot)}) = L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ and $\partial\Phi_{\theta(\cdot)} = Z_{\theta(\cdot)}$ is single-valued, functional $\Phi_{\theta(\cdot)}$ is Gâteaux differential, and its Gâteaux differential $\Phi'_{\theta(\cdot)}$ equals $Z_{\theta(\cdot)}$.*

Suppose that $\mathcal{B}$ is another $\mathcal{BTL}$, $\{\varphi_\beta : \beta \in \mathcal{B}\}$ is another $\mathcal{CMN}$ on $Y \subseteq X$ and $\{V_\beta : \beta \in \mathcal{B}\}$ is the corresponding $\mathcal{BSN}$ generated by $\{\varphi_\beta\}$. We say $\{\varphi_\beta\}$ is stronger than $\{\varrho_\alpha\}$, we mean that

- V_β is imbedded continuously and densely in X_α, and the dual product $\langle\xi, u\rangle$ has the same value in both $V^*_\beta \times V_\beta$ and $X^*_\alpha \times X_\alpha$ for all $u \in V_\beta$ and $\xi \in X^*_\alpha$, and all $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$.

Under this assumption, for all $C > 0$,

$$E_C = \{u \in X_\alpha : \varphi_\beta(u) \le C\}$$

is a bounded and weakly closed subset of V_β. Hence by the inclusion $V_\beta \hookrightarrow X_\alpha$, we have

Lemma 4.6. *If V_β is reflexive, then φ_β is also a lower semicontinuous and convex function on X_α.*

Corollary 4.7. *Under the reflexivity assumption of V_β, for every $u \in L^0(I, X_\alpha)$, the multifunction $t \mapsto \varphi_\beta(u(t))$ is measurable.*

Suppose that $\vartheta : I \to \mathcal{B}$ is also an order-continuous map, then based on the above results and the continuity of $\{\varphi_\beta\}$, we can check that

Proposition 4.8. *Assume that for every $\beta \in \mathcal{B}$, the modular space V_β is reflexive. Then for each $u \in L^0(I, X_{\theta(\cdot)})$, functions $\{\varphi_{\vartheta_n^-(\cdot)}(u(\cdot))\}_{n=1}^\infty$ are all measurable, hence as the limit function, $\varphi_{\vartheta(\cdot)}(u(\cdot))$ is also measurable.*

Now we can define the Nemytzkij functional of $\{\varphi_{\vartheta(t)} : t \in I\}$ through

$$\Phi_{\varphi_{\vartheta(\cdot)}}(u) = \int_I \varphi_{\vartheta(t)}(u(t))dt, \quad u \in L^0(I, X_{\theta(\cdot)}).$$

Lemma 4.9. *Suppose that φ_β satisfies the Δ_2–condition, and V_β is a reflexive and separable space. Then a function $u \in L^0(I, X_\alpha)$ is also a member of $L^{\varphi_\beta}(I, V_\beta)$ provided $\Phi_{\varphi_\beta}(u) < \infty$.*

Proof. By the condition $\Phi_{\varphi_\beta}(u) < \infty$, it suffices to show the inclusion $u \in L^0(I, V_\beta)$. Taking any $r > 0$ and $u_0 \in V_\beta$, denote by

$$B_{V_\beta}(u_0, r) = \{u \in V_\beta : \|u - u_0\|_{V_\beta} < r\},$$

and

$$B_{\varphi_\beta}(u_0, r) = \{u \in V_\beta : \varphi_\beta(u - u_0) < r\}.$$

By the unit ball property, we know that $B_{V_\beta}(u_0, r) \subseteq B_{\varphi_\beta}(u_0, r)$ provided $0 < r < 1$. Moreover, for each $r > 0$, by the Δ_2–condtion of φ_β, there is a $\delta > 0$ such that $B_{\varphi_\beta}(u_0, \delta) \subseteq B_{V_\beta}(u_0, r)$ (cf. [2, P. 43]). Thus for any subset E of V_β, one can check that

$$\overline{E} = \bigcap_{n\geq 1} \bigcup_{u\in E} B_{\varphi_\beta}\big(u, \frac{1}{n}\big).$$

Take an arbitrary nonempty closed subset F of V_β. By the separability of V_β, F has a countable dense subset $\{v_k\}$ making

$$F = \bigcap_{n\geq 1} \bigcup_{k\geq 1} B_{\varphi_\beta}\big(v_k, \frac{1}{n}\big),$$

which results in

$$\{t \in I : u(t) \in F\} = \bigcap_{n\geq 1} \bigcup_{k\geq 1} \Big\{t \in I : \varphi_\beta(u(t) - v_k) < \frac{1}{n}\Big\} =: \bigcap_{n\geq 1} \bigcup_{k\geq 1} E_{k,n}.$$

Evidently, for each $k \in \mathbb{N}$, function $t \mapsto u(t) - v_k$ belongs to $L^0(I, X_\alpha)$, so the set $E_{k,n}$ is measurable for all $n \in \mathbb{N}$. Consequently as the intersection and union of countable measurable sets, $\{t \in I : u(t) \in F\}$ is also measurable, which leads to the measurability of u as a V_β–valued function. Since V_β is separable, we have that $u \in L^0(I, V_\beta)$, and the proof has been completed. □

Proposition 4.10. *Suppose that the every φ_β satisfies the Δ_2–condition, and every V_β is reflexive and separable. Then for all functions $u \in L^0(I, X_{\theta(\cdot)})$ fulfilling $\Phi_{\varphi_{\vartheta(\cdot)}}(u) < \infty$, we have $u \in L^0(I, V_{\vartheta(\cdot)})$.*

This proposition can be proved with the aid of the interim spaces $L^0(I, V_{\vartheta_n^-(\cdot)})$ $(n = 1, 2, \cdots)$.

Proposition 4.10 shows that $\Phi_{\varphi_{\vartheta(\cdot)}}$ is also a modular defined on the linear space $L^0(I, X_{\theta(\cdot)})$, and the semimodular space derived by $\Phi_{\varphi_{\vartheta(\cdot)}}$ is exactly $L^{\varphi_{\vartheta(\cdot)}}(I, V_{\vartheta(\cdot)})$, a separable Banach space. Moreover, suppose that $\{\varphi_\beta\}$ satisfies $H(A)$ and $H(B)'$ with the supremum β^+ of $\mathcal{B}$ instead of α^+, then $L^{\varphi_{\vartheta(\cdot)}}(I, V_{\vartheta(\cdot)})$ is uniformly convex with the dual

$$L^{\varphi_{\vartheta(\cdot)}}(I, V_{\vartheta(\cdot)})^* \cong L^{\varphi^*_{\vartheta(\cdot)}}(I, V^*_{\vartheta(\cdot)}).$$

Moreover, similar to Remark 4.3, the restriction of $\Phi_{\varphi_{\vartheta(\cdot)}}$ on $L^{\varphi_{\vartheta(\cdot)}}(I, V_{\vartheta(\cdot)})$, denoted by $\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}$, is convex and locally Lipschitz everywhere. Its subdifferential operator has the form

$$\partial\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(u) = \big\{\xi \in L^{\varphi^*_{\vartheta(\cdot)}}(I, V^*_{\vartheta(\cdot)}) : \xi(t) \in \partial\varphi_{\vartheta(t)}(u(t)) \text{ a.e. on } I\big\}$$

for all $u \in L^{\varphi_{\vartheta(\cdot)}}(I, V_{\vartheta(\cdot)})$.

Example 4.11. *Let us pay attention to the Sobolev space of variable exponent type*

$$W^{1,q(x)}(\Omega) = \{u \in W^{1,1}(\Omega) : \ u, \ D_i u \in L^{q(x)}(\Omega), \ i = 1, 2, \cdots, N\},$$

where $q \in \mathcal{P}_b(\Omega)$ with the notation $\mathcal{P}_b(\Omega)$ introduced in Example 3.22, and $D_i u = \partial u/\partial x_i$ denotes the i–th weak derivative of u. Recall that endowed with the norm

$$\|u\|_{W^{1,q(x)}(\Omega)} = \inf\big\{\lambda > 0 : \varrho_q\big(\frac{u}{\lambda}\big) + \sum_{i=1}^{N} \varrho_q\big(\frac{D_i u}{\lambda}\big)\big\},$$

or equivalently

$$\|u\|_{L^{q(x)}(\Omega)} + \|\nabla u\|_{L^{q(x)}(\Omega)},$$

$W^{1,q(x)}(\Omega)$ turns to be a separable Banach space. It is uniformly convex, and of course, reflexive provided $q^- > 1$.

Assume that $\partial\Omega \in C^1$ and $q \in C(\overline{\Omega})$ is a log-Hölder continuous exponent, or $q \in \mathcal{P}^{\omega}_{log}(\Omega)$ in symbol, which means that

$$|q(x) - q(y)| \le \omega(|x - y|), \quad \text{for } |x - y| < 1, \tag{19}$$

where $\omega : [0, \infty) \to [0, \infty)$ is a nondecreasing function fulfilling $\omega(0) = 0$ and

$$C_\omega = \sup_{r>0} \omega(r) \log \frac{1}{r} < \infty. \tag{20}$$

Under this situation, $C^\infty(\overline{\Omega})$ is dense in $W^{1,q(\cdot)}(\Omega)$, and $W_0^{1,q(x)}(\Omega)$ can be defined as the complement of $C_0^\infty(\Omega)$ in $W^{1,q(x)}(\Omega)$. By this definition, $W_0^{1,q(\cdot)}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,q(x)}(\Omega)$, and Poincaré's inequality

$$\|u\|_{L^{q(x)}(\Omega)} \le C\|\nabla u\|_{L^{q(x)}(\Omega)}, \ \forall \ u \in W_0^{1,q(\cdot)}(\Omega)$$

remains true. Therefore $W_0^{1,q(\cdot)}(\Omega)$ is topologically equivalent to the homogeneous Sobolev space $D_0^{1,q(\cdot)}(\Omega)$ (see [2, Ch. 8, 9] for relative discussions), and it can be regarded as a semimodular space derived from $X = L^0(\Omega)$ by the modular

$$\varphi_q(u) = \begin{cases} \sum_{i=1}^N \varrho_q(D_i u), & \text{if } u \in W_0^{1,1}(\Omega), \\ \infty, & \text{if } u \in L^0(\Omega)\backslash W_0^{1,1}(\Omega). \end{cases}$$

Take $\mathcal{A}_b$ as in Example 3.22 and fix ω fulfilling (20). Then as an ordered topological subspace of $\mathcal{A}_b$, the intersection $\mathcal{B}_\omega = \mathcal{A}_b \cap \mathcal{P}^{\omega}_{log}(\Omega)$ is also a $\mathcal{BTL}$, on which $\{\varphi_q : q \in \mathcal{B}_\omega\}$ is a $\mathcal{CMN}$, and $\{W_0^{1,q(x)}(\Omega) : q \in \mathcal{B}_\omega\}$ is the corresponding regular $\mathcal{BSN}$.

For any $q \in \mathcal{P}^{\omega}_{log}(\Omega)$ with $q^- > 1$, if we take $W_0^{1,q(\cdot)}(\Omega)$ and $D_0^{1,q(\cdot)}(\Omega)$ as the same space, then every member of the dual space $W_0^{1,q(\cdot)}(\Omega)^* =: W^{-1,q'(\cdot)}(\Omega)$ has a representation in $L^{q'(\cdot)}(\Omega)^N$, i.e. for each $\xi \in W^{-1,q'(\cdot)}(\Omega)$, there is an $F = (f_1, \cdots, f_N) \in L^{q'(\cdot)}(\Omega)^N$ such that $\xi = -\sum_{i=1}^N D_i f_i$ in $\mathcal{D}'(\Omega)$, or equivalently,

$$\langle \xi, u\rangle = \sum_{i=1}^{N} \int_\Omega f_i(x) D_i u(x) dx$$

for all $u \in W_0^{1,q(\cdot)}(\Omega)$ (see [2], §12.3). Thus for the $\mathcal{BTL}$ $\mathcal{B}_\omega$ with $\underline{q} > 1$, we can fix a basis $\{\varepsilon_k\}$ of $W^{-1,\underline{q}'}(\Omega)$, whose representations $\{G_k\}$ are also fixed in $L^{\underline{q}'}(\Omega)^N$. Based on this selection, for any $q \in \mathcal{A}_0$, due to the density of $W^{-1,\underline{q}'}(\Omega)$ in $W^{-1,q'(\cdot)}(\Omega)$, every element $\xi \in W^{-1,q'(\cdot)}(\Omega)$ has a unique representation in $L^{q'(\cdot)}(\Omega)^N$. Conversely, every vector function of $L^{q'(\cdot)}(\Omega)^N$ represents a unique member of $W^{-1,q'(\cdot)}(\Omega)$. Naturally, $W^{-1,q'(\cdot)}(\Omega)$ can be viewed as a semimodular space deduced by the modular

$$\psi_{q'}(\xi) = \sum_{i=1}^{N} \varrho_{q'}(f_i),$$

where $F = (f_1, \cdots, f_N) \in L^{q'(\cdot)}(\Omega)^N$ is the representation of ξ determined by $\{G_k\}$. Notice that, for all $u \in W_0^{1,q(\cdot)}(\Omega)$ and $\xi = -\sum_{i=1}^N D_i f_i \in W^{-1,q'(\cdot)}(\Omega)$, we have

$$\langle \xi, u\rangle = \sum_{i=1}^N \int_\Omega f_i(x) D_i u(x) dx \le \varphi_q(u) + \psi_{q'}(\xi),$$

and equality holds if and only if $f_i \in \partial \varrho_q(Du_i)$ or equivalently $Du_i \in \partial \varrho_{q'}(f_i)$ for all $i \in \{1, 2, \cdots, N\}$. In this sense, $\psi_{q'}$ can be viewed as the dual modular of φ_q, and $\{\psi_{q'} : q \in \mathcal{B}_\omega\}$ can be regarded as the $\mathcal{DMN}$ of $\{\varphi_q : q \in \mathcal{B}_\omega\}$. By the continuity of $\{\varrho_{q'}\}$, one can easily check that $\{\psi_{q'}\}$ is also a $\mathcal{CMN}$, and correspondingly $\{V_q^* = W^{-1,q'(\cdot)}(\Omega) : q \in \mathcal{B}_\omega\}$ is a regular $\mathcal{BSN}$, which is the $\mathcal{DSN}$ of $\{V_q = W_0^{1,q(\cdot)}(\Omega) : q \in \mathcal{B}_\omega\}$.

If we set $\vartheta(t) = q(t, \cdot)$, we then obtain another order-continuous exponent $\vartheta : I \to \mathcal{B}_\omega$. The authors in [11] revealed that for a continuous exponent $q(t, x)$ satisfying (19) uniformly for all $t \in I$ with a fixed ω verifying (20) and $1 < \underline{q} \le \bar{q} < \infty$,

$$L^0(I; W_0^{1,q(x,\cdot)}(\Omega)) = L^0_+(I; W_0^{1,q(x,\cdot)}(\Omega))$$

and

$$L^0(I; W^{-1,q'(x,\cdot)}(\Omega)) = L^0_+(I; W^{-1,q'(x,\cdot)}(\Omega)).$$

Thus by invoking Corollary 3.18 and Theorem 3.19 again, we get the reflexivity of $L^{\varphi_{q(\cdot)}}(I; W_0^{1,q(x,\cdot)}(\Omega))$, and the representation

$$L^{\varphi_{q(\cdot)}}(I; W_0^{1,q(x,\cdot)}(\Omega))^* \cong L^{\psi_{q'(\cdot)}}(I; W^{-1,q'(x,\cdot)}(\Omega)).$$

Remark 4.12. *Given an exponent $p \in \mathcal{P}(\Omega)$ with $1 < p^- < p^+ < \infty$, it is easy to check that*

$$\varrho_p\Big(\frac{u}{t^{1/p^-}}\Big) \le \frac{\varrho_p(u)}{t}.$$

Hence, ϱ_p satisfies the strongly coercive property (14) with $\gamma(t) = t^{p^-}$ verifying (15). Furthermore, under the assumptions $1 < \underline{p} < \bar{p} < \infty$ and $1 < \underline{q} < \bar{q} < N$ together with

$$p(t, x) \le \frac{Nq(t, x)}{N - q(t, x)} =: q^*(t, x),$$

$\{\varphi_q : q \in \mathcal{B}_\omega\}$ is stronger than $\{\varrho_p : p \in \mathcal{A}_b\}$, and all the modular nets $\{\varrho_p\}$, $\{\varrho_{p'}\}$, $\{\varphi_q\}$ and $\{\psi_{q'}\}$ satisfy the strong coercivity with $\gamma(t) = t^{\underline{p}}$, $t^{\bar{p}'}$, $t^{\underline{q}}$ and $t^{\bar{q}'}$ respectively. Thus taking the strict convexity of $L^{p'(x)}(\Omega)$ and $W^{-1,q'(x)}(\Omega)$ into account, we can assert that hypotheses $H(A)$ and $H(B)$ are fulfilled by both $\{\varrho_p : p \in \mathcal{A}_b\}$ and $\{\varphi_q : q \in \mathcal{B}_\omega\}$. Therefore in our setting,

$$Z_{\theta(\cdot)}(u)(t) = |u|^{p(t,x)-2}u = \partial \varrho_{p(t)}(u)$$

defines a bounded, coercive and demicontinuous operator

$$Z_{\theta(\cdot)} : L^{\varrho_{p(\cdot)}}(I; L^{q(\cdot,x)}(\Omega)) \to L^{\varrho_{p'(\cdot)}}(I; L^{p'(\cdot,x)}(\Omega))$$

as in Theorem 4.4, and

$$\partial \widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(u)(t) = \partial \varphi_{\vartheta(t)}(u(t)) = -div(|\nabla u|^{q(t,x)-2}\nabla u)$$

defines the subdifferential of $\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}$ at $u \in L^{\psi_{q(\cdot)}}(I; W_0^{1,q(\cdot,x)}(\Omega))$.

Remark 4.13. *Replace q by a vector exponent $\mathbf{q} = (q_1, q_2, \cdots, q_N)$ in Example 4.11, and define the anisotropic space*

$$W_0^{1,\mathbf{q}(x)}(\Omega) = \{u \in W_0^{1,1}(\Omega) : D_i u \in L^{q_i(x)}(\Omega)\}.$$

Here $q_i \in \mathcal{P}^\omega_{log}$ with a common ω, and $q_i(x) \in [\underline{q}, \bar{q}] \subseteq (1, \infty)$ for all $x \in \Omega$, $i = 1, 2, \cdots, N$. Similar to the isotropic ones, $W_0^{1,\mathbf{q}(x)}(\Omega)$ can be viewed as a semimodular space derived by

$$\varphi_{\mathbf{q}}(u) = \begin{cases} \sum_{i=1}^N \varrho_{q_i}(D_i u), & \text{if } u \in W_0^{1,1}(\Omega), \\ \infty, & \text{if } u \in L^0(\Omega) \backslash W_0^{1,1}(\Omega), \end{cases}$$

and its dual space, denoted by $W^{-1,\mathbf{q}'(x)}(\Omega)$*, can also be represented by the product space* $\prod_{i=1}^{N} L^{q_i'(x)}(\Omega)$*. Therefore following the same discussions as in Example 4.11, we can conclude that* $\{\varphi_{\mathbf{q}} : \mathbf{q} \in \mathcal{B}_\omega^N\}$*, where* $\mathcal{B}_\omega^N$ *is equipped with the product topology and the order:* $\mathbf{q} \prec \mathbf{r}$ *if and only if* $q_i \prec r_i$ *for all* $i = 1, 2, \cdots, N$*, is a* $\mathcal{CMN}$ *fulfilling hypotheses* $H(A)$ *and* $H(B)$*, and stronger than* $\{\varrho_p : p \in \mathcal{A}_b\}$*.* $\{W_0^{1,\mathbf{q}(x)}(\Omega) : \mathbf{q} \in \mathcal{B}_\omega^N\}$ *is a* $\mathcal{BSN}$ *derived by* $\{\varphi_{\mathbf{q}}\}$*, its* $\mathcal{DSN}$ $\{W^{-1,\mathbf{q}'(x)}(\Omega)\}$ *is generated by* $\{\varrho_{\mathbf{q}'}\}$ *with a basis fixed in* $W^{-1,\underline{\mathbf{q}}'(x)}(\Omega)$ *and its representation fixed in* $\prod_{i=1}^{N} L^{\underline{q_i}'}(\Omega)$*, where* $\underline{\mathbf{q}} = (\underline{q_1}, \underline{q_2}, \cdots, \underline{q_N})$*, and*

$$\varrho_{\mathbf{q}'}(F) = \sum_{i=1}^{N} \varrho_{q_i'}(f_i),\ F = (f_1, f_2, \cdots, f_N) \in L^0(\Omega)^N.$$

Therefore, for a continuous vector exponent $\mathbf{q} : I \times \Omega \to [\underline{q}, \bar{q}]^N$ with every component q_i satisfying (19) uniformly for $t \in I$, the modular-modular space $L^{\varphi_{\mathbf{q}(\cdot)}}(I, W_0^{1,\mathbf{q}(\cdot,x)}(\Omega))$ deduced from $L^0(I, L^{p(\cdot,x)}(\Omega))$ by the modular

$$\Phi_{\varphi_{\mathbf{q}(\cdot)}}(u) = \int_I \varphi_{\mathbf{q}(t)}(u(t))dt$$

is a separable and uniformly convex space with the dual

$$L^{\varphi_{\mathbf{q}(\cdot)}}(I, W_0^{1,\mathbf{q}(\cdot,x)}(\Omega))^* \cong L^{\varphi_{\mathbf{q}'(\cdot)}}(I, W^{-1,\mathbf{q}'(\cdot,x)}(\Omega)).$$

Moreover, the restriction functional $\widetilde{\Phi}_{\varphi_{\mathbf{q}(\cdot)}}$ is convex and locally Lipschitz everywhere on $L^{\varphi_{\mathbf{q}(\cdot)}}(I, W_0^{1,\mathbf{q}(\cdot,x)}(\Omega))$, its subdifferential $\partial\widetilde{\Phi}_{\varphi_{\mathbf{q}(\cdot)}}$ at $u \in L^{\varphi_{\mathbf{q}(\cdot)}}(I, W_0^{1,\mathbf{q}(\cdot,x)}(\Omega))$ has the expression

$$\partial\widetilde{\Phi}_{\varphi_{\mathbf{q}(\cdot)}}(u) = \partial\varphi_{\mathbf{q}(t)}(u(t)) = -\sum_{i=1}^{N} D_i(|D_i u|^{q_i(t,x)-2} D_i u),$$

or equivalently,

$$\langle\langle \partial\widetilde{\Phi}_{\varphi_{\mathbf{q}(\cdot)}}(u), v\rangle\rangle_{\vartheta(\cdot)} = \sum_{i=1}^{N} \int_I \int_\Omega |D_i u(t,x)|^{q_i(t,x)-2} D_i u(t,x) D_i v(t,x) dx dt$$

for all $v \in L^{\varphi_{\mathbf{q}(\cdot)}}(I, W_0^{1,\mathbf{q}(\cdot,x)}(\Omega))$.

5 Bochner-Sobolev spaces of modular-modular type and applications in doubly nonlinear differential equations

We begin with the Bochner-Sobolev space of range-fixed type, that is

$$W^{1,\varrho_\alpha}(I, X_\alpha) = \left\{u \in L^{\varrho_\alpha}(I, X_\alpha) : u' \in L^{\varrho_\alpha}(I, X_\alpha)\right\},$$

where u' denotes the derivative of u in the sense of distribution, i.e. for all $\xi \in X_\alpha^*$ and all $\gamma \in C_0^\infty(I)$, equality

$$\int_I \langle u'(t), \gamma(t)\xi\rangle_\alpha dt = -\int_I \langle u(t), \gamma'(t)\xi\rangle_\alpha dt$$

holds. It is easy to check that, endowed with the norm

$$\|u\|_{W^{1,\varrho_\alpha}(I,X_\alpha)} = \|u\|_{L^{\varrho_\alpha}(I,X_\alpha)} + \|u'\|_{L^{\varrho_\alpha}(I,X_\alpha)},$$

which is equivalent to

$$\inf\left\{\lambda > 0 : \Phi_{\varrho_\alpha}(u/\lambda) + \Phi_{\varrho_\alpha}(u'/\lambda) \le 1\right\},$$

$W^{1,\varrho_\alpha}(I, X_\alpha)$ turns to be a Banach space.

Theorem 5.1. *Function space $W^{1,\varrho_\alpha}(I, X_\alpha)$ can be embedded into the space of continuous functions $C(I, X_\alpha)$. If in addition V_β is embedded into X_α compactly, then $W^{1,\varrho_\alpha}(I, X_\alpha) \cap L^{\varphi_\beta}(I, V_\beta)$ is embedded compactly into $L^{\varrho_\alpha}(I, X_\alpha)$.*

Proof. Firstly, from the inequality

$$\|u\|_{X_\alpha} \le \varrho_\alpha(u) + 1, \ \forall\ u \in X_\alpha,$$

we can deduce that $L^{\varrho_\alpha}(I, X_\alpha) \hookrightarrow L^1(I, X_\alpha)$. Similarly, we have $W^{1,\varrho_\alpha}(I, X_\alpha) \hookrightarrow W^{1,1}(I, X_\alpha)$ and $L^{\varphi_\beta}(I, V_\beta) \hookrightarrow L^1(I, V_\beta)$. The first conclusion comes since the embedding $W^{1,1}(I, X_\alpha)) \hookrightarrow C(I, X_\alpha)$ holds.

Given a bounded subset F of $W^{1,\varrho_\alpha}(I, X_\alpha) \cap L^{\varphi_\beta}(I, V_\beta)$, it is also bounded in $W^{1,1}(I, X_\alpha)) \cap L^1(I, V_\beta))$. Assume that

$$\Phi_{\varrho_\alpha}(u') + \Phi_{\varrho_\alpha}(u) + \Phi_{\varphi_\beta}(u) \le C$$

for some $C > 0$ independent of $u \in F$. Then for any $u \in F$ and $0 < h < \min\{1, T/2\}$ and $t, t + h \in I$, we have

$$\varrho_\alpha(u(t + h) - u(t)) \le \int_t^{t+h} \varrho_\alpha(u'(\tau))d\tau,$$

consequently

$$\begin{aligned}
&\int_0^{T-h} \varrho_\alpha(u(t + h) - u(t))dt \le \int_0^{T-h}\int_t^{t+h} \varrho_\alpha(u'(\tau))d\tau dt \qquad (21)\\
&= \Big(\int_0^h\int_0^\tau + \int_h^{T-h}\int_{\tau-h}^\tau + \int_{T-h}^T\int_{\tau-h}^{T-h}\Big)\varrho_\alpha(u'(\tau))dtd\tau\\
&= \int_0^h \tau\varrho_\alpha(u'(\tau))d\tau + \int_h^{T-h} h\varrho_\alpha(u'(\tau))d\tau + \int_{T-h}^T (T - \tau)\varrho_\alpha(u'(\tau))d\tau\\
&\le h\int_I \varrho_\alpha(u'(\tau))d\tau \le Ch.
\end{aligned}$$

Taking any $r \in (0, T)$, consider the average operator M_r on $L^{\varrho_\alpha}(I, X_\alpha)$ defined by

$$M_r u(t) = \frac{1}{r}\int_t^{t+r} u(\tau)d\tau, \ t \in [0, T - r].$$

Obviously, for all $u \in L^{\varphi_\beta}(I, V_\beta)$, $M_r u \in C([0, T - r], V_\beta)$ with the estimate

$$\varphi_\beta(M_r u(t)) \le \frac{1}{r}\int_t^{t+r} \varphi_\beta(u(\tau))d\tau, \ t \in [0, T - r].$$

Moreover, due to the boundedness of F in $W^{1,\varrho_\alpha}(I, X_\alpha)$ and the estimate (21), precompactness of the set $F_r = \{M_r u : u \in F\}$ in $C([0, T - r], X_\alpha)$ can be reached (refer to [16, 17]).

In addition, from (21), one can deduce that

$$\begin{aligned}
\int_0^{T-h} \varrho_\alpha(M_r u(t) - u(t))dt &= \int_0^{T-h} \varrho_\alpha\Big(\frac{1}{r}\int_0^r (u(t + \tau) - u(t))d\tau\Big)dt\\
&\le \frac{1}{r}\int_0^{T-h}\int_0^r \varrho_\alpha(u(t + \tau) - u(t))d\tau dt\\
&\le \frac{1}{r}\int_0^r\int_0^{T-\tau} \varrho_\alpha(u(t + \tau) - u(t))dtd\tau
\end{aligned}$$

$$\leq Ch$$

provided $0 < r \leq h$, which means that $M_r u \to u$ in $L_{\varrho_\alpha}([0, T-h], X_\alpha)$ as $r \to 0$ uniformly for $u \in F$. This fact, combined with the precompactness of F_r in $C([0, T-r], X_\alpha)$ for every fixed $r \in (0, h]$, leads to the precompactness of F in $L_{\varrho_{\alpha_1}}([0, T-h], X_\alpha)$. The final conclusion comes if we make the same discussions on the set $\widetilde{F} = \{\widetilde{u}(t) = u(T-t) : u \in F\}$ (see [16]). □

Using the facts $L^{\varrho_\alpha}(I, X_\alpha) \hookrightarrow L^1(I, X_\alpha)$ and $W^{1,\varrho_\alpha}(I, X_\alpha) \hookrightarrow C(I, X_\alpha)$, we can also deduce that

Corollary 5.2. *Under hypotheses of the above theorem, $W^{1,\varrho_\alpha}(I, X_\alpha) \cap L^{\varphi_\beta}(I, V_\beta)$ can be embedded compactly into $L^p(I, X_\alpha)$ for any $1 \leq p < \infty$, hence $L^{p(\cdot)}(I, X_\alpha)$ for all $p \in \mathcal{P}_b(I, X_\alpha)$.*

Given two $\mathcal{CMN}$s $\{\varrho_\alpha :\in \mathcal{A}\}$ and $\{\varphi_\beta : \beta \in \mathcal{B}\}$ satisfying $H(A) + H(B)$ and $H(A) + H(B)'$ respectively, and the latter stronger than the former, introduce the Bochner-Sobolev space of range-varying type

$$W^{1,\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) = \left\{u \in W^{1,\varrho_{\alpha^-}}(I, X_{\alpha^-}) : u, u' \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})\right\}.$$

Similarly, equipped with the norm

$$\|u\|_{W^{1,\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})} = \|u\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})} + \|u'\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})},$$

which is equivalent to

$$\inf\left\{\lambda > 0 : \Phi_{\varrho_{\theta(\cdot)}}(u/\lambda) + \Phi_{\varrho_{\theta(\cdot)}}(u'/\lambda) \leq 1\right\},$$

$W^{1,\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ becomes a Banach space.

Theorem 5.3. *Besides the assumptions upon $\{\varrho_\alpha\}$ and $\{\varphi_\beta\}$ as above, assume that there are scalar functions $\tilde{p}_i, \tilde{q}_i, \delta \in C(I)$ $(i = 1, 2)$ such that $1 \leq \tilde{p}_1(t) \leq \tilde{p}_2(t) < \infty$, $1 \leq \tilde{q}_1(t) \leq \tilde{q}_2(t) < \infty$, $\delta(t) \in (0, 1)$, and $\tilde{p}_2(t)\delta(t) < \tilde{q}_2(t)$ for all $t \in I$. Suppose also*

- *$V_{\vartheta^-} \hookrightarrow\hookrightarrow X_{\alpha^+}$ and there is a constant $C > 0$ such that*
- *$(X_{\alpha^-}, V_{\vartheta(t)})_{\delta(t)} \hookrightarrow X_{\theta(t)}$ uniformly for $t \in I$, in other words,*

$$\|u\|_{\theta(t)} \leq C\|u\|_{\alpha^-}^{1-\delta(t)}\|u\|_{\vartheta(t)}^{\delta(t)} \text{ for all } u \in V_{\vartheta(t)}, \tag{22}$$

  where notation $(X_{\alpha^-}, V_{\vartheta(t)})_{\delta(t)}$ represents the real or complex interpolation space between X_{α^-} and $V_{\vartheta(t)}$ with the index $\delta(t)$;
- *for all $t \in I$,*

$$\varrho_{\theta(t)}(u) \leq C \max\left\{\|u\|_{\theta(t)}^{\tilde{p}_1(t)}, \|u\|_{\theta(t)}^{\tilde{p}_2(t)}\right\}\ u \in X_{\theta(t)}, \tag{23}$$

  and

$$\min\left\{\|u\|_{\vartheta(t)}^{\tilde{q}_1(t)}, \|u\|_{\vartheta(t)}^{\tilde{q}_2(t)}\right\} \leq C\varphi_{\vartheta(t)}(u),\ u \in V_{\vartheta(t)}.$$

Then space $W^{1,\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) \cap L^{\varphi_{\vartheta(\cdot)}}(I, V_{\vartheta(\cdot)})$ is embedded into $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ compactly.

Proof. Firstly, by (23) and Remark 4.8 in [11], we have that

$$L^{\tilde{p}_2(\cdot)}(I, X_{\theta(\cdot)}) \hookrightarrow L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}).$$

Thus from Theorem 5.1 and its corollary, it suffices to show that a sequence $\{u_k\}$ bounded in $L^{\varphi_{\vartheta(\cdot)}}(I, V_{\vartheta(\cdot)})$ and convergent in $L^{p(\cdot)}(I, X_{\alpha^-})$ for all $p \in \mathcal{P}_b(I, X_\alpha)$ is convergent in $L^{\tilde{p}_2(\cdot)}(I, X_{\theta(\cdot)})$ definitely. Without loss of generality, assume that the limit of $\{u_k\}$ is 0. Take $K > 1$ so close to 1 that $\tilde{p}_2(t)\delta(t)K \leq \tilde{q}_2(t)$ and $\tilde{p}_2(t)(1 - \delta(t))K' \geq 1$ $(1/K + 1/K' = 1)$ for all $t \in I$, then by (22), we have

$$\begin{aligned} &\int_I \|u_k(t)\|_{\theta(t)}^{\tilde{p}_2(t)} dt \\ &\leq C \int_I \|u_k(t)\|_{\alpha^-}^{\tilde{p}_2(t)(1-\delta(t))} \|u_k(t)\|_{\vartheta(t)}^{\tilde{p}_2(t)\delta(t)} dt \end{aligned} \tag{24}$$

$$
\begin{aligned}
&\le C\Big(\int_I \|u_k(t)\|_{\alpha^-}^{\tilde{p}_2(t)(1-\delta(t))K'}dt\Big)^{1/K'}\Big(\int_I \|u_k(t)\|_{\vartheta(t)}^{\tilde{p}_2(t)\delta(t)K}dt\Big)^{1/K}\\
&\le C\Big(\int_I \|u_k(t)\|_{\alpha^-}^{\tilde{p}_2(t)(1-\delta(t))K'}dt\Big)^{1/K'}\\
&\quad\cdot\Big[T^{1/K}+\Big(\int_{\{\|u_k(t)\|_{\vartheta(t)}>1\}} \|u_k(t)\|_{\vartheta(t)}^{\tilde{p}_2(t)\delta(t)K}dt\Big)^{1/K}\Big]\\
&\le C\Big(\int_I \|u_k(t)\|_{\alpha^-}^{\tilde{p}_2(t)(1-\delta(t))K'}dt\Big)^{1/K'}\Big[T^{1/K}+\Big(\int_I \varphi_{\vartheta(t)}(u_k(t))dt\Big)^{1/K}\Big],
\end{aligned}
$$

which leads to the desired conclusion. □

Remark 5.4. *Under all the hypotheses of Theorem 5.3 with the compact embedding of V_{ϑ^-} into X_{α^+} replaced by the following condition*

$$\|u\|_{X_\alpha} \le C\|u\|_{V_\beta} \tag{25}$$

uniformly for $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, we have

$$\|u'\|_{L^1(I,X_{\alpha^-})} \le C\|u'\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})},\ \|u\|_{L^1(I,X_{\alpha^-})} \le C\|u\|_{L^{\varphi_{\vartheta(\cdot)}}(I,V_{\vartheta(\cdot)})}$$

for all $u \in W^{1,\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)}) \cap L^{\varphi_{\vartheta(\cdot)}}(I,V_{\vartheta(\cdot)})$, thus under the condition

$$\|u'\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})} + \|u\|_{L^{\varphi_{\vartheta(\cdot)}}(I,V_{\vartheta(\cdot)})} \le 1,$$

estimates (24) turn to be

$$
\begin{aligned}
\int_I \|u(t)\|_{\theta(t)}^{\tilde{p}_2(t)}dt &\le CT^{1/K'}\max\big\{(\max_{t\in I}\|u(t)\|_{\alpha^-})^{\tilde{p}_2^+(1-\delta^-)},(\max_{t\in I}\|u(t)\|_{\alpha^-})^{\tilde{p}_2^-(1-\delta^+)}\big\}\\
&\quad\cdot\Big(\int_I \|u(t)\|_{\vartheta(t)}^{\tilde{p}_2(t)\delta(t)K}dt\Big)^{1/K}\\
&\le CT^{1/K'}\max\big\{\|u\|_{W^{1,1}(I,X_{\alpha^-})}^{\tilde{p}_2^+(1-\delta^-)},\|u\|_{W^{1,1}(I,X_{\alpha^-})}^{\tilde{p}_2^-(1-\delta^+)}\big\}\\
&\quad\cdot\Big[T^{1/K}+\Big(\int_{\{\|u_k(t)\|_{V_{\vartheta(t)}}>1\}} \|u_k(t)\|_{V_{\vartheta(t)}}^{\tilde{p}_2(t)\delta(t)K}dt\Big)^{1/K}\Big]\\
&\le CT^{1/K'}\big(\|u'\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})}+\|u\|_{L^{\varphi_{\vartheta(\cdot)}}(I,V_{\vartheta(\cdot)})}\big)^{\tilde{p}_2^-(1-\delta^+)}\\
&\quad\cdot\Big[T^{1/K}+\Big(\int_I \varphi_{\vartheta(t)}(u(t))dt\Big)^{1/K}\Big]\le C,
\end{aligned}
$$

which in turn produces $\|u\|_{L^{\tilde{p}_2(t)}(I,X_{\theta(t)})} \le C$, and consequently $\|u\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})} \le C$ since $L^{\tilde{p}_2(\cdot)}(I,X_{\theta(\cdot)}) \hookrightarrow L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})$. Then by means of scaling transformation, we get

$$\|u\|_{W^{1,\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})} \le C(\|u'\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})} + \|u\|_{L^{\varphi_{\vartheta(\cdot)}}(I,V_{\vartheta(\cdot)})}). \tag{26}$$

This is an important inequality for later use.

In spite of the estimate (26), we do not expect the control of the modular $\Phi_{\varrho_{\theta(\cdot)}}(u)$ by the sum $\Phi_{\varrho_{\theta(\cdot)}}(u') + \widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(u)$. Conversely, we have

Proposition 5.5. *For every $r \ge 0$, there is an $\varepsilon_r > 0$ such that for all $0 \le \varepsilon \le \varepsilon_r$, the a priori estimate*

$$\Phi_{\varrho_{\theta(\cdot)}}(u') + \widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(u) \le \varepsilon_r\Phi_{\varrho_{\theta(\cdot)}}(u) + r$$

defines a bounded subset of $W^{1,\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})$.

Proof. We proceed by contradiction. Assume that there is a sequence $\{u_k\}$ in $W^{1,\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ verifying

$$\Phi_{\varrho_{\theta(\cdot)}}(u_k') + \widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(u_k) \le \frac{1}{k}\Phi_{\varrho_{\theta(\cdot)}}(u_k) + r,$$

but $\Phi_{\varrho_{\theta(\cdot)}}(u_k) \ge k$. Let $v_k = u_k/\Phi_{\varrho_{\theta(\cdot)}}(u_k)$. Then we have

$$\Phi_{\varrho_{\theta(\cdot)}}(v_k') + \widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(v_k) \le \frac{\Phi_{\varrho_{\theta(\cdot)}}(u_k') + \widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(u_k)}{\Phi_{\varrho_{\theta(\cdot)}}(u_k)} \le \frac{1+r}{k}.$$

Therefore $\Phi_{\varrho_{\theta(\cdot)}}(v_k') + \widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(v_k) \to 0$, and consequently $v_k \to 0$ in $W^{1,\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ as $k \to \infty$. Thus $\lim_{k\to\infty} \Phi_{\varrho_{\theta(\cdot)}}(v_k) = 0$, which contradicts to fact that $\Phi_{\varrho_{\theta(\cdot)}}(v_k) \ge 1/2$ for all $k \in \mathbb{N}$. □

Remark 5.6. *As a preparation for later arguments, following the same process as above, we can prove that for every $C > 0$, there exist two small numbers $\delta_0 > 0$ and $\mu_0 > 0$ such that all the functions satisfying the a priori estimate*

$$\Phi_{\varrho_{\theta(\cdot)}}(u') + \widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(u) \le (\delta + \mu\sigma(\delta^{-1}))\Phi_{\varrho_{\theta(\cdot)}}(u) + C\sigma(\delta^{-1})$$

comprise a bounded subset of $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ as long as $0 < \delta \le \delta_0$ and $0 < \mu \le \mu_0$.

Define two function spaces with periodic boundary condition, one is

$$W^{1,\varrho_\alpha}_{\mathrm{per}}(I, X_\alpha) = \big\{u \in W^{1,\varrho_\alpha}(I, X_\alpha) : u(0) = u(T)\big\},$$

the other is

$$W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)}) = W^{1,\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) \cap W^{1,\varrho_{\alpha^-}}_{\mathrm{per}}(I, X_{\alpha^-})$$

under the condition $\theta(0) = \theta(T)$. Evidently, the two spaces are closed subspaces of $W^{1,\varrho_\alpha}(I, X_\alpha)$ and $W^{1,\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ respectively.

Let us make some investigations on the operator $D^2_{\theta(\cdot)} : W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)}) \to W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)})^*$ defined through

$$\langle\langle D^2_{\theta(\cdot)}(u), v\rangle\rangle = \int_I \langle \partial\varrho_{\theta(t)}(u'(t)), v'(t)\rangle_{\theta(t)} dt,\ u, v \in W^{1,\varrho_\alpha}(I, X_\alpha). \tag{27}$$

By the definition, it is easy to check that $D^2_{\theta(\cdot)}$ is a single-valued, monotone and demicontinuous operator, hence it is maximal monotone. In addition, taking $v(t) = \gamma(t)w$ in (27) with $\gamma \in C_0^\infty(I)$ and $w \in X_{\alpha^+}$, we have

$$\begin{aligned}
\langle\langle D^2_{\theta(\cdot)}(u), v\rangle\rangle &= \int_I \langle \partial\varrho_{\theta(t)}(u'(t)), \gamma'(t)w\rangle_{\alpha^+} dt \\
&= -\int_I \langle \frac{d}{dt}\partial\varrho_{\theta(t)}(u'(t)), \gamma(t)w\rangle_{\alpha^+} dt \\
&= -\int_I \langle \frac{d}{dt}\partial\varrho_{\theta(t)}(u'(t)), \gamma(t)w\rangle_{\theta(t)} dt,
\end{aligned}$$

which means that $D^2_{\theta(\cdot)}$ is a second order nonlinear differential operator, i.e.

$$D^2_{\theta(\cdot)}(u) = -\frac{d}{dt}\partial\varrho_{\theta(\cdot)}(u'(\cdot))$$

in the sense of distribution.

Let $\widetilde{\Psi}(u) = \Phi_{\varrho_{\theta(\cdot)}}(u')$ at $u \in W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)})$, we obtain a continuous and convex functional. A direct calculation shows that $\widetilde{\Psi}$ is Gâteaux differentiable everywhere with the Gâteaux differential $\widetilde{\Psi}'(u) = D^2_{\theta(\cdot)}(u)$. Thus the subdifferential operator $\partial\widetilde{\Psi}$ is single-valued, and $\partial\widetilde{\Psi}(u) = D^2_{\theta(\cdot)}(u)$ for all $u \in W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)})$. Consider the extension of $\widetilde{\Psi}$ onto $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$

$$\Psi(u) = \begin{cases} \infty, & \text{if } u \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) \setminus W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)}) \\ \widetilde{\Psi}(u), & \text{if } u \in W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)}). \end{cases}$$

It is also easy to verify that Ψ is a semicontinuous and convex proper functional, whose subdifferential operator has the domain

$$\begin{aligned}\mathcal{D}(\partial\Psi) = \big\{u \in W^{1,\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) : &\text{ there is an } \xi \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)}) \text{ such that}\\ &\int_I \langle \xi(t), v(t)\rangle_{\theta(t)} dt = \int_I \langle Z_{\theta(\cdot)}(u')(t), v'(t)\rangle_{\theta(t)} dt \\ &\text{for all } v \in W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)})\big\},\end{aligned} \tag{28}$$

together with $\partial\Psi(u) = D^2_{\theta(\cdot)}(u) = \xi$ in $W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)})^*$ for the function $\xi \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$ satisfying (28) and all $u \in \mathcal{D}(\partial\Psi)$. In this sense, we can say that $\partial\Psi \subseteq D^2_{\theta(\cdot)}$.

Taking the intersection

$$\mathcal{W} = W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)}) \cap L^{\varphi_{\vartheta(\cdot)}}(I, V_{\vartheta(\cdot)})$$

as the work space, where the norm $\|\cdot\|_{\mathcal{W}}$ takes the value $\|u\|_{W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I,X_{\theta(\cdot)})} + \|u\|_{L^{\varphi_{\vartheta(\cdot)}}(I,V_{\vartheta(\cdot)})}$, and consider the sum

$$\widetilde{\Phi}(u) = \widetilde{\Psi}(u') + \widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(u)$$

as the potential functional. Evidently, $\widetilde{\Phi}(u)$ is a continuous modular on $\mathcal{W}$ with the effective domain $\mathcal{D}(\widetilde{\Phi}) = \mathcal{W}$, its subdifferential operator

$$\partial\widetilde{\Phi} = D^2_{\theta(\cdot)} + \partial\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}$$

is a maximal monotone subset of $W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)}) \times W^{1,\varrho_{\theta(\cdot)}}_{\mathrm{per}}(I, X_{\theta(\cdot)})^*$. Moreover, by (26) and the coercivity of $\Phi_{\varrho_{\theta(\cdot)}}$, $\Phi_{\varphi_{\vartheta(\cdot)}}$, we have

$$\frac{\langle\langle \partial\widetilde{\Phi}(u), u\rangle\rangle}{\|u\|_{\mathcal{W}}} \geq C \frac{\widetilde{\Psi}(u') + \widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(u)}{\|u'\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})} + \|u\|_{L^{\varphi_{\vartheta(\cdot)}}(I,V_{\vartheta(\cdot)})}} \to \infty \text{ as } \|u\|_{\mathcal{W}} \to \infty,$$

which yields the coercivity, hence surjectivity of $\partial\widetilde{\Phi}$. Furthermore, suppose that for every $\beta \in \mathcal{B}$, φ_β is strictly convex, then $\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}$ hence $\widetilde{\Phi}$ is also strictly convex, which in turn leads to the injectivity of $\partial\widetilde{\Phi}$. Summing up, we conclude that

Theorem 5.7. *Suppose all the hypotheses mentioned above are satisfied, then for every $f \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)}) \subseteq \mathcal{W}^*$, there is a unique $u \in \mathcal{W}$ and a corresponding selection $\xi \in \partial\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(u)$ such that*

$$D^2_{\theta(\cdot)}(u) + \xi = f \text{ in } \mathcal{W}^*. \tag{29}$$

In other words, for every $f \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$, second order differential inclusion

$$-\frac{d}{dt}\partial\varrho_{\theta(t)}(u'(t)) + \partial\varphi_{\vartheta(t)}(u(t)) \ni f(t) \text{ a.e. on } I \tag{30}$$

with periodic boundary values admits a unique weak solution u in the sense

$$\int_I \langle \partial\varrho_{\theta(\cdot)}(u'(t)), v'(t)\rangle_{\theta(t)} dt + \int_I \langle \xi(t), v(t)\rangle_{\vartheta(t)} dt = \int_I \langle f(t), v(t)\rangle_{\theta(t)} dt$$

for all $v \in \mathcal{W}$.

The above theorem shows that the sum operator $D^2_{\theta(\cdot)} + \partial\Phi_{\varphi_{\vartheta(\cdot)}}$ is both injective and surjective, its inverse $(D^2_{\theta(\cdot)} + \partial\Phi_{\varphi_{\vartheta(\cdot)}})^{-1}$ is existing and single-valued. Moreover, in light of Proposition 5.5, together with the compact imbedding $\mathcal{W} \hookrightarrow\hookrightarrow L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, we can prove that

Theorem 5.8. *If all the hypotheses of Theorem 5.3 together with (25) are satisfied, then the inverse operator*

$$(D^2_{\theta(\cdot)} + \partial\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}})^{-1} : L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)}) \to L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$$

is both bounded and strongly continuous in the sense that for any sequence $\{f_k\}$ convergent to f in $L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$ weakly, the corresponding sequence $\{(D^2_{\theta(\cdot)} + \partial\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}})^{-1} f_k\}$ converges to $(D^2_{\theta(\cdot)} + \partial\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}})^{-1} f$ in $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ strongly.

Let us consider the operator $f : I \times X \to X$. Assume that

- for any $J \in \Pi(I)$ and $u \in X_{\theta_J^+}$, $t \mapsto f(t,u)$ lies in $L^0(I, X^*_{\theta_J^+})$,
- for every $t \in I$, $f(t,\cdot) : X_{\theta(t)} \to X^*_{\theta(t)}$ is demicontinuous, and
- there is an $\mu > 0$ and a nonnegative integrable function h for which inequality

$$\varrho^*_{\theta(t)}(f(t,u)) \le \mu \varrho_{\theta(t)}(u) + h(t) \tag{31}$$

holds for all $u \in X_{\theta(t)}$.

Taken any $u \in L^0(I, X_{\theta(\cdot)})$, denote by $F(u) = f(t,u)$. It is easy to see that under the first assumption upon f, if $u \in \mathcal{S}(I, X_{\alpha^+})$, then $F(u) \in L^0_-(I, X^*_{\theta(\cdot)})$, and by (31), it comes $F(u) \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$. Consequently, from the density of $\mathcal{S}(I, X_{\alpha^+})$ in $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ and the demicontinuity assumption upon f, we can derive that $F(u) \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$ provided $u \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$. Moreover, we have

Proposition 5.9. *F is a bounded and demicontinuous operator from $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ to $L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$.*

Proof. Boundedness of F comes from (31) immediately. For the weak continuity, suppose that $\{u_n\}$ is a sequence of $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ converging to u strongly, that is

$$\int_I \varrho_{\theta(t)}(u_n(t) - u(t))dt \to 0 \text{ as } n \to \infty.$$

Thus there is a subsequence, say $\{u_n\}$ itself satisfying

$$\lim_{n\to\infty} \varrho_{\theta(t)}(u_n(t) - u(t)) = 0 \text{ a.e. on } I.$$

From the boundedness of $\{u_n\}$ in $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ and (31), we get the boundedness of $\{F(u_n)\}$ in $L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$. Consequently, there is a subsequence, without loss of generality, assuming also $\{u_n\}$ itself, and a function $\xi \in L^{\varrho^*_{\theta(\cdot)}}(I, X^*_{\theta(\cdot)})$, such that $F(u_n) \to \xi$ weakly, or equivalently

$$\lim_{n\to\infty} \int_I \langle f(t, u_n(t)) - \xi(t), v(t)\rangle_{\theta(t)} dt = 0$$

for all $v \in L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, which in turn yields

$$\lim_{n\to\infty} \int_E \langle f(t, u_n(t)) - \xi(t), v\rangle_{\theta(t)} dt = 0 \tag{32}$$

for all measurable subsets E of I and all elements v of X_{α^+} since $\mathcal{S}(I, X_{\alpha^+})$ is dense in $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ by $H(A)$.

On the other hand, since

$$\lim_{n\to\infty} \langle f(t, u_n(t)), v\rangle_{\theta(t)} = \langle f(t, u(t)), v\rangle_{\theta(t)}$$

a.e. on I, for each $\varepsilon > 0$, by Egorov's theorem, there is a measurable set $E_\varepsilon \subseteq I$ with $|I \setminus E_\varepsilon| < \varepsilon$ verifying

$$\langle f(t, u_n(t)), v\rangle_{\theta(t)} \to \langle f(t, u(t)), v\rangle_{\theta(t)} \text{ uniformly on } E_\varepsilon$$

as $n \to \infty$, consequently for any subset E of E_ε, we have

$$\lim_{n\to\infty} \int_E \langle f(t, u_n(t)) - f(t, u(t)), v\rangle_{\theta(t)} dt = 0. \tag{33}$$

Putting (32) and (33) together with the same subset E, we obtain

$$\int_E \langle f(t, u(t)) - \xi(t), v\rangle_{\theta(t)} dt = 0,$$

which implies

$$\langle f(t,u(t))-\xi(t),v\rangle_{\theta(t)}=0 \tag{34}$$

a.e. on E_ε and eventually a.e. on I by the arbitrariness of E and ε respectively.

Suppose that $\{v_k\}$ is a dense and countable subset of X_{α^+}, then there is a subset E_0 of I with zero complement on which (34) holds with v replaced by v_k for all $k\in\mathbb{N}$. Finally, using the density of X_{α^+} in $X_{\theta(t)}$, we deduce that $f(t,u(t))=\xi(t)$ in $X^*_{\theta(t)}$ on E_0. Therefore $F(u)=\xi$ and $F(u_n)\rightharpoonup F(u)$ in $L^{\varrho^*_{\theta(\cdot)}}(I,X^*_{\theta(\cdot)})$ as $n\to\infty$. Thus the proof has been completed. □

Putting Theorem 5.8 and Proposition 5.9 together, we can easily see that the composite operator

$$\mathcal{F}:=(D^2_{\theta(\cdot)}+\partial\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}})^{-1}\circ F: L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})\to L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})$$

is both continuous and compact. Moreover, we have

Theorem 5.10. *Under all the assumptions mentioned above, there is an $\mu_0>0$ such that as long as $0\le\mu\le\mu_0$, $\mathcal{F}$ has a fixed point in $L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})$, or in other words, second order differential inclusion*

$$-\frac{d}{dt}\partial\varrho_{\theta(t)}(u'(t))+\partial\varphi_{\vartheta(t)}(u(t))\ni f(t,u(t))\ \text{a.e. on } I \tag{35}$$

has a weak solution in $\mathcal{W}$.

Proof. Consider the set

$$S=\{u\in\mathcal{W}: u=\lambda\mathcal{F}(u)\ \text{for some } 0<\lambda<1\}.$$

Evidently, every member of S is a weak solution of the inclusion

$$-\frac{d}{dt}\partial\varrho_{\theta(t)}(\lambda^{-1}u'(t))+\partial\varphi_{\vartheta(t)}(\lambda^{-1}u(t))\ni f(t,u(t))\ \text{a.e. on } I.$$

By multiplying both sides of the above inclusion by $\lambda^{-1}u'(t)$, and integrate on I, then using the assumption (31), we can deduce that

$$\begin{aligned}\Phi_{\varrho_{\theta(\cdot)}}(\lambda^{-1}u')+\widetilde{\Phi}_{\varphi_{\vartheta(\cdot)}}(\lambda^{-1}u)&\le\langle\langle F(u),\lambda^{-1}u\rangle\rangle_{\theta(\cdot)}\\&\le\Phi_{\varrho^*_{\theta(\cdot)}}(\delta^{-1}F(u))+\Phi_{\varrho_{\theta(\cdot)}}(\delta\lambda^{-1}u)\\&\le\sigma(\delta^{-1})[\mu\Phi_{\varrho_{\theta(\cdot)}}(\lambda^{-1}u)+\|h\|_{L^1(I)}]+\delta\Phi_{\varrho_{\theta(\cdot)}}(\lambda^{-1}u)\\&=(\delta+\mu\sigma(\delta^{-1}))\Phi_{\varrho_{\theta(\cdot)}}(\lambda^{-1}u)+\|h\|_{L^1(I)}.\end{aligned}$$

By taking $\delta_0>0$ and $\mu_0>0$ as in Remark 5.6 and letting $0<\delta\le\delta_0$, $0\le\mu\le\mu_0$, we can claim that $\{\lambda^{-1}u: u\in S\}$ is bounded in $L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})$. Therefore, there is a constant $C>0$ independent of λ such that $\|u\|_{L^{\varrho_{\theta(\cdot)}}(I,X_{\theta(\cdot)})}\le C$ for all $u\in S$. Finally by invoking Leray-Schauder's alternative theorem for the compact and strongly continuous operators (refer to [18, Ch. 13] or [19]), we can assert the existence of the fixed point of $\mathcal{F}$. □

Remark 5.11. *From the demicontinuity of F and the compactness of the inverse $(D^2_{\theta(\cdot)}+\partial\Phi_{\varphi_{\vartheta(\cdot)}})^{-1}$, we can also check that solution set of (35) is a nonempty and compact subset of $\mathcal{W}$.*

Remark 5.12. *In our setting, periodic boundary condition can be replaced by the following one:*

$$u(T)=Ku(0),$$

where $K: X_{\theta(0)}\to X_{\theta(T)}$ is a bounded linear operator with other conditions unchanged.

At the end of this paper, let us choose an anisotropic elliptic partial differential equation of second order to illustrate our results,

$$\begin{cases} -D_t(|D_t u|^{p(x,t)-2}D_t u) - \sum_{i=1}^{N} D_i\big(|D_i u|^{q_i(x,t)-2}D_i u\big) \\ \quad = \mu g(t,x,u), \qquad (t,x) \in I \times \Omega, \\ u(0,x) = u(T,x), \quad x \in \Omega, \\ u(t,x) = 0, \qquad (t,x) \in I \times \partial\Omega. \end{cases} \tag{36}$$

Here D_t denotes the partial differential derivative with respect to t, p and q_i are doubly variable exponents introduced in Example 3.22 and Remark 4.13 respectively.

Suppose that $g : I \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function with a nonstandard growth, i.e. for a.e. $(t,x) \in I \times \Omega$, $u \mapsto g(t,x,u)$ is continuous, and for all $u \in \mathbb{R}$, $(t,x) \mapsto g(t,x,u)$ is measurable, together with

$$|g(t,x,u)| \le C(1 + |u|^{p(t,x)-1}) \tag{37}$$

holding for a.e. $(t,x) \in I \times \Omega$ and all $u \in \mathbb{R}$.

Let $f(t,u)(x) = g(t,x,u(t,x))$. Since $L^0(I, L^0(\Omega))$ is equivalent to $L^0(Q)$, one can easily check that $f(t,u)$ lies in $L^0(I, L^0(\Omega))$ provided u does. Moreover, combining (37) with the continuity of $g(t,x,u)$ with respect to u, we can find that $f(t,u) \in L^0(I, L^{\bar{p}^*}(\Omega))$ for all $u \in L^0(I, L^{\bar{p}}(\Omega))$, and $f(t,u)$ is weakly continuous from $L^{p(t,x)}(\Omega)$ to $L^{p'(t,x)}(\Omega)$ for a.e. $t \in I$, together with (31) verified.

Suppose that $p(t,x)$ and $q_i(t,x)$ $(i = 1, 2, \cdots, N)$ are variable exponents as in Example 3.22 and Remark 4.13 fulfilling

$$\inf_{x\in\Omega}(q_i^*(t,x) - p(t,x)) > 0, \ i = 1, 2, \cdots, N. \tag{38}$$

Using the notations and definitions in Examples 3.22, 4.11 and Remark 4.13, from equation (36), we then derive an abstract evolution equation (35) on the space

$$\mathcal{W} = W^{1,\varrho_{p(\cdot)}}_{\mathrm{per}}(I, L^{p(\cdot,x)}(\Omega)) \cap L^{\varrho_{\mathbf{q}(\cdot)}}(I, W^{1,\mathbf{q}(\cdot,x)}_0(\Omega)).$$

From all the assumptions of Example 3.22 and Remark 4.13 together with (38), one can easily verify all the conditions listed in Theorem 5.3 as well as (25). Thus in terms of Theorem 5.10, equation (35) has a solution u in $\mathcal{W}$, whose scalar version, still denoted by u, solves (36) in the sense of distribution, i.e.

$$\int_Q (|D_t u|^{p(x,t)-2}D_t u)D_t v dx dt + \sum_{i=1}^{N}\int_Q (|D_i u|^{q_i(x,t)-2}D_i u)D_i v dx dt = \int_Q f(t,u)v(t,x)dx dt$$

for all $v \in C^1_{\mathrm{per}}(I, C^\infty(\overline{\Omega}))$. In conclusion, under all the conditions upon h, p and q_i $(i = 1, 2, \cdots, N)$ including (38), there is a $\mu_0 > 0$ such that for all $0 \le \mu \le \mu_0$, Equation (36) has a weak solution.

Remark 5.13. *Unlike the traditional one, here we do not need $u(t,x) = 0$ on the whole boundary ∂Q and do not require the whole log-Hölder continuity of p and q_i $(i = 1, 2, \cdots, N)$. Hence Poincaré's inequality*

$$\|u\|_{L^{p(t,x)}(Q)} \le C\|\nabla u\|_{L^{\mathbf{q}(t,x)}(Q)}$$

could not be applied even if $p(t,x) \le q_i(t,x)$ for all $i \in \{1, 2, \cdots, N\}$. By these reasons, here we give up $W^{1,p(t,x)}_0(Q)$ as the work space, instead, an anisotropic space $W^{1,\varrho_{p(\cdot)}}_{per}(I, L^{p(\cdot,x)}(\Omega)) \cap L^{\varrho_{\mathbf{q}(\cdot)}}(I, W^{1,\mathbf{q}(\cdot,x)}_0(\Omega))$ is taken into account, while the elliptic equation (36) turns to be an abstract second order evolution equation. To our best knowledge, this way to deal with the anisotropic elliptic equations with nonstandard growth is new, and different to that applied in available literature.

Acknowledgement: This work is supported by Preresearch Project of Nantong University (17ZY01).

References

[1] Orlicz W., Über konjugierte exponentenfolgen, Studia Math., 1931, 3, 200-211.

[2] Diening L., Harjulehto P., Hästö P., and Růžička M., Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics, vol. 2017, 2011, Berlin: Springer-Verlag Press.
[3] Antontsev S. N. and Shmarev S. I., A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions, Nonlinear Anal., 2005, 60 (3), 515-545.
[4] Antontsev S. N., Shmarev S. I., Blow-up of solutions to parabolic equations with nonstandard growth conditions, J. Comput. Appl. Math., 2010, 234 (9), 2633-2645.
[5] Antontsev S. N., Shmarev S. I., Vanishing solutions of anisotropic parabolic equations with variable nonlinearity, J. Math. Anal. Appl., 2010, 361 (2), 371-391.
[6] Akagi G., Doubly nonlinear parabolic equations involving variable exponents, Discrete Contin. Dyn. Syst. Ser. S, 2014, 7 (1), 1-16.
[7] Akagi G., Schimperna G., Subdifferential calculus and doubly nonlinear evolution equations in L^p-spaces with variable exponents, J. Funct. Anal., 2014, 267 (1), 173-213.
[8] Wróblewska-Kamińska A., An application of Orlicz spaces in partial differential equations (PhD Thesis), 2012, Warsaw: University of Warsaw.
[9] Rodrigues J., Sanchón M., and Urbano J., The obstacle problem for nonlinear elliptic equations with variable growth and L^1-data, Monatsh Math., 2008, 154, 303-330.
[10] Zhang Q., Li G., On the $X_{\theta(\cdot)}-$valued function space: definition, property and applications, J. Math. Anal. Appl., 2016, 440 (1), 48-64.
[11] Zhang Q., Li G., Classification and geometrical properties of the $X_{\theta(\cdot)}-$valued function spaces, J. Math. Anal. Appl., 2017, 452 (1), 1359-1387.
[12] Barbu V., Nonlinear Semigroups and Differential Equations in Banach Spaces, 1976, Leyden: Noordhoff Press.
[13] Neerven J. V., Stochastic Evolution Equations, ISEM Lecture Notes, 2007/08.
[14] Aubin J.-P., Optima and Equilibria, An Introduction to Nonlinear Analysis (2nd Edition), 1998, New York: Springer-Verlag Press.
[15] Aubin J.-P., Frankowska H., Set-Valued Analysis, 1990, Boston: Birkhäuser Press.
[16] Simon J., Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl., 1987, 146, 65-96.
[17] Serrano R., An alternative proof of the Aubin-Lions lemma, Arch. Math., 2013, 101, 253-257.
[18] Zeidler E., Nonlinear Functional Analysis and Its Applications, Vol. I: Fixed-Point Theorems, 1990, New York: Springer-Verlag Press.
[19] Bader R., A topological fixed-point index theory for evolution inclusions, Z. Anal. Anwend, 2000, 20, 3-15.