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Research Article

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On some varieties of ai-semirings satisfying $x^{p+1} \approx x$

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Abstract: The aim of this paper is to study the lattice of subvarieties of the ai-semiring variety defined by the additional identities

$$x^{p+1} \approx x \text{ and } zxyz \approx (zxzyz)^p zyxz(zxzyz)^p,$$

where p is a prime. It is shown that this lattice is a distributive lattice of order 179. Also, each member of this lattice is finitely based and finitely generated.

Keywords: Burnside ai-semiring, Identity, Lattice, Variety

MSC: 08B15, 08B05, 16Y60, 20M07

1 Introduction

Semirings (see [9]) abound in the mathematical world around us. The set of natural numbers, the first mathematical structure we encounter, is a semiring. The intensive study of semiring theory was initiated during the late 1960's when their real and significant applications were found. Nowdays, semiring theory is an enormously broad topic and has advanced on a very broad front. Semirings $(S, +, \cdot)$ occurring in the literature satisfy at least the following axioms: $(S, +)$ and $(S, \cdot)$ are semigroups, and the multiplication distributes over addition from both sides. It is often assumed that $(S, +)$ is idempotent and/or commutative. A semiring S is an *additively idempotent semiring*, or shortly *ai-semiring*, if $(S, +)$ is a semilattice (it is also called a *semilattice-ordered semigroup* in [8, 10, 11]). It is well-known that the endomorphism semiring of a semilattice is an ai-semiring. Also every ai-semiring can be embeded into the endomorphism semiring of some semilattice (see [8, 12]). Important role in mathematics as well as broad applications (in theoretical computer science, optimization theory, quantum physics and many other areas of science [9, 13–15]) make ai-semirings and, especially, their varieties to be among the favourite subjects for the researchers in the algebraic theory of semirings.

The variety of all ai-semirings is denoted by $\mathbf{AI}$. Let X be a fixed countably infinite set of variables and X^+ the free semigroup on X. Then $P_f(X^+)$ is free in $\mathbf{AI}$ on X (see [8]). An $\mathbf{AI}$-*identity* means an identity $u \approx v$, where $u = u_1 + \cdots + u_k$, $v = v_1 + \cdots + v_\ell$, u_i, $v_j \in X^+$, $i \in \underline{k}$, $j \in \underline{\ell}$, $\underline{k} = \{1, 2, \ldots, k\}$. Recall that an ai-semiring is called a *Burnside ai-semiring* if its multiplicative reduct is a Burnside semigroup, i.e., it satisfies the identity $x^n \approx x^m$ with $m < n$ (see [16–18]). The variety of all Burnside ai-semirings (resp., Burnside semigroups) satisfying the identities $x^n \approx x^m$ will be denoted by $\mathbf{Sr}(n, m)$ (resp., $\mathbf{Sg}(n, m)$). There are many papers in the literature considering Burnside semigroups and Burnside ai-semirings (see [1, 2, 4–8, 10, 16–20]). In particular, in 1979,

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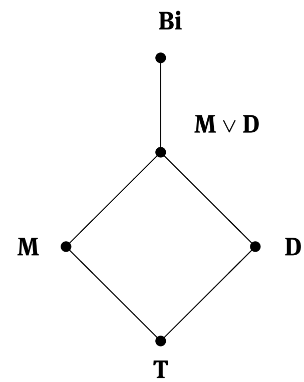
McKenzie and Romanowska [19] studied the lattice of subvarieties of the subvariety **Bi** of $\mathbf{Sr}(2,1)$ defined by the additional identity $xy \approx yx$. They showed that this lattice contains precisely 5 elements: the trivial variety **T**, the variety **D** of distributive lattices, the ai-semiring variety **M** defined by the additional identity $x + y \approx xy$, the varieties $\mathbf{D} \vee \mathbf{M}$ and **Bi** (see Figure 1).

Fig. 1. The lattice of subvarieties of **Bi**



In 2002, Zhao [7] studied the variety $\mathbf{Sr}(2,1)$, which need not satisfy $xy \approx yx$. They provided a model of the free object in this variety by introducing the notion of closed subsemigroup of a semigroup. In 2005, Ghosh *et al.* [1], Pastijn [2] and Pastijn and Zhao [3] studied the lattice of subvarieties of $\mathbf{Sr}(2,1)$. They showed that this lattice is a distributive lattice of order 78 and that every member of this lattice is finitely based and is generated by a finite number of finite ordered bands.

Along this research route, some authors studied the subvarieties of $\mathbf{Sr}(n,1)$. In 2005, Kuřil and Polák generalized the notion of closed subsemigroups introduced by Zhao [7] to that of n-closed subset of a semigroup. They provided a construction of the free object in $\mathbf{Sr}(n,1)$ by using n-closed subset of the free object in $\mathbf{Sg}(n,1)$. Moreover, Gajdoš and Kuřil [10] showed that $\mathbf{Sr}(n,1)$ is locally finite if and only if $\mathbf{Sg}(n,1)$ is locally finite. In 2015, Ren and Zhao [21] introduced the notion of (n,m)-closed subsets of a semigroup and gave a model of the free object in $\mathbf{Sr}(n,m)$ by using (n,m)-closed subset of the free object in $\mathbf{Sg}(n,m)$. In 2016, Ren, Zhao and Shao [20] proved that the multiplicative semigroup of each member of $\mathbf{Sr}(n,1)$ is a regular orthocryptogroup. As an application, a model of the free object in such a variety is given. In the same year, Ren and Zhao [4] studied the lattice of subvarieties of the subvariety of $\mathbf{Sr}(3,1)$ defined by the additional identity $xy \approx yx$. They showed that it is a 9-element distributive lattice. As a continuation of [4], Ren *et al.* [5] studied the lattice of subvarieties of the subvariety of $\mathbf{Sr}(n,1)$ defined by the additional identity $xy \approx yx$. They showed that if $n-1$ is square-free, then this lattice is a $2+2^{r+1}+3^r$-element distributive lattice, where r denotes the number of prime divisors of $n-1$. They also proved that this lattice is finitely based and finitely generated. In 2017, Ren *et al.* [6] studied that the lattice of the subvarieties of $\mathbf{Sr}(3,1)$. They showed that this lattice is a 179-element distributive lattice. They also showed that every member of this lattice is finitely based and finitely generated. This paper is another contribution to this line of investigation. We shall characterize the lattice of subvarieties of the subvariety of $\mathbf{Sr}(p+1,1)$ defined by the additional identities $zxyz \approx (zxzyz)^p zyxz(zxzyz)^p$.

This paper is organized as follows. After this introductory section, in Sect. 2 we shall give some auxiliary results and notations that are needed in the sequel. In Sect. 3 we shall study the subvariety of $\mathbf{Sr}(p+1,1)$ defined by the additional identity $zxyz \approx (zxzyz)^p zyxz(zxzyz)^p$. We shall show that its lattice of subvarieties is a distributive lattice of order 179. Also, all members of this lattice are finitely based and finitely generated. In particular, $\mathbf{Sr}(3,1)$ is just the case of above variety when $p = 2$. Thus our main results generalize and extend the main results in [1–4, 6, 19].

For notation and terminology not given in this paper, the reader is referred to [22–25].

2 Preliminaries and some notations

Let $\mathbf{A}_{p+1}$ denote the group variety defined by the identities

$$x^{p+1} \approx x, \tag{1}$$
$$xy \approx yx. \tag{2}$$

For any $S \in \mathbf{Sr}(3,1)$, every subgroup of $(S,\cdot)$ satisfies the identity $x^3 \approx x$. This shows that every subgroup of $(S,\cdot)$ is an abelian group in $\mathbf{A}_3$. That is to say, $(S,\cdot)$ is a union of abelian groups which belong to $\mathbf{A}_3$.

For any semigroup $S \in \mathbf{Sg}(p+1,\ 1)$, by [22, Proposition II.7.1 and Exercises V.5.8 (V)]), we have

Lemma 2.1. *If S satisfies the identity* $zxyz \approx (zxzyz)^p zyxz(zxzyz)^p$, *then S is a union of abelian groups which belong to* $\mathbf{A}_{p+1}$.

We denote by $\mathbf{ROBA}_{p+1}$ the subvariety of $\mathbf{Sg}(p+1,1)$ defined by the additional identity $zxyz \approx (zxzyz)^p zyxz(zxzyz)^p$. As usual, for a semigroup variety $\mathbf{V}$, we denote by $\mathbf{V}^\circ$ the semiring variety consisting of all ai-semirings whose multiplicative reduct belongs to $\mathbf{V}$. It is easy to see that $\mathbf{ROBA}_{p+1}^\circ$ is just the variety of all ai-semirings whose multiplicative reduct is a union of abelian groups which belong to $\mathbf{A}_{p+1}$, i.e., the ai-semiring variety defined by the additional identities

$$x^{p+1} \approx x \text{ and } zxyz \approx (zxzyz)^p zyxz(zxzyz)^p.$$

In particular, $\mathbf{Sr}(3,1)$ is equal to $\mathbf{ROBA}_3^\circ$.

Let $\mathbf{SA}_{p+1}$ denote the semigroup variety defined by the identities (1) and (2), and $\mathbf{S}\ell$ denote the variety of semilattices. By [22, Lemma IV.2.3, Theorem IV. 2.4 and Exercise IV.2.16 (xii(γ))], we have

Lemma 2.2. $\mathbf{SA}_{p+1} = \mathbf{S}\ell \vee \mathbf{A}_{p+1}$.

Let ω be an element of X^+. The following notions and notation are needed for solving the word problem for $\mathbf{SA}_{p+1}$ and $\mathbf{ROBA}_{p+1}$:

- $i(\omega)$ denotes the *initial part* of ω, i.e., the word obtained from ω by retaining only the first occurrence of each variable.
- $f(\omega)$ denotes the *final part* of ω, i.e., the word obtained from ω by retaining only the last occurrence of each variable.
- $c(\omega)$ denotes the *content* of ω, i.e., the set of all variables occurring in ω.
- $m(x,\ \omega)$ denotes the *multiplicity* of x in ω, i.e., the number of occurrence of x in ω.
- $r_i(\omega)$ denotes the set $\{x \in c(\omega) \mid i \equiv m(x,\ \omega) \pmod p\}$, where $i \in \{0,1,\dots,p-1\}$.
- $r_{p+1}(\omega)$ denotes the set $\{x^k \mid x \in c(\omega), x \in \underline{p-1}, k \equiv m(x,\ \omega) \pmod p\}$.
- $\overline{\omega}$ denotes the word obtaining from ω by deleting all occurrences of variables which belong to $r_0(\omega)$.

It is easy to check that a semigroup identity $u \approx v$ is satisfied by $\mathbf{A}_{p+1}$ if and only if $r_i(u) = r_i(v)$ for all $i \in \underline{p-1}$. Thus, for any semigroup identity $u \approx v$,

$$\mathbf{SA}_{p+1} \vDash u \approx v \Leftrightarrow c(u) = c(v),\ r_i(u) = r_i(v)\ (\forall\ i \in \underline{p-1}), \tag{3}$$
$$\mathbf{ROBA}_{p+1} \vDash u \approx v \Leftrightarrow i(u) = i(v),\ f(u) = f(v),\ r_i(u) = r_i(v)\ (\forall\ i \in \underline{p-1}), \tag{4}$$

where $\mathbf{ROBA}_{p+1} = \mathbf{ReB} \vee \mathbf{A}_{p+1}$ (see [22, Theorem V.5.3]), and $\mathbf{ReB}$ the variety of regular bands. Now we have

Lemma 2.3. $\mathbf{ROBA}_{p+1}^\circ$ *satisfies the following identities*

$$(xy)^p \approx x^p y^p, \tag{5}$$

$$x^p yx \approx xyx^p, \tag{6}$$
$$xyzx \approx xyx^p zx, \tag{7}$$
$$xy + z \approx xy + z + xz^p y, \tag{8}$$
$$x_1 + x_2 + \cdots + x_{p+1} \approx x_1 + x_2 + \cdots + x_{p+1} + x_1x_2\cdots x_{p+1}. \tag{9}$$

Proof. By (4), it follows immediately that (5), (6) and (7) hold in $\mathbf{ROBA}^\circ_{p+1}$. Also,

$$\begin{aligned}
&x_1 + x_2 + \cdots + x_{p+1} \\
\approx\ &(x_1 + x_2 + \cdots + x_{p+1})^{p+1} && (\text{since } \mathbf{ROBA}^\circ_{p+1} \models (1)) \\
\approx\ &\textstyle\sum_{i_1,i_2,\ldots,i_{p+1}\in \underline{p+1}} x_{i_1}x_{i_2}\cdots x_{i_{p+1}} \\
\approx\ &\textstyle\sum_{i_1,i_2,\ldots,i_{p+1}\in \underline{p+1}} x_{i_1}x_{i_2}\cdots x_{i_{p+1}} + x_1x_2\cdots x_{p+1} && (\text{since } \mathbf{ROBA}^\circ_{p+1} \models x + x \approx x) \\
\approx\ &(x_1 + x_2 + \cdots + x_{p+1})^{p+1} + x_1x_2\cdots x_{p+1} \\
\approx\ &x_1 + x_2 + \cdots + x_{p+1} + x_1x_2\cdots x_{p+1}.
\end{aligned}$$

Thus, (9) holds in $\mathbf{ROBA}^\circ_{p+1}$. In the remainder we need only to prove that (8) holds in $\mathbf{ROBA}^\circ_{p+1}$. The following is derivable from the identities determining $\mathbf{ROBA}^\circ_{p+1}$ and identities (4), (5), (9):

$$\begin{aligned}
xy + z &\approx xy + z + xyz^p + z^{p-1}xyz && (\text{by } (9)) \\
&\approx xy + z + x^{p+1}yz^p + z^{p-1}xyz(xy)^p z^p && (\text{by } (5)) \\
&\approx xy + z + (x^p + z^{p-1}xyz(xy)^{p-1})xyz^p \\
&\approx xy + z + (x^p + z^p x^p y^p + x^p z^p x^p y^p)xyz^p && (\text{by } (4), (9)) \\
&\approx xy + z + x^{p+1}yz^p + z^p xyz^p + x^p z^p xyz^p.
\end{aligned}$$

Thus, $\mathbf{ROBA}^\circ_{p+1}$ satisfies the identity

$$xy + z \approx xy + z + x^p z^p xyz^p.$$

In a left-right dual way we can show that $\mathbf{ROBA}^\circ_{p+1}$ satisfies the identity

$$xy + z \approx xy + z + z^p xyz^p y^p.$$

Furthermore, we have

$$\begin{aligned}
&xy + z \\
\approx\ &xy + z + x^p z^p xyz^p + z^p xyz^p y^p + xyz^p && (\text{by } (9)) \\
\approx\ &xy + z + x^p z^p xyz^p + z^p xyz^p y^p + xyz^p \\
&+ x^p z^p xyz^p \cdot (xyz^p)^{p-1} \cdot z^p xyz^p y^p && (\text{by } (9)) \\
\approx\ &xy + z + x^p z^p xyz^p + z^p xyz^p y^p + xyz^p + x^p z^p xyz^p y^p && (\text{by } (4)) \\
\approx\ &xy + z + x^p z^p xyz^p + z^p xyz^p y^p + xyz^p + x^p z^p xz^p yz^p y^p && (\text{by } (7)) \\
\approx\ &xy + z + x^p z^p xyz^p + z^p xyz^p y^p + xyz^p + (xz)^p xz^p y(zy)^p && (\text{by } (5)) \\
\approx\ &xy + z + x^p z^p xyz^p + z^p xyz^p y^p + xyz^p + xz^p y.
\end{aligned}$$

Thus, $\mathbf{ROBA}^\circ_{p+1}$ satisfies identity (8). □

Lemma 2.4. *Let S be an ai-semiring in $\mathbf{Sr}(p+1, 1)$. Then the following is true,*

$$(\forall a \in S)\,(\forall i, j \in \underline{p},\ i \neq j)\ a^i + a^j = a + a^2 + \cdots + a^p. \tag{10}$$

Proof. Suppose that S is an ai-semiring in $\mathbf{Sr}(p+1, 1)$. Then for any $a \in S$, $a^{p+1} = a$. Without loss of generality, suppose that $i, j \in \underline{p}$ and $i < j$. We have

$$\begin{aligned}
a^i + a^j &= a^i(a^p + a^{j-i}) \\
&= a^i(a^p + a^{j-i})^{p+1} && (\text{since } S \models x^{p+1} \approx x) \\
&= a^i(a^{j-i} + a^{2(j-i)} + \cdots + a^{p(j-i)}) \\
&= a^i(a + a^2 + \cdots + a^p) && (\text{since } p \text{ is a prime}) \\
&= a + a^2 + \cdots + a^p.
\end{aligned}$$

This completes the proof. □

Let $u \approx v$ be an **AI**-identity and Σ a set of identities which include the identities determining **AI**. Under the presence of the identities determining **AI**, it is easy to verify that $u \approx v$ gives rise to the identities $u \approx u + v_j$, $v \approx v + u_i$, $i \in \underline{k}, j \in \underline{\ell}$. Conversely, the latter $k + \ell$ identities give rise to $u \approx u + v \approx v$. Thus, to show that $u \approx v$ is derivable from Σ, we only need to show that the simpler identities $u \approx u + v_j$, $v \approx v + u_i$, $i \in \underline{k}, j \in \underline{\ell}$ are derivable from Σ.

3 The lattice $\mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1})$

In the current section, we shall characterize the lattice $\mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1})$ of subvarieties of $\mathbf{ROBA}^{\circ}_{p+1}$, and prove that each member of this lattice is finitely based and finitely generated.

Let $(Z_p, \cdot)$ be the cyclic group of order p and Z_p^0 the 0-group obtained from Z_p by adjoining an extra element 0, where $a \cdot 0 = 0 \cdot a = 0$ for every $a \in Z_p \cup \{0\}$. Define an additive operation on Z_p^0 as follows:

$$a + b = \begin{cases} a, & \text{if } a = b; \\ 0, & \text{otherwise.} \end{cases}$$

Then $(Z_p^0, +, \cdot)$ forms an ai-semiring. It is easy to check that Z_p^0 is in $\mathbf{ROBA}^{\circ}_{p+1}$, but not in $\mathbf{Sr}(2, 1)$. In fact, we have

Lemma 3.1. *Let S be an ai-semiring in $\mathbf{ROBA}^{\circ}_{p+1}$. Then S is a member of $\mathbf{Sr}(2, 1)$ if and only if it does not contain a copy of Z_p^0.*

Proof. The direct part is obvious. Conversely, suppose that S is an ai-semiring in $\mathbf{ROBA}^{\circ}_{p+1}$, but not in $\mathbf{Sr}(2, 1)$. Then there exists $a \in S$ such that a is not equal to a^p. Thus by Lemma 2.4, we can show that $\{a, a^2, \dots, a^p, a + a^2 + \cdots + a^p\}$ is a subsemiring of S and is a copy of Z_p^0. This completes the proof. □

Let $\overline{\mathbf{M}_p}$ denote $\mathbf{SA}^{\circ}_{p+1} \cap [x^p + y^p \approx x^p y^p]$. From [5, Lemma 4.12], $\mathbf{HSP}(Z_p^0) = \overline{\mathbf{M}_p}$. Also, by [5, Proposition 4.13], the interval $[\mathbf{M}, \overline{\mathbf{M}_p}]$ consists of the two varieties $\mathbf{M}$ and $\overline{\mathbf{M}_p}$. Thus we have

Lemma 3.2. *The interval $[\mathbf{T}, \overline{\mathbf{M}_p}]$ consists of the three varieties $\mathbf{T}$, $\mathbf{M}$ and $\overline{\mathbf{M}_p}$.*

Proof. Let $\mathbf{V} \in [\mathbf{T}, \overline{\mathbf{M}_p}]$ and $\mathbf{V} \neq \mathbf{T}$. Then $\mathbf{M} \subseteq \mathbf{V}$ since $\mathbf{M}$ is minimal nontrivial subvariety of $\overline{\mathbf{M}_p}$. It follows that $\mathbf{V} \in [\mathbf{M}, \overline{\mathbf{M}_p}]$ and so $\mathbf{V} = \mathbf{M}$ or $\mathbf{V} = \overline{\mathbf{M}_p}$. □

Let $u = u_1 + \cdots + u_k$, where $u_i \in X^+, i \in \underline{k}$. We put $C(u) = \bigcup_{i \in \underline{k}} c(u_i)$. To solve the word problem of the ai-semiring Z_p^0 we need the following Lemmas, which are the special cases of [5, Corollary 4.15 and Lemma 5.1] when $n = p + 1$.

Lemma 3.3. *Let $u \approx u + q$ be an **AI**-identity, where $u = u_1 + \cdots + u_k, u_i, q \in X^+, i \in \underline{k}$. Then Z_p^0 satisfies $u \approx u + q$ if and only if $c(q) \subseteq C(u)$ and $r_{p+1}(q) = r_{p+1}(u_{i_1} \cdots u_{i_{(p+1)\ell}})$ for some positive integer ℓ and some $u_{i_1}, \dots, u_{i_{(p+1)\ell}} \in \{u_i \mid i \in \underline{k}\}$.*

Lemma 3.4. *Let $u \approx u + q$ be an **AI**-identity, where $u = u_1 + \cdots + u_k$, u_i, $q \in X^+$, $i \in \underline{k}$. If Z_p^0 satisfies $u \approx u + q$, then there exists q_1 in X^+ with $r_{p+1}(q_1) = r_{p+1}(q)$ and $c(q) \subseteq c(q_1) \subseteq C(u)$ such that $u \approx u + q_1$ is satisfied in $\mathbf{ROBA}^{\circ}_{p+1}$.*

For a semiring variety $\mathbf{V}$, we denote by $\mathcal{L}(\mathbf{V})$ the lattice of all subvarieties of $\mathbf{V}$. In the following we shall characterize the lattice $\mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1})$. Suppose that S be a member of $\mathbf{ROBA}^{\circ}_{p+1}$. It is shown in [20, Lemma 2.1] that $E(S)$ forms a member of $\mathbf{Sr}(2, 1)$. Define a mapping φ as follows

$$\varphi : \mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1}) \to \mathcal{L}(\mathbf{Sr}(2, 1)), \ \mathbf{V} \mapsto \mathbf{V} \cap \mathbf{Sr}(2, 1).$$

Then it is easy to see that φ is subjective. For any $\mathbf{V} \in \mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1})$, if $S \in \varphi(\mathbf{V})$, then $S \in \mathbf{V} \cap \mathbf{Sr}(2,1)$ and so $S \in \{E(S)|S \in \mathbf{V}\}$. It follows that $\varphi(\mathbf{V}) \subseteq \{E(S)|S \in \mathbf{V}\}$. On the other hand, if $E(S) \in \{E(S)|S \in \mathbf{V}\}$, then $S \in \mathbf{V}$. Since $E(S)$ is a subsemiring of S, we have that $E(S) \in \mathbf{V}$ and so $E(S) \in \mathbf{V} \cap \mathbf{Sr}(2,1)$. It follows that $\{E(S)|S \in \mathbf{V}\} \subseteq \varphi(\mathbf{V})$. Therefore, $\varphi(\mathbf{V}) = \{E(S)|S \in \mathbf{V}\}$. We use $t(x_1, \dots, x_n)$ to denote the **AI**-term t which contains no other variables than $x_1, \dots, x_n$ (but not necessarily all of them). If $\mathbf{W}$ is the subvariety of $\mathbf{Sr}(2,1)$ determined by the additional identities

$$u(x_1, \dots, x_n) \approx v(x_1, \dots, x_n),$$

then we denote by $\widehat{\mathbf{W}_p}$ the subvariety of $\mathbf{ROBA}^{\circ}_{p+1}$ determined by the additional identities

$$u(x_1^p, \dots, x_n^p) \approx v(x_1^p, \dots, x_n^p).$$

Then for any $\mathbf{V} \in [\mathbf{W}, \widehat{\mathbf{W}}_p]$,

$$\mathbf{W} = \varphi(\mathbf{W}) \subseteq \varphi(\mathbf{V}) \subseteq \varphi(\widehat{\mathbf{W}}_p) = \widehat{\mathbf{W}}_p \cap (\mathbf{Sr}(2,1)) = \mathbf{W},$$

it follows that $\mathbf{V} \in \varphi^{-1}(\mathbf{W})$ and so $[\mathbf{W}, \widehat{\mathbf{W}}_p] \subseteq \varphi^{-1}(\mathbf{W})$. Conversely, if $\mathbf{V} \in \varphi^{-1}(\mathbf{W})$, then $\mathbf{V} \in [\mathbf{W}, \widehat{\mathbf{W}}_p]$. Otherwise, $\varphi(\mathbf{V}) \neq \mathbf{W}$, a contradiction. Thus, for any $\mathbf{W} \in \mathcal{L}(\mathbf{Sr}(2,1))$, $\varphi^{-1}(\mathbf{W})$ is the interval $[\mathbf{W}, \widehat{\mathbf{W}_p}]$ of $\mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1})$. Moreover, if $\mathbf{W}_1, \mathbf{W}_2 \in \mathcal{L}(\mathbf{Sr}(2,1))$ such that $\mathbf{W}_1 \subseteq \mathbf{W}_2$, then it is easy to check that $\widehat{\mathbf{W}}_1 \subseteq \widehat{\mathbf{W}}_2$. Suppose that $(\mathbf{V}_i)_{i\in I}$ is a family of varieties in $\mathbf{ROBA}^{\circ}_{p+1}$. Then $\varphi(\mathbf{V}_i) \subseteq \mathbf{V}_i \subseteq \widehat{\varphi(\mathbf{V}_i)}$ for all $i \in I$. Furthermore,

$$\bigvee_{i\in I}\varphi(\mathbf{V}_i) \subseteq \bigvee_{i\in I}\mathbf{V}_i \subseteq \bigvee_{i\in I}\widehat{\varphi(\mathbf{V}_i)} \subseteq \widehat{\bigvee_{i\in I}\varphi(\mathbf{V}_i)}.$$

This implies that $\varphi(\bigvee_{i\in I}\mathbf{V}_i) = \bigvee_{i\in I}\varphi(\mathbf{V}_i)$ and so φ is a complete $\vee$-epimorphism. Moreover, it is clear that φ is a complete $\wedge$-epimorphism. It follows that φ is a complete epimorphism. Thus, $\mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1}) = \bigcup_{\mathbf{W}\in\mathcal{L}(\mathbf{Sr}(2,1))}[\mathbf{W}, \widehat{\mathbf{W}_p}]$.

Notice that $\mathbf{M}$ is the subvariety of $\mathbf{Sr}(2,1)$ determined by the additional identities $xy \approx yx$ and $x + y \approx xy$. Thus $\widehat{\mathbf{M}_p}$ is the subvariety of $\mathbf{ROBA}^{\circ}_{p+1}$ determined by the additional identities $x^py^p \approx y^px^p$ and $x^p + y^p \approx x^py^p$. By [22, Lemma IV.2.3], $\mathbf{ROBA}^{\circ}_{p+1} \cap [x^py^p \approx y^px^p] = \mathbf{SA}^{\circ}_{p+1}$ and so $\widehat{\mathbf{M}_p} = \overline{\mathbf{M}_p}$.

Recall (see [1]) that the lattice $\mathcal{L}(\mathbf{Sr}(2,1))$ can be divided into five intervals: $[\mathbf{T}, \mathbf{N} \cap \mathbf{P} \cap \mathbf{Sr}(2,1)]$, $[\mathbf{D}, \mathbf{N} \cap \mathbf{Sr}(2,1)]$, $[\mathbf{M}, \mathbf{P} \cap \mathbf{Sr}(2,1)]$, $[\mathbf{D} \vee \mathbf{M}, \mathbf{K} \cap \mathbf{Sr}(2,1)]$ and $[\mathbf{Bi}, \mathbf{Sr}(2,1)]$, where $\mathbf{N} \cap \mathbf{Sr}(2,1)$ is the subvariety of $\mathbf{Sr}(2,1)$ determined by the identity

$$x \approx x + xyx, \tag{11}$$

$\mathbf{P} \cap \mathbf{Sr}(2,1)$ is the subvariety of $\mathbf{Sr}(2,1)$ determined by the identity

$$xyx \approx x + xyx, \tag{12}$$

and $\mathbf{K} \cap \mathbf{Sr}(2,1)$ is the subvariety of $\mathbf{Sr}(2,1)$ determined by the identity

$$x + xyx + xyzyx \approx x + xyzyx. \tag{13}$$

Thus we have

Theorem 3.5. *Let* $\mathbf{V} \in [\mathbf{T}, \mathbf{N} \cap \mathbf{P} \cap \mathbf{Sr}(2,1)] \cup [\mathbf{D}, \mathbf{N} \cap \mathbf{Sr}(2,1)]$. *Then* $\widehat{\mathbf{V}_p} = \mathbf{V}$.

Proof. From $\mathbf{N} \cap \mathbf{Sr}(2,1) \models (11)$, we have that $\mathbf{N} \cap \widehat{\mathbf{Sr}(2,1)}_p$ satisfies the identity

$$x^p \approx x^p + x^py^px^p. \tag{14}$$

Notice that $Z_p^0 \not\models (14)$. Then $Z_p^0 \notin \mathbf{N} \cap \widehat{\mathbf{Sr}(2,1)}_p$ and so by Lemma 3.1, $\mathbf{N} \cap \widehat{\mathbf{Sr}(2,1)}_p \subseteq \mathbf{Sr}(2,1)$. It follows from $\widehat{\mathbf{V}_p} \subseteq \mathbf{N} \cap \widehat{\mathbf{Sr}(2,1)}_p$ that $\widehat{\mathbf{V}_p} \subseteq \mathbf{Sr}(2,1)$. □

Assume that $\mathbf{V} \in \mathcal{L}(\mathbf{Sr}(2,1))$ and $\mathbf{V} \models u \approx u + q$, where $u = u_1 + \cdots + u_k$, u_i, $q \in X^+$, $i \in \underline{k}$. From $\mathbf{ROBA}^{\circ}_{p+1} \models (5)$ it follows that $\widehat{\mathbf{V}_p}$ satisfies the identity

$$u_1^p + \cdots + u_k^p \approx u_1^p + \cdots + u_k^p + q^p. \tag{15}$$

Lemma 3.6. *Let* $\mathbf{V} \in [\mathbf{M}, \mathbf{P} \cap \mathbf{Sr}(\mathbf{2}, \mathbf{1})]$. *Then* $\mathbf{V} \vee \mathbf{HSP}(Z_p^0) = \widehat{\mathbf{V}_p}$.

Proof. Since $\mathbf{P} \cap \mathbf{Sr}(2, 1) \models (12)$, it follows that $\mathbf{P} \cap \widehat{\mathbf{Sr}(2, 1)}_p$ satisfies the identity

$$x^p y^p x^p \approx x^p + x^p y^p x^p. \tag{16}$$

We immediately have that this is the case for $\widehat{\mathbf{V}_p}$, too. Notice that $\mathbf{HSP}(Z_p^0) = \widehat{\mathbf{M}_p}$. Thus $\mathbf{V} \vee \mathbf{HSP}(Z_p^0) \subseteq \widehat{\mathbf{V}_p}$. It remains to show that every identity which is satisfied in $\mathbf{V} \vee \mathbf{HSP}(Z_p^0)$ can be deduced from the identities which hold in $\widehat{\mathbf{V}_p}$.

Let $u \approx u + q$ be an **AI**-identity which is satisfied in $\mathbf{V} \vee \mathbf{HSP}(Z_p^0)$, where $u = u_1 + \cdots + u_k$, u_i, $q \in X^+$, $i \in \underline{k}$. Since Z_p^0 satisfies this identity, by Lemma 3.4 there exists q_1 in X^+ with $r_{p+1}(q_1) = r_{p+1}(q)$ (and so $r_i(q_1) = r_i(q)$ for all $i \in \underline{p-1}$) and $c(q) \subseteq c(q_1)$ such that $u \approx u + q_1$ is satisfied in $\mathbf{ROBA}_{p+1}^{\circ}$. Thus the following are derivable from the identities which hold in $\widehat{\mathbf{V}_p}$:

$$\begin{aligned} u &\approx u + q_1 \\ &\approx u + q_1 + u_1^p q_1 + \cdots + u_k^p q_1 && \text{(by (9))} \\ &\approx u + q_1 + (u_1^p + \cdots + u_k^p) q_1 \\ &\approx u + q_1 + (u_1^p + \cdots + u_k^p + q^p) q_1 && \text{(by (15))} \\ &\approx u + q_1 + (u_1^p + \cdots + u_k^p) q_1 + q^p q_1. \end{aligned}$$

This derives the identity

$$u \approx u + q^p q_1.$$

In a left-right dual way we have

$$u \approx u + q_1 q^p.$$

Therefore the following are satisfied in $\widehat{\mathbf{V}_p}$:

$$\begin{aligned} u &\approx u + q^p q_1 + q_1 q^p \\ &\approx u + q^p q_1 + q_1 q^p + q^p q_1 (q_1 q^p)^p && \text{(by (9))} \\ &\approx u + q^p q_1 + q_1 q^p + q^p q_1 q^p && \text{(by (1), (5))} \\ &\approx u + q^p q_1 q^p. \end{aligned}$$

Furthermore, we have the following are derivable from the identities which are satisfied in $\widehat{\mathbf{V}_p}$:

$$\begin{aligned} u &\approx u + q^p q_1 q^p \\ &\approx u + q q_1^p q^p && (\widehat{\mathbf{V}_p} \models q^p q_1 q^p \approx q q_1^p q^p) \\ &\approx u + q(q^p q_1^p q^p + q^p) && \text{(by (16))} \\ &\approx u + q q^p q_1^p q^p + q q^p \\ &\approx u + q q^p q_1^p q^p + q. \end{aligned}$$

This shows that $u \approx u + q$ is satisfied in $\widehat{\mathbf{V}_p}$ and so $\mathbf{V} \vee \mathbf{HSP}(Z_p^0) = \widehat{\mathbf{V}_p}$. □

By Lemma 3.1 and 3.6, we can establish the following result.

Theorem 3.7. *Let* $\mathbf{V} \in [\mathbf{M}, \mathbf{P} \cap \mathbf{Sr}(\mathbf{2}, \mathbf{1})]$. *Then the interval* $[\mathbf{V}, \widehat{\mathbf{V}_p}]$ *of* $\mathcal{L}(\mathbf{ROBA}_{p+1}^{\circ})$ *consists of two varieties* $\mathbf{V}$ *and* $\mathbf{V} \vee \mathbf{HSP}(Z_p^0)$.

For an ai-semiring S we denote by S^0 the ai-semiring obtained from S by adding an extra element 0, where $a + 0 = a$, $a0 = 0a = 0$ for every $a \in S$. Written B_p as Z_p^0, then we have

Lemma 3.8. *Let* $S \in \mathbf{ROBA}_{p+1}^{\circ}$. *Then* S *satisfies the identity*

$$x + (x + x^p) y^p (x + x^p) \approx x^p + (x + x^p) y^p (x + x^p) \tag{17}$$

if and only if it does not contain a copy of B_p^0.

Proof. *Necessary*. It is obvious.

Sufficiency. Suppose that $S \in \mathbf{ROBA}^{\circ}_{p+1}$, but that S does not satisfy identity (17). Then there exist $a, b \in S$ such that

$$a + (a + a^p)b^p(a + a^p) \neq a^p + (a + a^p)b^p(a + a^p).$$

It is easily verified that $c,\ a + c,\ a^2 + c, \dots, a^p + c$ and $a + a^p + c$ are not equal to each other, where $c = (a + a^p)b^p(a + a^p)$. Put $S_1 = \{c,\ a + c,\ a^2 + c, \dots, a^p + c,\ a + a^p + c\}$. By identities (1) and (10), it is routine to verify that S_1 forms a subsemiring of S, which is a copy of B^0_p. □

Let $u = u_1 + \cdots + u_m$, where $u_i \in X^+,\ i \in \underline{m}$, and $Z \subseteq \bigcup_{i \in m} c(u_i)$. If $c(u_i) \cap Z \neq \varnothing$ for every i, then we write $D_Z(u) = \varnothing$. Otherwise, $D_Z(u)$ is the sum of terms u_i for which $c(u_i) \cap Z = \varnothing$. The proof of the following result is easy and omitted.

Lemma 3.9. *Let $S \in \mathbf{ROBA}^{\circ}_{p+1}$ and $u \approx v$ be an **AI**-identity, where $u = u_1 + \cdots + u_m,\ v = v_1 + \cdots + v_n,\ u_i,\ v_j \in X^+,\ i \in \underline{m},\ j \in \underline{n}$. Then S^0 satisfies $u \approx v$ if and only if for every $Z \subseteq C(u) \bigcup C(v)$, either $D_Z(u) = D_Z(v) = \varnothing$ or $D_Z(u) \neq \varnothing \neq D_Z(v)$ and $D_Z(u) \approx D_Z(v)$ is satisfied in S.*

Lemma 3.10. $\mathbf{HSP}(B^0_p) = \mathbf{SA}^{\circ}_{p+1}$.

Proof. It is obvious that $\mathbf{HSP}(B^0_p) \subseteq \mathbf{SA}^{\circ}_{p+1}$. To prove that $\mathbf{SA}^{\circ}_{p+1} \subseteq \mathbf{HSP}(B^0_p)$, we need only to show that every identity satisfied in $\mathbf{HSP}(B^0_p)$ can be derived from the identities determining $\mathbf{SA}^{\circ}_{p+1}$.

Let $u \approx u + q$ be any identity which is satisfied in $\mathbf{HSP}(B^0_p)$, where $u = u_1 + \cdots + u_n,\ u_i,\ q \in X^+,\ i \in \underline{n}$. Choose $Z = C(u) \backslash c(q)$. Since $D_Z(u + q) \neq \varnothing$, by Lemma 3.9, $D_Z(u) \neq \varnothing$. Assume that $D_Z(u) = u_1 + \cdots + u_k$, $k \leq n$. By using Lemma 3.9 again, $B_p \vDash D_Z(u) \approx D_Z(u) + q$. It follows from Lemma 3.3 that $C(D_Z(u)) = c(q)$. By Lemma 3.4, there exists q_1 in X^+ with $r_{p+1}(q_1) = r_{p+1}(q)$ (and so $r_i(q_1) = r_i(q)$ for all $i \in \underline{p-1}$) and $c(q_1) \subseteq C(D_Z(u))$ such that $D_Z(u) \approx D_Z(u) + q_1$ is satisfied in $\mathbf{SA}^{\circ}_{p+1}$. Thus, it is easy to check that

$$D_Z(u) \approx D_Z(u) + q_1 \approx D_Z(u) + q_1 + u_1^p \cdots u_k^p q_1 \quad \text{(by (9))}$$

also is satisfied in $\mathbf{SA}^{\circ}_{p+1}$. Notice that $u_1^p \cdots u_k^p q_1 \approx q$ is satisfied in $\mathbf{SA}_{p+1}$. Thus $D_Z(u) \approx D_Z(u) + q$ is satisfied in $\mathbf{SA}^{\circ}_{p+1}$ and so $u \approx u + q$ is also satisfied in $\mathbf{SA}^{\circ}_{p+1}$. This shows that $\mathbf{HSP}(B^0_p) = \mathbf{SA}^{\circ}_{p+1}$. □

Suppose that $S \in \widehat{\mathbf{Bi}_p}$. Then $S \in \mathbf{ROBA}^{\circ}_{p+1}$ and $E(S) \in \mathbf{Bi}$. This means that $(S, \cdot)$ is a semilattice of abelian groups which belong to $\mathbf{A}_{p+1}$ and so by [22, Exercise IV.2.16 (v), p177], $S \in \mathbf{SA}^{\circ}_{p+1}$. It follows that $\widehat{\mathbf{Bi}_p} \subseteq \mathbf{SA}^{\circ}_{p+1}$. Thus, $\widehat{\mathbf{Bi}_p} = \mathbf{SA}^{\circ}_{p+1}$ since $\mathbf{SA}^{\circ}_{p+1} \subseteq \widehat{\mathbf{Bi}_p}$ is clear.

Lemma 3.11. *Let* $\mathbf{V} \in [\mathbf{Bi}, \mathbf{Sr}(\mathbf{2}, \mathbf{1})]$. *Then* $\mathbf{V} \vee \mathbf{HSP}(B^0_p) = \widehat{\mathbf{V}_p}$.

Proof. Let $\mathbf{V} \in [\mathbf{Bi}, \mathbf{Sr}(2, 1)]$. Since it is easily seen that $\mathbf{V} \vee \mathbf{HSP}(B^0_p) \subseteq \widehat{\mathbf{V}_p}$, we only need to show that $\widehat{\mathbf{V}_p} \subseteq \mathbf{V} \vee \mathbf{HSP}(B^0_p)$. Let $u \approx u + q$ be an **AI**-identity which is satisfied in $\mathbf{V} \vee \mathbf{HSP}(B^0_p)$, where $u = u_1 + \cdots + u_n,\ u_i,\ q \in X^+,\ i \in \underline{n}$. In the following we show that this identity is derivable from the identities determining $\widehat{\mathbf{V}_p}$.

Let $Z = C(u) \backslash c(q)$. Since $D_Z(u + q) \neq \varnothing$, by Lemma 3.9, $D_Z(u) \neq \varnothing$. Assume that $D_Z(u) = u_1 + \cdots + u_k$, $k \leq n$. By Lemma 3.9, $B_p \vDash D_Z(u) \approx D_Z(u) + q$ and so $\bigcup_{i \in \underline{k}} c(u_i) = c(q)$. By Lemma 3.4, there exists q_1 in X^+ with $r_{p+1}(q_1) = r_{p+1}(q)$ (and so $r_i(q_1) = r_i(q)$ for all $i \in \underline{p-1}$) and $c(q) \subseteq c(q_1) \subseteq \bigcup_{i \in \underline{k}} c(u_i)$ such that $D_Z(u) \approx D_Z(u) + q_1$ is satisfied in $\mathbf{ROBA}^{\circ}_{p+1}$. Therefore, $u \approx u + q_1$ is also satisfied in $\mathbf{ROBA}^{\circ}_{p+1}$. Proceeding as in the proof of Lemma 3.6, we have that $u \approx u + q^p q_1 q^p$ is derivable from the identities determining $\widehat{\mathbf{V}_p}$. Notice that $\widehat{\mathbf{V}_p} \vDash q^p q_1 q^p \approx q$. Thus $u \approx u + q$ is derivable from the identities determining $\widehat{\mathbf{V}_p}$. Thus, $\widehat{\mathbf{V}_p} \subseteq \mathbf{V} \vee \mathbf{HSP}(B^0_p)$ and so $\widehat{\mathbf{V}_p} = \mathbf{V} \vee \mathbf{HSP}(B^0_p)$. □

Lemma 3.12. *The subvariety of* $\mathbf{ROBA}^{\circ}_{p+1}$ *determined by* (17) *satisfies the identity*

$$xy^p z^p + x^p z \approx x^p y^p z + xz^p. \tag{18}$$

Proof. We only need to show that identity (18) is derivable from (17) and the identities determining $\mathbf{ROBA}^{\circ}_{p+1}$. On one hand, we have

$$
\begin{aligned}
& x^p y^p z^p + x^{p-1} z \\
\approx\ & x^p y^p z^p + x^{p-1} z + x^{p-1}(x^p y^p z^p)^p z + (x^{p-1}z)^p x^p y^p z^p (x^{p-1}z)^p && \text{(by (8), (9))} \\
\approx\ & x^p y^p z^p + x^{p-1} z + x^{p-1} y^p z + (x^{p-1}z)^p y^p (x^{p-1}z)^p && \text{(by (1), (5), (7))} \\
\approx\ & x^p y^p z^p + x^{p-1} z + x^{p-1} y^p z + (x^{p-1}z)^p y^p (x^{p-1}z)^p \\
& +(x^{p-1}z)((x^{p-1}z)^p y^p (x^{p-1}z)^p)^p + ((x^{p-1}z)^p y^p (x^{p-1}z)^p)^p (x^{p-1}z) \\
& +(x^{p-1}z)(x^p y^p z^p)^{p-1}(x^{p-1}z) && \text{(by (9))} \\
\approx\ & x^p y^p z^p + x^{p-1} z + x^{p-1} y^p z + (x^{p-1}z)^p y^p (x^{p-1}z)^p \\
& +(x^{p-1}z) y^p (x^{p-1}z)^p + (x^{p-1}z)^p y^p (x^{p-1}z) + (x^{p-1}z) y^p (x^{p-1}z) && \text{(by (1), (5), (7))} \\
\approx\ & x^p y^p z^p + x^{p-1} y^p z + x^{p-1} z \\
& +(x^{p-1}z + (x^{p-1}z)^p) y^p (x^{p-1}z + (x^{p-1}z)^p) \\
\approx\ & x^p y^p z^p + x^{p-1} y^p z + (x^{p-1}z)^p \\
& +(x^{p-1}z + (x^{p-1}z)^p) y^p (x^{p-1}z + (x^{p-1}z)^p). && \text{(by (17))}
\end{aligned}
$$

This derives the identity

$$
x y^p z^p + x^p z \approx x(x^p y^p z^p + x^{p-1} z) \approx x(x^p y^p z^p + x^{p-1} z + x^{p-1} y^p z + (x^{p-1}z)^p). \tag{19}
$$

On the other hand, we also have

$$
\begin{aligned}
& x^{p-1} y^p z + x^p z^p \\
\approx\ & x^{p-1} y^p z + x^p z^p + x^p (x^{p-1} y^p z)^p z^p && \text{(by (8))} \\
\approx\ & x^{p-1} y^p z + x^p z^p + x^p y^p z^p && \text{(by (1), (5))} \\
\approx\ & x^{p-1} y^p z + (x^{p-1}z)^p + x^p y^p z^p + (x^{p-1}z)^p x^p y^p z^p (x^{p-1}z)^p && \text{(by (1), (5), (9))} \\
\approx\ & x^{p-1} y^p z + (x^{p-1}z)^p + x^p y^p z^p + (x^{p-1}z)^p y^p (x^{p-1}z)^p && \text{(by (1), (5), (7))} \\
\approx\ & x^{p-1} y^p z + (x^{p-1}z)^p + x^{p-1}((x^{p-1}z)^p)^p y^p ((x^{p-1}z)^p)^p z \\
& +x^p y^p z^p + (x^{p-1}z)^p y^p (x^{p-1}z)^p && \text{(by (8))} \\
\approx\ & x^{p-1} y^p z + (x^{p-1}z)^p + x^{p-1} z^p y^p x^p z + x^p y^p z^p \\
& +(x^{p-1}z)^p y^p (x^{p-1}z)^p && \text{(by (1), (5)} \\
\approx\ & x^{p-1} y^p z + (x^{p-1}z)^p + x^{p-1} z^p y^p x^p z + x^{p-1} z y^p x^p z^p \\
& +x^p y^p z^p + (x^{p-1}z)^p y^p (x^{p-1}z)^p && \text{(by (6))} \\
\approx\ & x^{p-1} y^p z + (x^{p-1}z)^p + x^{p-1} z^p y^p x^p z + (x^{p-1}z) y^p (x^{p-1}z)^p \\
& +x^p y^p z^p + (x^{p-1}z)^p y^p (x^{p-1}z)^p && \text{(by (1), (5))} \\
\approx\ & x^{p-1} y^p z + (x^{p-1}z)^p + x^{p-1} z^p y^p x^p z + (x^{p-1}z) y^p (x^{p-1}z)^p \\
& +(x^{p-1}z)^p y^p (x^{p-1}z) + x^p y^p z^p + (x^{p-1}z)^p y^p (x^{p-1}z)^p && \text{(by (6))} \\
\approx\ & x^{p-1} y^p z + (x^{p-1}z)^p + x^{p-1} z^p y^p x^p z + (x^{p-1}z) y^p (x^{p-1}z)^p \\
& +(x^{p-1}z)^p y^p (x^{p-1}z) + x^p y^p z^p + (x^{p-1}z)^p y^p (x^{p-1}z)^p \\
& +(x^{p-1}z) y^p (x^{p-1}z)^p (x^p y^p z^p)^{p-1} (x^{p-1}z)^p y^p (x^{p-1}z) && \text{(by (9))} \\
\approx\ & x^{p-1} y^p z + (x^{p-1}z)^p + x^{p-1} z^p y^p x^p z + (x^{p-1}z) y^p (x^{p-1}z)^p \\
& +(x^{p-1}z)^p y^p (x^{p-1}z) + (x^{p-1}z) y^p (x^{p-1}z) \\
& +x^p y^p z^p + (x^{p-1}z)^p y^p (x^{p-1}z)^p && \text{(by (1), (5), (7))} \\
\approx\ & x^{p-1} y^p z + (x^{p-1}z)^p + x^{p-1} z^p y^p x^p z + x^p y^p z^p \\
& +(x^{p-1}z + (x^{p-1}z)^p) y^p (x^{p-1}z + (x^{p-1}z)^p) \\
\approx\ & x^{p-1} y^p z + x^{p-1} z + x^{p-1} z^p y^p x^p z + x^p y^p z^p \\
& +(x^{p-1}z + (x^{p-1}z)^p) y^p (x^{p-1}z + (x^{p-1}z)^p). && \text{(by (17))}
\end{aligned}
$$

This deduces the identity

$$
x^p y^p z + x z^p \approx x(x^{p-1} y^p z + x^p y^p z^p) \approx x(x^{p-1} y^p z + x^p y^p z^p + x^{p-1} z + (x^{p-1}z)^p). \tag{20}
$$

This shows that the subvariety of $\mathbf{ROBA}^{\circ}_{p+1}$ determined by (17) satisfies identity (18). □

Lemma 3.13. *Let* $\mathbf{V} \in \mathcal{L}(\mathbf{Sr}(\mathbf{2},\mathbf{1}))$ *such that* $\mathbf{D} \vee \mathbf{M} \subseteq \mathbf{V}$. *Then* $\mathbf{V} \vee \mathbf{HSP}(B_p)$ *is the subvariety of* $[\mathbf{V}, \widehat{\mathbf{V}_p}]$ *determined by the identity (18).*

Proof. It is easily seen that $\mathbf{V} \vee \mathbf{HSP}(B_p) \subseteq \widehat{\mathbf{V}_p}$. Since both $\mathbf{V}$ and B_p satisfy (18), $\mathbf{V} \vee \mathbf{HSP}(B_p)$ also satisfies this identity. It remains to show that every **AI**-identity satisfied in $\mathbf{V} \vee \mathbf{HSP}(B_p)$ is derivable from (18) and the identities determining $\widehat{\mathbf{V}_p}$.

Let $u \approx u + q$ be an **AI**-identity which is satisfied in $\mathbf{V} \vee \mathbf{HSP}(B_p)$, where $u = u_1 + \cdots + u_n$, u_i, $q \in X^+$, $i \in \underline{n}$. Since $D_2 \vDash u \approx u + q$, there exists u_i such that $c(u_i) \subseteq c(q)$, where $i \in \underline{n}$. Since $B_p \vDash u \approx u + q$, by Lemma 3.4 there exists q_1 in X^+ with $r_{p+1}(q_1) = r_{p+1}(q)$ (and so $r_i(q_1) = r_i(q)$ for all $i \in \underline{p-1}$) and $c(q) \subseteq c(q_1)$ such that $u \approx u + q_1$ is satisfied in $\mathbf{ROBA}^{\circ}_{p+1}$. Proceeding as in the proof of Lemma 3.6,

$$\widehat{\mathbf{V}_p} \vDash u \approx u + q^p q_1 q^p.$$

Similarly,

$$\widehat{\mathbf{V}_p} \vDash u \approx u + q^p u_i q^p.$$

Thus the following are derivable from identity (18) and the identities which hold in $\widehat{\mathbf{V}_p}$:

$$\begin{aligned} u &\approx u + q^p q_1 q^p + q^p u_i q^p \\ &\approx u + q q_1^p q^p + q^p u_i q^p \\ &\approx u + q q_1^p (u_i q^p)^p + q^p u_i q^p \quad (\text{since } \widehat{\mathbf{V}_p} \vDash q^p q_1 q^p \approx q q_1^p (u_i q^p)^p) \\ &\approx u + q(u_i q^p)^p + q^p q_1^p u_i q^p \quad (\text{by } (18)) \\ &\approx u + q + q^p q_1^p u_i q^p. \end{aligned}$$

Hence, $u \approx u + q$ is derivable from (18) and the identities determining $\widehat{\mathbf{V}_p}$. □

Theorem 3.14. *Let* $\mathbf{V} \in [\mathbf{Bi}, \mathbf{Sr}(\mathbf{2},\mathbf{1})]$. *Then the interval* $[\mathbf{V}, \widehat{\mathbf{V}_p}]$ *of* $\mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1})$ *consists of the three varieties* $\mathbf{V}$, $\mathbf{V} \vee \mathbf{HSP}(B_p)$ *and* $\mathbf{V} \vee \mathbf{HSP}(B_p^0)$.

Proof. By Lemma 3.11, $\mathbf{V} \vee \mathbf{HSP}(B_p^0) = \widehat{\mathbf{V}_p}$ and so $\mathbf{V} \vee \mathbf{HSP}(B_p^0) \in [\mathbf{V}, \widehat{\mathbf{V}_p}]$. Since $B_p^0 \in \mathbf{V} \vee \mathbf{HSP}(B_p^0)$ and $B_p^0 \notin \mathbf{V} \vee \mathbf{HSP}(B_p)$, $\mathbf{V} \vee \mathbf{HSP}(B_p) \neq \mathbf{V} \vee \mathbf{HSP}(B_p^0)$. Further, it is easy to see that $\mathbf{V}$, $\mathbf{V} \vee \mathbf{HSP}(B_p)$ and $\mathbf{V} \vee \mathbf{HSP}(B_p^0)$ are different members of $[\mathbf{V}, \widehat{\mathbf{V}_p}]$. Suppose that $\mathbf{W} \in [\mathbf{V}, \widehat{\mathbf{V}_p}]$ and $\mathbf{W} \neq \mathbf{V}$. Then it follows from Lemma 3.1 that $\mathbf{V} \vee \mathbf{HSP}(B_p) \subseteq \mathbf{W}$. If $\mathbf{W}$ is a proper subvariety of $\mathbf{V} \vee \mathbf{HSP}(B_p^0)$, then, by Lemma 3.8 and 3.12, $\mathbf{W}$ satisfies identity (18). It follows from Lemma 3.13 that $\mathbf{V} \vee \mathbf{HSP}(B_p) = \mathbf{W}$. □

Theorem 3.15. *Let* $\mathbf{V} \in [\mathbf{D} \vee \mathbf{M}, \mathbf{K} \cap \mathbf{Sr}(\mathbf{2},\mathbf{1})]$. *Then the interval* $[\mathbf{V}, \widehat{\mathbf{V}_p}]$ *consists of the two varieties* $\mathbf{V}$ *and* $\mathbf{V} \vee \mathbf{HSP}(B_p)$.

Proof. Since $\mathbf{K} \cap \mathbf{Sr}(2,1) \vDash (13)$, it follows immediately that $\mathbf{K} \cap \widehat{\mathbf{Sr}(2,1)}_p$ satisfies the identity

$$x^p + x^p y^p x^p + x^p y^p z^p y^p x^p \approx x^p + x^p y^p z^p y^p x^p. \tag{21}$$

Notice that $B_p^0 \nvDash (21)$. Then, by Lemma 3.8, $\mathbf{K} \cap \widehat{\mathbf{Sr}(2,1)}_p \vDash (17)$. From Lemma 3.12 we have that $\mathbf{K} \cap \widehat{\mathbf{Sr}(2,1)}_p \vDash (18)$. Suppose now that $\mathbf{V} \in [\mathbf{D} \vee \mathbf{M}, \mathbf{K} \cap \mathbf{Sr}(\mathbf{2},\mathbf{1})]$. Then $\widehat{\mathbf{V}_p} \vDash (18)$. Thus, by Lemma 3.13, $\mathbf{V} \vee \mathbf{HSP}(B_p) = \widehat{\mathbf{V}_p}$. Thus, by Lemma 3.1, the result holds. □

Theorem 3.16. *Each member of* $\mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1})$ *is finitely based and finitely generated.*

Proof. Since each member of $\mathcal{L}(\mathbf{Sr}(2,1))$ is finitely based and finitely generated, it follows from Lemmas 3.6, 3.11, 3.13, Theorems 3.5, 3.7, 3.14 and 3.15 that this is the case for each member of $\mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1})$, too. □

Theorem 3.17. $\mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1})$ *is a 179-element distributive lattice.*

Proof. Notice that both $[\mathbf{T}, \mathbf{N} \cap \mathbf{P} \cap \mathbf{Sr}(2,1)]$ and $[\mathbf{M}, \mathbf{P} \cap \mathbf{Sr}(2,1)]$ have 4 elements, $[\mathbf{D}, \mathbf{N} \cap \mathbf{Sr}(2,1)]$ has 9 elements, $[\mathbf{D} \vee \mathbf{M}, \mathbf{K} \cap \mathbf{Sr}(2,1)]$ has 25 elements, and $[\mathbf{Bi}, \mathbf{Sr}(2,1)]$ has 36 elements (see [2, Section 4]). By Theorems 3.5, 3.7, 3.14 and 3.15, we have that $\mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1})$ has 179 elements.

Assume that $\mathbf{V_1}, \mathbf{V_2}, \mathbf{V_3} \in \mathcal{L}(\mathbf{ROBA}^{\circ}_{\mathbf{p+1}})$ such that $\mathbf{V_1} \vee \mathbf{V_2} = \mathbf{V_1} \vee \mathbf{V_3}$ and $\mathbf{V_1} \wedge \mathbf{V_2} = \mathbf{V_1} \wedge \mathbf{V_3}$. Then we have that $\varphi(\mathbf{V_1}) \vee \varphi(\mathbf{V_2}) = \varphi(\mathbf{V_1}) \vee \varphi(\mathbf{V_3})$ and $\varphi(\mathbf{V_1}) \wedge \varphi(\mathbf{V_2}) = \varphi(\mathbf{V_1}) \wedge \varphi(\mathbf{V_3})$ and so $\varphi(\mathbf{V_2}) = \varphi(\mathbf{V_3})$ since $\mathcal{L}(\mathbf{Sr}(\mathbf{2}, \mathbf{1}))$ is distributive. Let $\mathbf{V}$ denote the variety $\varphi(\mathbf{V_2})$. Then $\mathbf{V_2}$ and $\mathbf{V_3}$ are members of $[\mathbf{V}, \widehat{\mathbf{V}_p}]$. If $\mathbf{V} \in [\mathbf{T}, \mathbf{N} \cap \mathbf{P} \cap \mathbf{Sr}(\mathbf{2}, \mathbf{1})] \cup [\mathbf{D}, \mathbf{N} \cap \mathbf{Sr}(\mathbf{2}, \mathbf{1})]$. Then, by Theorem 3.5, $\mathbf{V_2} = \mathbf{V_3}$. If $\mathbf{V} \in [\mathbf{M}, \mathbf{P} \cap \mathbf{Sr}(\mathbf{2}, \mathbf{1})] \cup [\mathbf{D} \vee \mathbf{M}, \mathbf{K} \cap \mathbf{Sr}(\mathbf{2}, \mathbf{1})]$. Then, by Theorem 3.7 and 3.15, we have that $[\mathbf{V}, \widehat{\mathbf{V}_p}] = \{\mathbf{V}, \mathbf{V} \vee \mathbf{HSP}(B_p)\}$. Suppose that $\mathbf{V_2} \neq \mathbf{V_3}$. Then $\mathbf{V_1} \vee \mathbf{V_2} = \mathbf{V_1} \vee \mathbf{V_3}$ and $\mathbf{V_1} \wedge \mathbf{V_2} = \mathbf{V_1} \wedge \mathbf{V_3}$ can not hold at the same time, by Lemma 3.1. This implies that $\mathbf{V_2} = \mathbf{V_3}$. If $\mathbf{V} \in [\mathbf{Bi}, \mathbf{Sr}(\mathbf{2}, \mathbf{1})]$. Then, by Theorem 3.14, we have that $[\mathbf{V}, \widehat{\mathbf{V}_p}] = \{\mathbf{V}, \mathbf{V} \vee \mathbf{HSP}(B_p), \mathbf{V} \vee \mathbf{HSP}(B_p^0)\}$. Suppose that $\mathbf{V_2} \neq \mathbf{V_3}$. Then $\mathbf{V_1} \vee \mathbf{V_2} = \mathbf{V_1} \vee \mathbf{V_3}$ and $\mathbf{V_1} \wedge \mathbf{V_2} = \mathbf{V_1} \wedge \mathbf{V_3}$ can not hold at the same time, by Lemma 3.1 and 3.8. Thus, $\mathbf{V_2} = \mathbf{V_3}$. This shows that $\mathcal{L}(\mathbf{ROBA}^{\circ}_{p+1})$ is a distributive lattice. □

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