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A note on the three-way generalization of the Jordan canonical form

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Abstract: The limit point $\mathcal{X}$ of an approximating rank-R sequence of a tensor $\mathcal{Z}$ can be obtained by fitting a decomposition $(\mathbf{S}, \mathbf{T}, \mathbf{U}) \cdot \mathcal{G}$ to $\mathcal{Z}$. The decomposition of the limit point $\mathcal{X} = (\mathbf{S}, \mathbf{T}, \mathbf{U}) \cdot \mathcal{G}$ with $\mathcal{G} = blockdiag(\mathcal{G}_1, \ldots, \mathcal{G}_m)$ can be seen as a three order generalization of the real Jordan canonical form. The main aim of this paper is to study under what conditions we can turn $\mathcal{G}_j$ into canonical form if some of the upper triangular entries of the last three slices of $\mathcal{G}_j$ are zeros. In addition, we show how to turn $\mathcal{G}_j$ into canonical form under these conditions.

Keywords: Low-rank tensor approximations, Jordan canonical form, Tensor decomposition

MSC: 15A18, 15A69

1 Introduction

A tensor can be regarded as a higher-order generalization of a matrix, which takes the form

$$\mathcal{A} = (a_{i_1,\ldots,i_m}) \qquad a_{i_1,\ldots,i_m} \in R \qquad 1 \le i_1, \ldots, i_m \le n$$

Such a multi-array $\mathcal{A}$ is said to be an mth-order n-dimensional square real tensor with n^m entries $a_{i_1,\ldots,i_m}$. In this paper, we only consider the case $m = 3$ and real-valued three-way arrays.

Definition 1.1 ([1]). *Let $\mathcal{A}$ be a mth-order n-dimensional tensor. The mode-k matrix (or k-th matrix unfolding) $A_{(k)} \in \mathbb{R}^{n \times n^{m-1}}$ is a matrix containing the element $a_{i_1 i_2 \ldots i_m}$.*

When $\mathcal{A}$ is a 3rd-order n-dimensional tensor, its mode-k matrices are:

$$\begin{aligned}
\mathcal{A}_{(1)} &= [\mathcal{A}(:,1,1)\ldots\mathcal{A}(:,n,1)\mathcal{A}(:,1,2)\ldots\mathcal{A}(:,n,2)\ldots\mathcal{A}(:,n,n)];\\
\mathcal{A}_{(2)} &= [\mathcal{A}(1,:,1)\ldots\mathcal{A}(1,:,n)\mathcal{A}(2,:,1)\ldots\mathcal{A}(2,:,n)\ldots\mathcal{A}(n,:,n)];\\
\mathcal{A}_{(3)} &= [\mathcal{A}(1,1,:)\ldots\mathcal{A}(n,1,:)\mathcal{A}(1,2,:)\ldots\mathcal{A}(n,2,:)\ldots\mathcal{A}(n,n,:)].
\end{aligned}$$

Definition 1.2 ([2]). *The multilinear rank of an $I \times J \times K$ array is defined as the triplet(mode-1 rank, mode-2 rank, mode-3 rank). The mode-k rank of a tensor $\mathcal{A}$ is defined as the rank of mode-k matrix.*

Obviously, a three order tensor has 3 mode-k ranks and the different mode-k ranks of tensor are not necessarily the same [3]. In addition, the rank and the mode-k rank of a same tensor are not necessarily equal even though all the mode-k ranks are equal.

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Let

$$S_R(I,J,K) = \{\mathcal{Y} \in \mathbb{R}^{I\times J\times K} | rank(\mathcal{Y}) \le R\} \tag{1}$$

Fitting the CP decomposition [4] to $\mathcal{Z}$ boils down to solving the following minimization problem:

$$Minimize \parallel \mathcal{Z} - \mathcal{Y} \parallel \; subject\ to\ \mathcal{Y} \in S_R(I,J,K) \tag{2}$$

Hence, we are looking for a best rank-R approximation [5] to $\mathcal{Z}$. In [6], A.Stegeman has shown that such a best rank-R approximation may not exist due to the set $S_R(I,J,K)$ not being closed for $R \ge 2$. In this case, we are trying to compute the approximation results in diverging rank-1 terms [7]. This phenomenon can be seen as a three-way generalization of approximate diagonalization of a nondiagonalizable matrix. In [6, 8], A.Stegeman has shown that, analogous to the matrix case, the limit point of the approximating rank-R sequence satisfies a three-way generalization of the real Jordan canonical form. [6, 9] show that the limit point $\mathcal{X}$ is a boundary point of $S_R(I,J,K)$ and can be obtained by fitting a decomposition $(S,T,U)\cdot\mathcal{G}$ to $\mathcal{Z}$, with $\mathcal{G} = blockdiag(\mathcal{G}_1,\dots,\mathcal{G}_m)$ and core block $\mathcal{G}_j$ of size $d_j \times d_j \times d_j$ and in sparse canonical form. The decomposition of $\mathcal{X}$ has been introduced in [10–12], where the block terms are $(S_j,T_j,U_j)\cdot\mathcal{G}_j$. Nondiverging rank-1 terms have an associated core block with $d_j = 1$, and core blocks with $d_j \ge 2$ are the limit of a group of d_j diverging rank-1 terms.

For groups of two, or three, or four diverging rank-1 terms, [6, 8] have shown limit point $\mathcal{X} = \sum_{j=1}^m \mathcal{X}_j$ and its decomposition $\mathcal{X} = (S,T,U)\cdot\mathcal{G} = \sum_{j=1}^m (S_j,T_j,U_j)\cdot\mathcal{G}_j$ have the following results.

Lemma 1.3 ([13]). *For a group of $d_j = 2$ diverging rank-1 terms, the limit $\mathcal{X}_j$ can be written as $\mathcal{X}_j = (\mathbf{S}_j,\mathbf{T}_j,\mathbf{U}_j)\cdot\mathcal{G}_j$ with $\mathbf{S}_j,\mathbf{T}_j,\mathbf{U}_j$ of rank 2, and $2\times2\times2$ array $\mathcal{G}_j$ given by*

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]. \tag{3}$$

we have $rank(\mathcal{G}_j) = 3$. Here, we denote the $2\times2\times2$ array $\mathcal{G}_j$ with 2×2 slices $\mathbf{G}_1$ and $\mathbf{G}_2$ as $[\mathbf{G}_1|\mathbf{G}_2]$. (3) is referred to as the canonical form of a boundary array of $S_2(2,2,2)$.

Lemma 1.4 ([6]). *For a group of $d_j = 3$ diverging rank-1 terms, and $min(I,J,K) \ge 3$, almost all limits $\mathcal{X}_j$ with multilinear $rank(3,3,3)$ can be written as $\mathcal{X}_j = (\mathbf{S}_j,\mathbf{T}_j,\mathbf{U}_j)\cdot\mathcal{G}_j$ with $\mathbf{S}_j,\mathbf{T}_j,\mathbf{U}_j$ of rank 3, and $3\times3\times3$ array $\mathcal{G}_j$ given by*

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & * & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{4}$$

where $*$ denotes a nonzero entry. We have $rank(\mathcal{X}_j) = rank(\mathcal{G}_j) = 5$.

Lemma 1.5 ([8]). *For a group of $d_j = 4$ diverging rank-1 terms, and $min(I,J,K) \ge 4$, almost all limits $\mathcal{X}_j$ with multilinear $rank(4,4,4)$ can be written as $\mathcal{X}_j = (\mathbf{S}_j,\mathbf{T}_j,\mathbf{U}_j)\cdot\mathcal{G}_j$ with $\mathbf{S}_j,\mathbf{T}_j,\mathbf{U}_j$ of rank 4, and $4\times4\times4$ array $\mathcal{G}_j$ given by*

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{5}$$

where $*$ denotes a nonzero entry. We have $rank(\mathcal{X}_j) = rank(\mathcal{G}_j) \ge 7$.

Remark 1.6. *The proof of Lemma 1.5 in [8] has shown that $\mathcal{G}_j$ has multilinear rank (4,4,4).*

However, the proof of Lemma 1.5 in [8] does not take account of the cases that some of the upper triangular entries of the last three slices of $\mathcal{G}_j$ are zeros. To make up for this defect, we assume that some of the upper triangular entries of the last three slices of $\mathcal{G}_j$ are zeros and study whether we can turn $\mathcal{G}_j$ into the canonical form (5). Firstly, we consider the following two examples.

Example 1.7. *Let $\mathcal{G}_j$ be a $4 \times 4 \times 4$ array that satisfies the conditions of Lemma 1.5, and the (1,4) entries of the last three slices are equal to zeros, i.e.,*

$$\mathcal{G}_j = \left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0 & 3&1&2&0 & 3&2&9&0 & 3&3&16&0 \\ 0&1&0&0 & 0&3&2&5 & 0&3&4&11 & 0&3&6&17 \\ 0&0&1&0 & 0&0&3&3 & 0&0&3&6 & 0&0&3&9 \\ 0&0&0&1 & 0&0&0&3 & 0&0&0&3 & 0&0&0&3 \end{array}\right].$$

By subtracting 3 times the first slice of $\mathcal{G}_j$ from slice 2,3,4, then we obtain an all-zero diagonal in slice 2,3,4. Similarly, by subtracting 2 times the second slice from the third slice and 3 times the second slice from the fourth slice, $\mathcal{G}_j$ is of the form

$$\mathcal{G}_j = \left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0 & 0&1&2&0 & 0&0&5&0 & 0&0&10&0 \\ 0&1&0&0 & 0&0&2&5 & 0&0&0&1 & 0&0&0&2 \\ 0&0&1&0 & 0&0&0&3 & 0&0&0&0 & 0&0&0&0 \\ 0&0&0&1 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 \end{array}\right].$$

Next, by subtracting 2 times the third slice from the fourth slice, we can turn the (1,3) and (2,4) entries of slice four into zeros. By subtracting 5 times the third slice from the second slice, we can turn the $(2,4)$ entry of slice two into zero. Then we obtain the following form

$$\mathcal{G}_j = \left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0 & 0&1&-23&0 & 0&0&5&0 & 0&0&0&0 \\ 0&1&0&0 & 0&0&2&0 & 0&0&0&1 & 0&0&0&0 \\ 0&0&1&0 & 0&0&0&3 & 0&0&0&0 & 0&0&0&0 \\ 0&0&0&1 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 \end{array}\right].$$

In each slice of the above $\mathcal{G}_j$, we add $23/2$ times row 2 to row 1 and subtract $23/2$ times column 1 from column 2. Then we obtain the following form

$$\mathcal{G}_j = \left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0 & 0&1&0&0 & 0&0&5&23/2 & 0&0&0&0 \\ 0&1&0&0 & 0&0&2&0 & 0&0&0&1 & 0&0&0&0 \\ 0&0&1&0 & 0&0&0&3 & 0&0&0&0 & 0&0&0&0 \\ 0&0&0&1 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 \end{array}\right].$$

It is apparent from this example that if some of the upper triangular entries of the last three slices of $\mathcal{G}_j$ are zeros, we can't turn $\mathcal{G}_j$ into canonical form (5).

Example 1.8. *Let $\mathcal{G}_j$ be a $4 \times 4 \times 4$ array that satisfies the conditions of Lemma 1.5, and the (1,4) entries of the second and the third slices are equal to zeros, i.e.,*

$$\mathcal{G}_j = \left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0 & 3&1&2&0 & 3&2&9&0 & 3&3&16&5 \\ 0&1&0&0 & 0&3&2&5 & 0&3&4&11 & 0&3&6&17 \\ 0&0&1&0 & 0&0&3&3 & 0&0&3&6 & 0&0&3&9 \\ 0&0&0&1 & 0&0&0&3 & 0&0&0&3 & 0&0&0&3 \end{array}\right].$$

Using the same method as Example 1.7, we can obtain

$$\mathcal{G}_j = \left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0 & 0&1&0&0 & 0&0&5&23/2 & 0&0&0&5 \\ 0&1&0&0 & 0&0&2&0 & 0&0&0&1 & 0&0&0&0 \\ 0&0&1&0 & 0&0&0&3 & 0&0&0&0 & 0&0&0&0 \\ 0&0&0&1 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 \end{array}\right].$$

Now, the fourth slice only has its (1,4) entry nonzero, we normalize it to one. Then by subtracting the $23/2$ times the fourth slice from the third slice, the (1,4) entry of the third slice can be turned into zero. Then we obtain

$$\mathcal{G}_j = \left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0 & 0&1&0&0 & 0&0&5&0 & 0&0&0&1 \\ 0&1&0&0 & 0&0&2&0 & 0&0&0&1 & 0&0&0&0 \\ 0&0&1&0 & 0&0&0&3 & 0&0&0&0 & 0&0&0&0 \\ 0&0&0&1 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 \end{array}\right].$$

According to this example, we see that if some of the upper triangular entries of the last three slices of $\mathcal{G}_j$ are zeros, we can turn $\mathcal{G}_j$ into canonical form (5).

A natural question is under what conditions we can turn $\mathcal{G}_j$ into canonical form (5) if some of the upper triangular entries of the last three slices of $\mathcal{G}_j$ are zeros. The answer to this question is the main contribution of this paper.

Remark 1.9. *It is worth noting that if some of the upper triangular entries of the last three slices of $\mathcal{G}_j$ are zeros, the entry $*$ of canonical form (5) may be zero. Therefore, in this paper, we mainly consider the following canonical form*

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&\bar{e}&0&0&0&0&\bar{h}&0&0&0&0&1\\ 0&1&0&0&0&0&\bar{f}&0&0&0&0&\bar{i}&0&0&0&0\\ 0&0&1&0&0&0&0&\bar{g}&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right], \tag{6}$$

where $\bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}$ may be zero, and there must be at least one nonzero entry in every slice.

Now, we prove why we require that there must be at least one nonzero entry in every slice. According to the the definition of the mode-k matrix of a tensor, (5) is actually the mode-1 matrix of $\mathcal{G}_j$. Because the first slice of $\mathcal{G}_j$ is an identity matrix, it follows that the mode-1 rank of $\mathcal{G}_j$ is always 4, no matter what values $\bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}$ take.

The mode-2 matrix of $\mathcal{G}_j$ is

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&\bar{e}&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&\bar{h}&0&0&\bar{f}&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&\bar{i}&0&0&\bar{g}&0&0&1&0&0&0 \end{array}\right].$$

Since each row of this matrix has a nonzero entry 1, it follows that mode-2 rank of $\mathcal{G}_j$ is always 4, no matter what values $\bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}$ take.

The mode-3 matrix of $\mathcal{G}_j$ is

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&\bar{e}&0&0&0&0&\bar{f}&0&0&0&0&\bar{g}&0\\ 0&0&0&0&0&0&0&0&\bar{h}&0&0&0&0&\bar{i}&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \end{array}\right].$$

For this matrix, if there exists a row that its elements are all zeros, then the mode-3 rank of $\mathcal{G}_j$ does not equal to 4. This contradicts the fact that the mode-3 rank of $\mathcal{G}_j$ is 4. Consequently, $\bar{e}, \bar{f}, \bar{g}$ can't be zero at the same time, and $\bar{h}, \bar{i}$ can't be zero at the same time. This implies that there must be at least one nonzero entry in every slice.

Definition 1.10. *For a matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$, we call the main diagonal elements of matrix A as the 1st diagonal elements , and a_{12}, a_{23}, a_{34} as the 2nd diagonal elements, and a_{13}, a_{24} as the 3rd diagonal elements, and a_{14} as the 4th diagonal element. The kth diagonal element of A is called zero if the elements of the kth diagonal are all zeros. The kth diagonal elements of A is called nonzero if the kth diagonal of A has a nonzero element.*

The remaining of this paper is organized as follows. In section 2, we study some properties related to $\mathcal{G}_j$. In section 3, we discuss under what conditions can we turn the 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ into zeros. In section 4, based on the results of the third section, we analyze under what conditions can we turn the 3rd and 4th diagonal elements of the last three slices of $\mathcal{G}_j$ into zeros. Finally, some concluding remarks are given in section 5.

2 Some properties related to $\mathcal{G}_j$

Before we discuss what conditions we need to turn $\mathcal{G}_j$ into canonical form (6), we first give some properties related to $\mathcal{G}_j$.

Property 2.1. *If $\mathcal{G}_j$ satisfies the conditions of Lemma 1.5, then it can be turned into*

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&a_2&e_2&h_2&j_2&a_3&e_3&h_3&j_3&a_4&e_4&h_4&j_4\\ 0&1&0&0&0&b_2&f_2&i_2&0&b_3&f_3&i_3&0&b_4&f_4&i_4\\ 0&0&1&0&0&0&c_2&g_2&0&0&c_3&g_3&0&0&c_4&g_4\\ 0&0&0&1&0&0&0&d_2&0&0&0&d_3&0&0&0&d_4 \end{array}\right], \tag{7}$$

and we can obtain the following three conclusions: (1) $a_p = b_p = c_p = d_p$ $(p = 2,3,4)$ hold for almost all $\mathcal{G}_j$; (2) The equations

$$\begin{aligned} e_3f_2 &= f_3e_2, f_3g_2 = g_3f_2\\ e_4f_2 &= f_4e_2, f_4g_2 = g_4f_2\\ e_4f_3 &= f_4e_3, f_4g_3 = g_4f_3 \end{aligned} \tag{8}$$

hold for almost all $\mathcal{G}_j$; (3) The equations

$$\begin{aligned} e_3i_2 - e_2i_3 + h_3g_2 - g_3h_2 &= 0\\ e_4i_2 - e_2i_4 + h_4g_2 - g_4h_2 &= 0\\ e_4i_3 - e_3i_4 + h_4g_3 - g_4h_3 &= 0 \end{aligned} \tag{9}$$

holds for almost all $\mathcal{G}_j$.

Proof. The proof of Lemma 3.3 in [8] has shown that $\mathcal{G}_j$ can be turned into (7) if it satisfies the conditions of Lemma 3.3.

The first conclusion has been proved in Lemma 3.3 of [8].

Now we show the proof of the second conclusion. Similarly to the proof of the vectors, (e_p, f_p, g_p) are proportional for $p = 2,3,4$ in Lemma 3.3 of the [8]. We write $A^{(n)}$ in terms of $p = 3$ and compute $Y_2^{(n)} = A^{(n)}C_2^{(n)}(A^{(n)})^{-1}$, which yields matrix (A.10). The entries in this matrix equal those of $Y_2^{(n)}$ in (A.2). It follows that $e_3f_2 = f_3e_2, f_3g_2 = g_3f_2$. When we write $A^{(n)}$ in terms of $p = 4$ and compute $Y_2^{(n)} = A^{(n)}C_2^{(n)}(A^{(n)})^{-1}$, this yields a matrix similar with (A.10). The entries in this matrix equal those of $Y_3^{(n)}$ in (A.2). It follows that $e_4f_2 = f_4e_2, f_4g_2 = g_4f_2$. Analogously, when writing $A^{(n)}$ in terms of $p = 3$ and compute $Y_4^{(n)} = A^{(n)}C_4^{(n)}(A^{(n)})^{-1}$, we obtain that $e_4f_3 = f_4e_3, f_4g_3 = g_4f_3$.

Now we prove the third conclusion. Similarly to the proof of the vectors $(h_3 - \alpha h_2, i_3 - \alpha i_2)$ and $(h_4 - \beta h_2, i_4 - \beta i_2)$ are proportional for $\alpha = e_3/e_2, \beta = e_4/e_2$ in Lemma 3.3 of the [8], we write $A^{(n)}$ in terms of $p = 3$ and compute $Y_2^{(n)} = A^{(n)}C_2^{(n)}(A^{(n)})^{-1}$, which yields matrix (A.10). The entries in this matrix equal those of $Y_2^{(n)}$ in (A.2). It follows that $e_3i_2 - e_2i_3 + h_3g_2 - g_3h_2 = 0$. We write $A^{(n)}$ in terms of $p = 4$ and compute $Y_2^{(n)} = A^{(n)}C_2^{(n)}(A^{(n)})^{-1}$, which yields a matrix similar with (A.10). The entries in this matrix equal those of $Y_3^{(n)}$ in (A.2). It follows that $e_4i_2 - e_2i_4 + h_4g_2 - g_4h_2 = 0$. Analogously, when writing $A^{(n)}$ in terms of $p = 3$ and compute $Y_4^{(n)} = A^{(n)}C_4^{(n)}(A^{(n)})^{-1}$, we obtain that $e_4i_3 - e_3i_4 + h_4g_3 - g_4h_3 = 0$. □

Remark 2.2. *According to the first conclusion of Property 2.1, if $\mathcal{G}_j$ satisfies the conditions of Lemma 1.5, by subtracting a_p times the first slice of $\mathcal{G}_j$ from slice p $(p = 2,3,4)$, then it can be further turned into*

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&e_2&h_2&j_2&0&e_3&h_3&j_3&0&e_4&h_4&j_4\\ 0&1&0&0&0&0&f_2&i_2&0&0&f_3&i_3&0&0&f_4&i_4\\ 0&0&1&0&0&0&0&g_2&0&0&0&g_3&0&0&0&g_4\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

Remark 2.3. *According to the second conclusion of the Property 2.1, if $f_2f_3f_4 \neq 0$, the vectors (e_p, f_p, g_p) $(p = 2,3,4)$ are proportionate. If $f_2 = f_3 = f_4 = 0$, the vectors (e_p, f_p, g_p) $(p = 2,3,4)$ are disproportionate.*

By the first conclusion of Property 2.1, we have turned the first diagonal elements of the last three slices of $\mathcal{G}_j$ into zeros. In the next section, we mainly consider turning the 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ into zeros. Noting that if $f_2f_3f_4 \neq 0$, the vectors (e_p, f_p, g_p) $(p = 2, 3, 4)$ are proportionate. This implies that we can directly turn the 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ into zeros if $f_2f_3f_4 \neq 0$. However, in this paper we assume some of the upper triangular entries of the last three slices of $\mathcal{G}_j$ are zeros. This means the vectors (e_p, f_p, g_p) $(p = 2, 3, 4)$ may not be proportionate. Consequently, we first discuss all the combinations of e_p, f_p, g_p $(p = 2, 3, 4)$ when some of them are equal to zeros. On the other hand, noting that e_p, f_p, g_p $(p = 2, 3, 4)$ satisfies equations (8), it is easy to get the following conclusion.

Property 2.4. *There are 91 combinations of e_p, f_p, g_p $(p = 2, 3, 4)$ when some of them are equal to zeros and satisfy the equations (8).*

Proof. We traverse e_p, f_p, g_p $(p = 2, 3, 4)$ based on the number of nonzero entries, and remove some of the combinations that do not meet the conditions (8).

If one of e_p, f_p, g_p $(p = 2, 3, 4)$ is not equal to zero, and the remaining eight of them are equal to zeros, this yields 9 combinations. For convenience, we only write nonzero entry in each combination, i.e., e_2, e_3, e_4, f_2, f_3, f_4, g_2, g_3, g_4. For example, e_2 represents $e_2 \neq 0$, $e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$.

If two of e_p, f_p, g_p $(p = 2, 3, 4)$ are not equal to zeros, and the remaining seven of them are equal to zeros, this yields C_9^2 combinations. However, some combinations do not satisfy conditions (8). For example, combination $e_2 \neq 0, f_3 \neq 0, e_3 = e_4 = f_2 = f_4 = g_2 = g_3 = g_4 = 0$ contradicts the condition $f_3e_2 = e_3f_2$ of (8). Thus, we remove this combination. Similarly, we remove other combinations that do not meet conditions (8). Finally, by traversing we obtain the following 24 combinations that satisfy conditions (8): e_2e_3, e_2e_4, e_3e_4, f_2f_3, f_2f_4, f_3f_4, g_2g_3, g_2g_4, g_3g_4, e_2f_2, e_2g_2, f_2g_2, e_3f_3, e_3g_3, f_3g_3, e_4f_4, e_4g_4, f_4g_4, e_2g_3, e_2g_4, e_3g_2, e_3g_4, e_4g_2, e_4g_3.

Analogously, if three of e_p, f_p, g_p $(p = 2, 3, 4)$ are not equal to zeros, and the remaining six of them are equal to zeros, there are 24 combinations that satisfy conditions (8), i.e., $e_2e_3e_4$, $f_2f_3f_4$, $g_2g_3g_4$, $e_2f_2g_2$, $e_3f_3g_3$, $e_4f_4g_4$, $e_2g_2g_3$, $e_2g_2g_4$, $e_2g_3g_4$, $e_3g_2g_3$, $e_3g_2g_4$, $e_3g_3g_4$, $e_4g_2g_3$, $e_4g_2g_4$, $e_4g_3g_4$, $e_2e_3g_2$, $e_2e_3g_3$, $e_2e_3g_4$, $e_2e_4g_2$, $e_2e_4g_3$, $e_2e_4g_4$, $e_3e_4g_2$, $e_3e_4g_3$, $e_3e_4g_4$.

If four of e_p, f_p, g_p $(p = 2, 3, 4)$ are not equal to zeros, and the remaining five of them are equal to zeros, there are 21 combinations that satisfy conditions (8), i.e., $e_2e_3f_2f_3$, $e_2e_4f_2f_4$, $e_3e_4f_3f_4$, $e_2e_3g_2g_3$, $e_2e_4g_2g_4$, $e_3e_4g_3g_4$, $f_2f_3g_2g_3$, $f_2f_4g_2g_4$, $f_3f_4g_3g_4$, $e_2e_3g_2g_4$, $e_2e_3g_3g_4$, $e_2e_4g_2g_3$, $e_2e_4g_3g_4$, $e_3e_4g_2g_3$, $e_3e_4g_2g_4$, $e_2g_2g_3g_4$, $e_3g_2g_3g_4$, $e_4g_2g_3g_4$, $e_2e_3e_4g_2$, $e_2e_3e_4g_3$, $e_2e_3e_4g_4$.

If five of e_p, f_p, g_p $(p = 2, 3, 4)$ are not equal to zeros, and the remaining four of them are equal to zeros, there are 6 combinations that satisfy conditions (8), i.e., $e_2e_3g_2g_3g_4$, $e_2e_4g_2g_3g_4$, $e_3e_4g_2g_3g_4$, $e_2e_3e_4g_2g_3$, $e_2e_3e_4g_2g_4$, $e_2e_3e_4g_3g_4$.

If six of e_p, f_p, g_p $(p = 2, 3, 4)$ are not equal to zeros, and the remaining three of them are equal to zeros, there are 6 combinations that satisfy conditions (8), i.e., $e_2e_3e_4f_2f_3f_4$, $e_2e_3e_4g_2g_3g_4$, $f_2f_3f_4g_2g_3g_4$, $e_2f_2g_2e_3f_3g_3$, $e_2f_2g_2e_4f_4g_4$, $e_3f_3g_3e_4f_4g_4$.

If seven (or eight) of e_p, f_p, g_p $(p = 2, 3, 4)$ are not equal to zeros, the remaining two (or one) of them are equal to zeros (or zero), there is no combination that satisfies conditions (8).

The last one combination is that e_p, f_p, g_p $(p = 2, 3, 4)$ are all nonzero, i.e., $e_2e_3e_4f_2f_3f_4g_2g_3g_4$. □

In the next section, we will study in which combinations of the 91 combinations in Property 2.4 we turn the 2nd diagonal elements of any two slices of the last three slices into zeros.

Remark 2.5. *Noting that if e_p, f_p, g_p $(p = 2, 3, 4)$ are all zeros, this combination also satisfies the conditions (8). However, another question arises: Under this combination, we can turn $\mathcal{G}_j$ into canonical form? Our answer is negative. In fact, if e_p, f_p, g_p $(p = 2, 3, 4)$ are all zeros, then the 2nd diagonal elements of the last three slices of $\mathcal{G}_j$ are all zeros. This means in the process of transforming $\mathcal{G}_j$ into canonical form, the 2nd diagonal elements of the last three slices of $\mathcal{G}_j$ are always zeros. However, in canonical form (6), there exist a slice whose 2nd diagonal elements are nonzero. Therefore, if e_p, f_p, g_p $(p = 2, 3, 4)$ are all zeros, we can't turn $\mathcal{G}_j$ into canonical form.*

3 Make the 2nd diagonal of any two slices of the last three slices of $\mathcal{G}_j$ zero

In this section we analyze under what conditions we can turn the 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ into zeros.

It is worth noting that if 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ can be turned into zeros, then we can continue to analyze the 3rd and 4th diagonal elements. If the 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ cannot be turned into zeros, it is meaningless to continue to analyze the 3rd and 4th diagonal elements. Therefore, in this section, we only consider the 2nd diagonal elements of the last three slices of $\mathcal{G}_j$.

Another thing we should pay attention to is why we discuss the conditions that turn the 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ into zeros, instead of the conditions that turn the 2nd diagonal elements of the 3rd and the 4th slices into zeros. In fact, if the 2nd diagonal elements of the second slice and the fourth slice (or the second slice and the third slice) are equal to zeros, we can exchange the second slice and the third slice (or the second slice and the fourth slice). The specific operation will be discussed in detail in the next section and so is omitted here.

Now, based on the Property 2.4, we discuss under what conditions we can turn 2nd diagonal elements of any two slices of the last slices of $\mathcal{G}_j$ into zeros. For convenience of the following discussion, we divide the 91 combinations of Property 2.4 into three categories. We regard each category as a set. Each element of the set represents a combination, which can be represented by the nonzero entries of e_p, f_p, g_p $(p = 2, 3, 4)$. For example, the element e_2 of set T_1 represents $e_2 \neq 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$. The element e_2g_3 of set T_2 represents $e_2 \neq 0, g_3 \neq 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_4 = 0$.

$$\begin{aligned}
T_1 = \{&e_2, e_3, e_4, g_2, g_3, g_4, e_2e_3, e_2e_4, e_3e_4, g_2g_3, g_2g_4, g_3g_4, e_2f_2, e_2g_2, f_2g_2, e_3f_3, e_3g_3, f_3g_3, e_4f_4,\\
&e_4g_4, f_4g_4, e_2e_3e_4, g_2g_3g_4, e_2f_2g_2, e_3f_3g_3, e_4f_4g_4, e_2e_3f_2f_3, e_2e_4f_2f_4, e_3e_4f_3f_4, f_2f_3g_2g_3,\\
&f_2f_4g_2g_4, f_3f_4g_3g_4, e_2e_3e_4f_2f_3f_4, f_2f_3f_4g_2g_3g_4, e_2f_2g_2e_3f_3g_3, e_2f_2g_2e_4f_4g_4,\\
&e_3f_3g_3e_4f_4g_4, e_2e_3e_4f_2f_3f_4g_2g_3g_4\}\\
T_2 = \{&e_2g_3, e_2g_4, e_3g_2, e_3g_4, e_4g_2, e_4g_3, e_2g_2g_3, e_2g_2g_4, e_2g_3g_4, e_3g_2g_3, e_3g_2g_4, e_3g_3g_4, e_4g_2g_3,\\
&e_4g_2g_4, e_4g_3g_4, e_2e_3g_2, e_2e_3g_3, e_2e_3g_4, e_2e_4g_2, e_2e_4g_3, e_2e_4g_4, e_3e_4g_2, e_3e_4g_3, e_3e_4g_4,\\
&e_2e_3g_2g_4, e_2e_3g_3g_4, e_2e_4g_2g_3, e_2e_4g_3g_4, e_3e_4g_2g_3, e_3e_4g_2g_4, e_2g_2g_3g_4, e_3g_2g_3g_4,\\
&e_4g_2g_3g_4, e_2e_3e_4g_2, e_2e_3e_4g_3, e_2e_3e_4g_4, e_2e_3g_2g_3g_4, e_2e_4g_2g_3g_4, e_3e_4g_2g_3g_4,\\
&e_2e_3e_4g_2g_3, e_2e_3e_4g_2g_4, e_2e_3e_4g_3g_4\}\\
T_3 = \{&f_2, f_3, f_4, f_2f_3, f_2f_4, f_3f_4, f_2f_3f_4, e_2e_3g_2g_3, e_2e_4g_2g_4, e_3e_4g_3g_4, e_2e_3e_4g_2g_3g_4\}
\end{aligned}$$

Now we show that under each combination of T_1 we can turn 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ into zeros. For each combination in T_1, nonzero entries in e_p, f_p, g_p $(p = 2, 3, 4)$ are either in the same slice or in the same position of different slices.

In fact, if nonzero entries are in the same slice, there are just two slices of the last three slices of $\mathcal{G}_j$ whose 2nd diagonal elements are zeros. For example, the element $e_3f_3g_3$ of T_1 represents $e_3 \neq 0, f_3 \neq 0, g_3 \neq 0, e_2 = e_4 = f_2 = f_4 = g_2 = g_4 = 0$. From Remark 2.2, $\mathcal{G}_j$ is of the form

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
1&0&0&0&0&0&h_2&j_2&0&e_3&h_3&j_3&0&0&h_4&j_4\\
0&1&0&0&0&0&0&i_2&0&0&f_3&i_3&0&0&0&i_4\\
0&0&1&0&0&0&0&0&0&0&0&g_3&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right].$$

From the above $\mathcal{G}_j$, we can see that 2nd diagonal elements of the second and fourth slices are zeros.

If nonzero entries lie in the same position of different slices, it will yield two possibilities. The first case is that there exist two slices whose 2nd diagonal elements are nonzero. For these two slices, by subtracting one slice from the other slice we can turn the 2nd diagonal elements of one of them into zeros. After this, we can obtain two slices of the last three slices of $\mathcal{G}_j$ whose 2nd diagonal elements are equal to zeros. For example, the element $e_3e_4f_3f_4$ in T_1 represents $e_3 \neq 0, e_4 \neq 0, f_3 \neq 0, f_4 \neq 0, e_2 = f_2 = g_2 = g_3 = g_4 = 0$. From Remark 2.2, $\mathcal{G}_j$ is of the form

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&0&h_2&j_2&0&e_3&h_3&j_3&0&e_4&h_4&j_4\\ 0&1&0&0&0&0&0&i_2&0&0&f_3&i_3&0&0&f_4&i_4\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

For this $\mathcal{G}_j$, through subtracting e_4/e_3 times the third slice from the fourth slice (or e_3/e_4 times the fourth slice from the third slice), we can turn e_4 and f_4 (or e_3 and f_3) into zeros because $e_4f_3 = f_4e_3$ holds for almost all $\mathcal{G}_j$ in (8). Then we get the second and fourth slices (or the second and third slices) whose 2nd diagonal elements are zeros. The second case is that there exist three slices whose 2nd diagonal elements are nonzero. For these three slices, through subtracting one slice from another two slices, then we can turn the 2nd diagonal elements of two of them into zeros. For example, the element $e_2e_3e_4f_2f_3f_4$ in T_1 represents $e_2 \neq 0$, $e_3 \neq 0$, $e_4 \neq 0, f_2 \neq 0, f_3 \neq 0, f_4 \neq 0, g_2 = g_3 = g_4 = 0$. From Remark 2.2, $\mathcal{G}_j$ is of the form

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&e_2&h_2&j_2&0&e_3&h_3&j_3&0&e_4&h_4&j_4\\ 0&1&0&0&0&0&f_2&i_2&0&0&f_3&i_3&0&0&f_4&i_4\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

For this $\mathcal{G}_j$, through subtracting e_3/e_2 times the second slice from the third slice and e_4/e_2 times the second slice from the fourth slice (or e_2/e_3 times the third slice from the second slice and e_4/e_3 times the third slice from the fourth slice or e_2/e_4 times the fourth slice from the second slice and e_3/e_4 times the fourth slice from the third slice), we can turn e_3, f_3, e_4, f_4 (or e_2, f_2, e_4, f_4 or e_2, f_2, e_3, f_3) into zeros because $e_3f_2 = f_3e_2$, $e_4f_2 = f_4e_2$, $e_4f_3 = f_4e_3$ holds for almost all $\mathcal{G}_j$ in (8). Then we get the third and fourth (or the second and fourth or the second and third) slices whose 2nd diagonal elements are zeros.

Now we show that under each combination of T_2, we can't turn 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ into zeros. For each combination in T_2, we find that nonzero entries in e_p, f_p, g_p ($p = 2, 3, 4$) are in different position of different slices, which results in at least two slices whose 2nd diagonal elements can't be turned into zeros. For example, the element e_2g_3 in T_2 represents $e_2 \neq 0, g_3 \neq 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_4 = 0$. From Remark 2.2, $\mathcal{G}_j$ is of the form

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&e_2&h_2&j_2&0&0&h_3&j_3&0&0&h_4&j_4\\ 0&1&0&0&0&0&0&i_2&0&0&0&i_3&0&0&0&i_4\\ 0&0&1&0&0&0&0&0&0&0&0&g_3&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

For this $\mathcal{G}_j$, if we can turn e_2 into zero, then we get the second and fourth slices whose 2nd diagonal elements are all zeros; if we can turn g_3 into zero, then we get the third and fourth slices whose 2nd diagonal elements are all zeros. On the other hand, e_2 can be turned into zero only by the (1,2) entry of the third slice or the (1,2) entry of the fourth slice; g_3 can be turned into zero only by the (3,4) entry of the second slice or the (3,4) entry of the fourth slice. However, (1,2) entries of the third and fourth slices are all zeros; (3,4) entries of the second and fourth slices are all zeros; so e_2 and g_3 can't be turned into zeros. This implies that if nonzero entries are in different position of different slices, we can't turn any two slices of the last three slices whose 2nd diagonal elements into zeros. Therefore, it makes no sense to continue to analyze the 3rd and 4th diagonal elements. Consequently, in the next section, we no longer consider all the combinations in T_2.

For each combination of T_3, we have not found the relationship between e_p, f_p, g_p and h_p, i_p, j_p for $p = 2, 3, 4$. Consequently, in the next section analysis we no longer consider all the combinations in T_3.

4 Make the 3rd and 4th diagonal of the last three slices of $\mathcal{G}_j$ zero

In this section we analyze under what conditions we can turn the 3rd and the 4th diagonal elements of the last three slices of $\mathcal{G}_j$ into zeros.

Our main idea is as follows. For each combination in T_1, combining with expression (9), we can obtain the relationship between e_p, g_p and h_p, i_p ($p = 2, 3, 4$). Through the relationship between them, we give conditions that can turn the 3rd and 4th diagonal elements of the last three slices of $\mathcal{G}_j$ into zeros. Under these conditions, we can turn $\mathcal{G}_j$ into canonical form (6).

Theorem 4.1. *If one of e_p, f_p, g_p ($p = 2, 3, 4$) is not equal to zero, the remaining eight of them are equal to zeros, there are 6 combinations in T_1, i.e., e_p, g_p ($p = 2, 3, 4$). If $e_x \neq 0$, the other of e_p, f_p, g_p ($p = 2, 3, 4$) are equal to zeros, under conditions $i_x = i_y = i_z = 0$, $h_y j_z \neq j_y h_z$, or if $g_x \neq 0$, the other of e_p, f_p, g_p ($p = 2, 3, 4$) are equal to zeros, under conditions $h_x = h_y = h_z = 0$, $j_y i_z \neq i_y j_z$, where $x, y, z \in \{2, 3, 4\}$ and $x \neq y \neq z$, we can turn $\mathcal{G}_j$ into canonical form (6).*

Proof. Here, we only consider the combination that $e_2 \neq 0$, the remaining eight of e_p, f_p, g_p ($p = 2, 3, 4$) are equal to zeros. The combinations that $e_3 \neq 0$ (or $e_4 \neq 0$ or $g_2 \neq 0$ or $g_3 \neq 0$ or $g_4 \neq 0$), the remaining eight of e_p, f_p, g_p ($p = 2, 3, 4$) are equal to zeros can be proved in a similar way as the combination that $e_2 \neq 0$, the remaining eight of e_p, f_p, g_p ($p = 2, 3, 4$) are equal to zeros.

If $e_2 \neq 0$, $e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, according to (9), we have $i_3 = i_4 = 0$, then $\mathcal{G}_j$ is of the form

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&e_2&h_2&j_2&0&0&h_3&j_3&0&0&h_4&j_4\\ 0&1&0&0&0&0&0&i_2&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

Because the mode-3 rank of $\mathcal{G}_j$ is 4, then $h_3 j_4 \neq j_3 h_4$. Next, we show why the mode-3 rank of $\mathcal{G}_j$ is not equal to 4 if $h_3 j_4 = j_3 h_4$. Suppose $h_3 = 0$, then $j_3 h_4 = 0$. This yields three possibilities. The first case is $j_3 = 0, h_4 \neq 0$, then all the entries of the third slice are equal to zeros. The second case is $j_3 \neq 0, h_4 = 0$, if $j_4 = 0$, then all the entries of the fourth slice are equal to zeros; if $j_4 \neq 0$, by subtracting j_4/j_3 times the third slice from the fourth slice (or j_3/j_4 times the fourth slice from the third slice), we can turn j_4 (or j_3) into zero, then all the entries of the fourth (or third) slice are equal to zeros. The third case is $j_3 = 0, h_4 = 0$, then all the entries of the third slice are equal to zeros. The situations where we suppose that $j_4 = 0$ or $j_3 = 0$ or $h_4 = 0$ can be dealt with analogously. Suppose h_3, j_4, j_3, h_4 are nonzero, by subtracting h_4/h_3 times the third slice from the fourth slice (or h_3/h_4 times the fourth slice from the third slice), we can turn h_4 and j_4 (or h_3 and j_3) into zeros. Then all the entries of the third slice (or the fourth slice) are equal to zeros. From the above discussion we can see that, if $h_3 j_4 = j_3 h_4$, there always exists a slice whose entries are all equal to zeros. This implies that the mode-3 rank of $\mathcal{G}_j$ is 3. Thus we draw the conclusion that $h_3 j_4 \neq j_3 h_4$.

Now we prove that i_2 must be equal to zero, because i_2 can be turned into zero only through (2,3) or (3,4) entry of the second slice or through (2,4) entry of the third or (2,4) entry of the fourth slice. However, (2,3) and (3,4) entries of the second slice, (2,4) entry of the third and fourth slices are all equal to zeros. This means in the process of transforming $\mathcal{G}_j$ into canonical form, i_2 can't be turned into zero. Therefore, we must have i_2 equal to zero.

Now we prove how we turn $\mathcal{G}_j$ into canonical form if $e_2 \neq 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, under conditions that $i_2 = i_3 = i_4 = 0$ and $h_3 j_4 \neq j_3 h_4$. In fact, under these conditions, $\mathcal{G}_j$ have the following form

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&e_2&h_2&j_2&0&0&h_3&j_3&0&0&h_4&j_4\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

Firstly, we discuss how to standardize the last two slices. Because $h_3 j_4 \neq j_3 h_4$, suppose $h_3 = 0$, then $j_3 h_4 \neq 0$. If $j_4 = 0$, then the third slice only has its $(1, 4)$ entry j_3 nonzero, the fourth slice only has its $(2, 3)$ entry h_4 nonzero. We normalize them to one. By exchanging the third slice and the fourth slice, then we obtain the following form

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&e_2&h_2&j_2&0&0&1&0&0&0&0&1\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right]. \tag{10}$$

If $j_4 \neq 0$, by subtracting j_4/j_3 times the third slice from the fourth slice, we can turn j_4 into zero. After this, the third slice only has j_3 is nonzero, the fourth slice only has h_4 is nonzero. Similarly, we normalize them to one. By exchanging the third slice and the fourth slice, then we obtain (10). The situations where we suppose that $j_4 = 0$ or $j_3 = 0$ or $h_4 = 0$ can be dealt with analogously.

Next, we discuss how to standardize the second slice. If $h_2 = j_2 = 0$, then we have transformed $\mathcal{G}_j$ into canonical form. If h_2, j_2 are nonzero, we can turn them into zeros. The specific method is: h_2 can be turned into zero by subtracting h_2 times the third slice from the second slice, j_2 can be turned into zero by subtracting j_2 times the fourth slice from the second slice.

Consequently, if $e_x \neq 0$, the remaining eight of e_p, f_p, g_p $(p = 2, 3, 4)$ are equal to zeros, under conditions $i_x = i_y = i_z = 0$ and $h_y j_z \neq j_y h_z$, where $x = 2, y = 3, z = 4$ or $x = 2, y = 4, z = 3$, we can turn $\mathcal{G}_j$ into canonical form. □

Remark 4.2. *In fact, theorem 4.1 contains the following six cases.*

Let $x = 2, y = 3, z = 4$ or $x = 2, y = 4, z = 3$.

Case 1: If $e_2 \neq 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, under conditions $i_2 = i_3 = i_4 = 0$ and $h_3 j_4 \neq j_3 h_4$, we can turn $\mathcal{G}_j$ into canonical form.

Case 2: If $g_2 \neq 0, e_2 = e_3 = e_4 = f_2 = f_3 = f_4 = g_3 = g_4 = 0$, under conditions $h_2 = h_3 = h_4 = 0$ and $j_3 i_4 \neq i_3 j_4$, we can turn $\mathcal{G}_j$ into canonical form.

Let $x = 3, y = 2, z = 4$ or $x = 3, y = 4, z = 2$.

Case 3: If $e_3 \neq 0, e_2 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, under conditions $i_2 = i_3 = i_4 = 0$ and $h_2 j_4 \neq j_2 h_4$, we can turn $\mathcal{G}_j$ into canonical form.

Case 4: If $g_3 \neq 0, e_2 = e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_4 = 0$, under conditions $h_2 = h_3 = h_4 = 0$ and $j_2 i_4 \neq i_2 j_4$, we can turn $\mathcal{G}_j$ into canonical form.

Let $x = 4, y = 2, z = 3$ or $x = 4, y = 3, z = 2$.

Case 5: If $e_4 \neq 0, e_2 = e_3 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, under conditions $i_2 = i_3 = i_4 = 0$ and $h_2 j_3 \neq j_2 h_3$, we can turn $\mathcal{G}_j$ into canonical form.

Case 6: If $g_4 \neq 0, e_2 = e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = 0$, under conditions $h_2 = h_3 = h_4 = 0$ and $j_2 i_3 \neq i_2 j_3$, we can turn $\mathcal{G}_j$ into canonical form.

For the sake of simplicity, we write the above six cases as follows:

$$e_x \neq 0, \textit{the other are zero}, \; i_x = i_y = i_z = 0, h_y j_z \neq j_y h_z;$$
$$g_x \neq 0, \textit{the other are zero}, \; h_x = h_y = h_z = 0, i_y j_z \neq j_y i_z;$$

where $x, y, z \in \{2, 3, 4\}$ and $x \neq y \neq z$.

The following theorem is also written in a similar way, and we don't repeat it. For example, $e_x e_y \neq 0$, $i_x = i_y = i_z = 0$, $\tilde{h}_y j_z \neq \tilde{j}_y h_z$, where $x = 2, y = 3, z = 3$, in Theorem 4.3 represents if $e_2 \neq 0$, $e_3 \neq 0$, $e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, under conditions $i_2 = i_3 = i_4 = 0$ and $\tilde{h}_3 j_4 \neq \tilde{j}_3 h_4$, can we turn $\mathcal{G}_j$ into canonical form.

Theorem 4.3. *If two of e_p, f_p, g_p $(p = 2, 3, 4)$ are not equal to zeros, the remaining seven of them are equal to zeros, there are 15 combinations in T_1, i.e. e_2e_3, e_2e_4, e_3e_4, g_2g_3, g_2g_4, g_3g_4, e_2f_2, e_3f_3, e_4f_4, f_2g_2, f_3g_3, f_4g_4,*

e_2g_2, e_3g_3, e_4g_4. For each combinations, we can turn $\mathcal{G}_j$ into canonical form under the following conditions:

$$\begin{aligned}
&e_xe_y \neq 0, i_x = i_y = i_z = 0, \tilde{h}_y j_z \neq \tilde{j}_y h_z;\\
&g_xg_y \neq 0, h_x = h_y = h_z = 0, \tilde{i}_y j_z \neq \tilde{j}_y i_z;\\
&e_xf_x \neq 0, i_y = i_z = 0, h_y j_z \neq j_y h_z;\\
&f_xg_x \neq 0, h_y = h_z = 0, i_y j_z \neq j_y i_z;\\
&e_xg_x \neq 0, i_y = h_y = 0, i_z h_z \neq 0, j_y \neq 0, h_x i_z = i_x h_z;\\
&(e_xg_x \neq 0, i_y h_y i_z h_z \neq 0, h_x i_y = i_x h_y, h_y j_z \neq j_y h_z);\\
&\textit{where } x, y, z \in \{2, 3, 4\} \textit{ and } x \neq y \neq z.
\end{aligned}$$

Proof. Firstly, we discuss the combination that $e_2e_3 \neq 0$, the remaining seven of e_p, f_p, g_p $(p = 2, 3, 4)$ are equal to zeros. The combinations e_2e_4, e_3e_4, g_2g_3, g_2g_4, g_3g_4 can be similar as in the discussion with e_2e_3.

If $e_2e_3 \neq 0$, $e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, according to (9), we can obtain $i_4 = 0$ and $e_2i_3 = e_3i_2$. Because $e_2e_3 \neq 0$, then i_2 and i_3 are zero or nonzero at the same time. This yields two possibilities.

The first case is $i_2 = i_3 = 0$. Then $\mathcal{G}_j$ is of the form

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
1&0&0&0&0&e_2&h_2&j_2&0&e_3&h_3&j_3&0&0&h_4&j_4\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right].$$

By subtracting e_3/e_2 times the second slice from the third slice, we can turn e_3 into zero. Then we obtain the following form:

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
1&0&0&0&0&e_2&h_2&j_2&0&0&\tilde{h}_3&\tilde{j}_3&0&0&h_4&j_4\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right],$$

where $\tilde{h}_3 = h_3 - e_3h_2/e_2, \tilde{j}_3 = j_3 - e_3j_2/e_2$. According to Theorem 4.1, under condition $\tilde{h}_3 j_4 \neq \tilde{j}_3 h_4$, we can turn $\mathcal{G}_j$ into canonical form. Analogously, by subtracting e_2/e_3 times the third slice from the second slice, we can turn e_2 into zero, under condition $\tilde{h}_2 j_4 \neq \tilde{j}_2 h_4$, where $\tilde{h}_2 = h_2 - e_2h_3/e_3, \tilde{j}_2 = j_2 - e_2j_3/e_3$, we can also turn $\mathcal{G}_j$ into canonical form.

The second case is $i_2i_3 \neq 0$. By subtracting e_3/e_2 times the second slice from the third slice, we can turn e_3, i_3 into zeros according to $e_2i_3 = e_3i_2$. Then we obtain the following form:

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
1&0&0&0&0&e_2&h_2&j_2&0&0&\tilde{h}_3&\tilde{j}_3&0&0&h_4&j_4\\
0&1&0&0&0&0&0&i_2&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right],$$

where $\tilde{h}_3 = h_3 - e_3h_2/e_2, \tilde{j}_3 = j_3 - e_3j_2/e_2$. According to Theorem 4.1, only under conditions $i_2 = 0$ and $\tilde{h}_3 j_4 \neq \tilde{j}_3 h_4$, we can turn $\mathcal{G}_j$ into canonical form. This contradicts the fact that $i_2i_3 \neq 0$.

Thus we conclude that if $e_xe_y \neq 0$, the remaining seven of $e_p, f_p, g_p (p = 2, 3, 4)$ are equal to zeros, under conditions $i_x = i_y = i_z = 0$, $\tilde{h}_y j_z \neq \tilde{j}_y h_z$, where $x = 2, y = 3, z = 4$ or $x = 3, y = 2, z = 4$, we can turn $\mathcal{G}_j$ into canonical form (6).

Next, we discuss the combination that $e_2f_2 \neq 0$, the remaining seven of e_p, f_p, g_p $(p = 2, 3, 4)$ are equal to zeros. The combinations f_2g_2, e_3f_3, f_3g_3, e_4f_4, f_4g_4 can be similar as in the discussion with e_2f_2.

If $e_2f_2 \neq 0$, $e_3 = e_4 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, according to (9), we can obtain $i_3 = i_4 = 0$. Because the mode-3 rank of $\mathcal{G}_j$ is 4, then $h_3j_4 \neq j_3h_4$. The way to standardize the last two slices is the same as in Theorem 4.1, so we don't repeat it here. Now, we show how to standardize the second slice. The way to turn h_2 and j_2 into zeros is the same as in Theorem 4.1. It remains to consider how to turn i_2 into zero. In fact, if $i_2 = 0$, then we have transformed $\mathcal{G}_j$ into canonical form. If $i_2 \neq 0$, for every slice of $\mathcal{G}_j$, by subtracting i_2/f_2

times column 3 from column 4 and adding i_2/f_2 times row 4 to row 3 , we can turn i_2 into zero. Then the (1,4) entry of slice three is turned into $-i_2/f_2$. Next, by subtracting $-i_2/f_2$ times the fourth slice form the third slice, we can turn $(1,4)$ entry of slice three into zero.

Consequently, if $e_xf_x \neq 0$, the remaining seven of $e_p, f_p, g_p (p = 2,3,4)$ are equal to zeros, under conditions $i_y = i_z = 0$ and $h_yj_z \neq j_xh_z$, where $x = 2, y = 3, z = 4$ or $x = 2, y = 4, z = 3$, we can turn $\mathcal{G}_j$ into canonical form.

Next, we discuss the combination that $e_2g_2 \neq 0$, the remaining seven of e_p, f_p, g_p $(p = 2,3,4)$ are equal to zeros. The combinations e_3g_3, e_4g_4 can be similar as in the discussion with e_2g_2.

If $e_2g_2 \neq 0$, $e_3 = e_4 = f_2 = f_3 = f_4 = g_3 = g_4 = 0$, according to (9), we have $e_2i_3 = h_3g_2$, $e_2i_4 = h_4g_2$. Because $e_2g_2 \neq 0$, so we must have i_3 and h_3 being zero or nonzero at the same time, i_4 and h_4 being zero or nonzero at the same time. This yields four possibilities.

The first case is $i_3 = h_3 = 0$, $i_4h_4 \neq 0$. Then $\mathcal{G}_j$ is of the form

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&e_2&h_2&j_2&0&0&0&j_3&0&0&h_4&j_4\\ 0&1&0&0&0&0&0&i_2&0&0&0&0&0&0&0&i_4\\ 0&0&1&0&0&0&0&g_2&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

Because the mode-3 rank of $\mathcal{G}_j$ is 4, then $j_3 \neq 0$. Next, we normalize j_3 to one, and it can be used to turn j_2 and j_4 into zeros if j_2 and j_4 are nonzero. After this, by exchanging the third slice and the fourth slice, then we can obtain the following form

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&e_2&h_2&0&0&0&h_4&0&0&0&0&1\\ 0&1&0&0&0&0&0&i_2&0&0&0&i_4&0&0&0&0\\ 0&0&1&0&0&0&0&g_2&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

If h_2 and i_2 are equal to zeros, then we have turned $\mathcal{G}_j$ into canonical form. If one of h_2, i_2 is equal to zero, we can't turn $\mathcal{G}_j$ into canonical form. In fact, if $h_2 = 0, i_2 \neq 0$, for every slice, by subtracting i_2/g_2 times row 3 from row 2 and adding i_2/g_2 times column 2 to column 3, we can turn i_2 into zero. But meanwhile, h_2 is turned into nonzero. Similarly, if $h_2 \neq 0, i_2 = 0$, while turning h_2 into zero, i_2 is turned into nonzero. Similarly to the discussion of the above, if h_2 and i_2 are nonzero, after turn h_2 into zero, i_2 is turned into $\tilde{i}_2 = i_2 + h_2g_2/e_2$. While turning $\tilde{i}_2$ into zero, h_2 is turned into nonzero. If we add a restriction condition $h_2i_4 = i_2h_4$, by subtracting h_4/h_2 times the third slice from the second slice, we can turn h_2 and i_2 into zeros. Hence, we have turned $\mathcal{G}_j$ into canonical form. From the above discussion we can see that, if $i_3 = h_3 = 0$, $i_4h_4 \neq 0$, only under condition $h_2i_4 = i_2h_4$ we can turn $\mathcal{G}_j$ into canonical form.

The second case is $i_3h_3 \neq 0$, $i_4 = h_4 = 0$. This situation can be similar as in the discussion with $i_3 = h_3 = 0$, $i_4h_4 \neq 0$.

The third case is $i_3 = h_3 = i_4 = h_4 = 0$. Then $\mathcal{G}_j$ is of the form

$$\left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&e_2&h_2&j_2&0&0&0&j_3&0&0&0&j_4\\ 0&1&0&0&0&0&0&i_2&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&g_2&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

If one of j_3, j_4 is equal to zero, then the mode-3 rank of $\mathcal{G}_j$ is equal to 3. If j_3 and j_4 are all equal to zeros, then the mode-3 rank of $\mathcal{G}_j$ is equal to 2. If j_3 and j_4 are nonzero, by subtracting j_4/j_3 times the third slice from the fourth slice, we can turn j_4 into zero. Then the mode-3 rank of $\mathcal{G}_j$ is equal to 3. Consequently, if $i_3 = h_3 = i_4 = h_4 = 0$, we can't turn $\mathcal{G}_j$ into canonical form.

The fourth case is $i_3h_3i_4h_4 \neq 0$. It follows from $e_2i_3 = h_3g_2$, $e_2i_4 = h_4g_2$ that $i_3h_4 = i_4h_3$. This means the vectors (h_3, i_3) and (h_4, i_4) are proportional. If the vectors (h_3, j_3) and (h_4, j_4) are also proportional, then the mode-3 rank of $\mathcal{G}_j$ is not equal to 4. Therefore, we conclude that $h_3j_4 \neq h_4j_3$. Similarly to the discussion of the case that $i_3 = h_3 = 0$, $i_4h_4 \neq 0$. If we add a restriction condition $h_2i_4 = i_2h_4$ or $h_2i_3 = i_2h_3$, we can turn h_2 and i_2 into zero. Hence, if $i_3h_3i_4h_4 \neq 0$, under conditions $h_3j_4 \neq h_4j_3$, $h_2i_4 = i_2h_4$ or $h_3j_4 \neq h_4j_3$, $h_2i_3 = i_2h_3$, we can turn $\mathcal{G}_j$ into canonical form.

Based on the above argument, we draw the conclusion that if $e_x g_x \neq 0$, the remaining seven of e_p, f_p, g_p (p=2,3,4) are equal to zeros, under conditions $i_y = h_y = 0$, $i_z h_z \neq 0$, $j_y \neq 0$, $h_x i_z = i_x h_z$ or $i_y h_y i_z h_z \neq 0$, $h_x i_y = i_x h_y$, $h_y j_z \neq j_y h_z$, where $x = 2, y = 3, z = 4$ or $x = 2, y = 4, z = 3$, we can turn $\mathcal{G}_j$ into canonical form. □

Theorem 4.4. *If three of e_p, f_p, g_p ($p = 2, 3, 4$) are not equal to zeros, the remaining six of them are equal to zeros, there have 5 combinations in T_1, i.e. $e_2e_3e_4$, $g_2g_3g_4$, $e_2f_2g_2$, $e_3f_3g_3$, $e_4f_4g_4$. For each combinations, $\mathcal{G}_j$ can be turned into canonical form under the following conditions:*

$$\begin{aligned}
&e_x e_y e_z \neq 0, i_x = i_y = i_z = 0, \tilde{h}_y \tilde{j}_z \neq \tilde{j}_y \tilde{h}_z;\\
&g_x g_y g_z \neq 0, h_x = h_y = h_z = 0, \tilde{i}_y \tilde{j}_z \neq \tilde{j}_y \tilde{i}_z;\\
&e_x f_x g_x \neq 0, i_y = h_y = 0, i_z h_z \neq 0, j_y \neq 0;\\
&(e_x f_x g_x \neq 0, i_y h_y i_z h_z \neq 0, j_y h_z \neq h_y j_z);\\
&\textit{where } x, y, z \in \{2, 3, 4\} \textit{ and } x \neq y \neq z.
\end{aligned}$$

Proof. Firstly, we discuss the combination that $e_2e_3e_4 \neq 0$, the remaining six of e_p, f_p, g_p ($p = 2, 3, 4$) are equal to zeros. The combination $g_2g_3g_4$ can be similar as in the discussion with $e_2e_3e_4$.

If $e_2e_3e_4 \neq 0, f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, according to (9), we have $e_3 i_2 = i_3 e_2$, $e_4 i_2 = e_2 i_4$, $e_4 i_3 = e_3 i_4$. Because $e_2e_3e_4 \neq 0$, so we must have i_2, i_3, i_4 being zero or nonzero at the same time. This yields two possibilities. The first case is $i_2 = i_3 = i_4 = 0$. The second case is $i_2 i_3 i_4 \neq 0$. Similarly to the discussion of the situation $e_2 e_3 \neq 0$ in Theorem 4.3, only for the case that $i_2 = i_3 = i_4 = 0$ we can turn $\mathcal{G}_j$ into canonical form. Now, we show if $i_2 = i_3 = i_4 = 0$, under what conditions we can turn $\mathcal{G}_j$ into canonical form. In fact, if $i_2 = i_3 = i_4 = 0$, then $\mathcal{G}_j$ is of the form

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
1&0&0&0&0&e_2&h_2&j_2&0&e_3&h_3&j_3&0&e_4&h_4&j_4\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right].$$

Through subtracting e_3/e_2 times the second slice from the third slice and e_4/e_2 times the second slice from the fourth slice, we can turn e_3 and e_4 into zeros. Then $\mathcal{G}_j$ is of the form

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
1&0&0&0&0&e_2&h_2&j_2&0&0&\tilde{h}_3&\tilde{j}_3&0&0&\tilde{h}_4&\tilde{j}_4\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right],$$

where $\tilde{h}_3 = h_3 - e_3h_2/e_2$, $\tilde{j}_3 = j_3 - e_3j_2/e_2$, $\tilde{h}_4 = h_4 - e_4h_2/e_2$, $\tilde{j}_4 = j_4 - e_4j_2/e_2$. According to the mode-3 rank of $\mathcal{G}_j$ is 4, then $\tilde{h}_3\tilde{j}_4 \neq \tilde{j}_3\tilde{h}_4$. Thus we draw the conclusion that if $e_x e_y e_z \neq 0$, the remaining six of $e_p, f_p, g_p (p = 2, 3, 4)$ are equal to zeros, under the conditions $i_x = i_y = i_z = 0$, $\tilde{h}_y\tilde{j}_z \neq \tilde{j}_y\tilde{h}_z$, where $x, y, z \in \{2, 3, 4\}$ and $x \neq y \neq z$, we can turn $\mathcal{G}_j$ into canonical form.

Next, we discuss the combination that $e_2f_2g_2 \neq 0$, the remaining six of e_p, f_p, g_p ($p = 2, 3, 4$) are equal to zeros. The combinations $e_3f_3g_3$, $e_4f_4g_4$ can be similar as in the discussion with $e_2f_2g_2$.

If $e_2f_2g_2 \neq 0$, $e_3 = e_4 = f_3 = f_4 = g_3 = g_4 = 0$, according to (9), we have $e_2 i_3 = h_3 g_2$, $e_2 i_4 = h_4 g_2$. Because $e_2 g_2 \neq 0$, then i_3 and h_3 are zero or nonzero at the same time, i_4 and h_4 are zero or nonzero at the same time. This yields four possibilities. The proof of these four cases is almost identical as the case that $e_2 g_2 \neq 0$ in Theorem 4.3, the major change is that h_2 can be turned into zero by e_2, i_2 can be turned into zero by f_2. Consequently, if $e_x f_x g_x \neq 0$, the remaining six of $e_p, f_p, g_p (p = 2, 3, 4)$ are zeros, under conditions $i_y = h_y = 0$, $i_z h_z \neq 0$, $j_y \neq 0$ or $i_y h_y i_z h_z \neq 0$, $h_y j_z \neq j_y h_z$, where $x = 2, y = 3, z = 4$ or $x = 2, y = 4, z = 3$, we can turn $\mathcal{G}_j$ into canonical form. □

Theorem 4.5. *If four of e_p, f_p, g_p ($p = 2, 3, 4$) are not equal to zero, the remaining five of them are equal to zeros, there have 6 combinations in T_1, i.e. $e_2e_3f_2f_3$, $e_2e_4f_2f_4$, $e_3e_4f_3f_4$, $f_2f_3g_2g_3$, $f_2f_4g_2g_4$, $f_3f_4g_3g_4$. For each* *combination, $\mathcal{G}_j$ can be turned into canonical form under the following conditions:*

$$e_x e_y f_x f_y \neq 0, i_z = 0, i_x, i_y \text{ being zero or nonzero at same time}, \tilde{h}_y j_z \neq \tilde{j}_y h_z;$$
$$f_x f_y g_x g_y \neq 0, h_z = 0, h_x, h_y \text{ being zero or nonzero at same time}, \tilde{j}_y i_z \neq \tilde{i}_y j_z;$$
$$\text{where } x, y, z \in \{2, 3, 4\} \text{ and } x \neq y \neq z.$$

Proof. Here, we only discuss the combination that $e_2e_3f_2f_3 \neq 0$, the remaining five of e_p, f_p, g_p $(p = 2, 3, 4)$are equal to zeros. The combinations $e_2e_4f_2f_4$, $e_3e_4f_3f_4$, $f_2f_3g_2g_3$, $f_2f_4g_2g_4$, $f_3f_4g_3g_4$ can be similar as in the discussion with $e_2e_3f_2f_3$.

If $e_2e_3f_2f_3 \neq 0$, $e_4 = f_4 = g_2 = g_3 = g_4 = 0$, according to (9), we have $i_4 = 0$ and $e_2i_3 = e_3i_2$. The proof of this case is almost identical as the combination that $e_2e_3 \neq 0$ in Theorem 4.3, the major change is that i_2 (or i_3) can be turned into zero by f_2 (or f_3) if i_2 (or i_3) is nonzero.

Thus we conclude that if $e_2e_3f_2f_3 \neq 0$, the remaining five of $e_p, f_p, g_p (p = 2, 3, 4)$ are equal to zeros, under conditions $i_z = 0$, i_x, i_y being zero or not at the same time, $\tilde{h}_y j_z \neq \tilde{j}_y h_z$, where $x = 2, y = 3, z = 4$ or $x = 3, y = 2, z = 4$, we can turn $\mathcal{G}_j$ into canonical form. □

Theorem 4.6. *If six of e_p, f_p, g_p $(p = 2, 3, 4)$ are not equal to zero, the remaining three of them are equal to zeros, there we have 5 combinations in T_1, i.e. $e_2e_3e_4f_2f_3f_4$, $f_2f_3f_4g_2g_3g_4$, $e_2f_2g_2e_3f_3g_3$, $e_2f_2g_2e_4f_4g_4$, $e_3f_3g_3e_4f_4g_4$. For each combination, $\mathcal{G}_j$ can be turned into canonical form under the following conditions:*

$$e_x e_y e_z f_x f_y f_z \neq 0, i_x, i_y, i_z \text{ being zero or nonzero at same time}, \tilde{h}_y \tilde{j}_z \neq \tilde{j}_y \tilde{h}_z;$$
$$f_x f_y f_z g_x g_y g_z \neq 0, h_x, h_y, h_z \text{ being zero or nonzero at same time}, \tilde{j}_y \tilde{i}_z \neq \tilde{i}_y \tilde{j}_z;$$
$$e_x f_x g_x e_y f_y g_y \neq 0, \tilde{h}_y = \tilde{i}_y = 0, h_z i_z \neq 0, \tilde{j}_y \neq 0;$$
$$(e_x f_x g_x e_y f_y g_y \neq 0, \tilde{h}_y \tilde{i}_y \neq 0, h_z = i_z = 0, j_z \neq 0);$$
$$(e_x f_x g_x e_y f_y g_y \neq 0, \tilde{h}_y \tilde{i}_y h_z i_z \neq 0, \tilde{h}_y j_z \neq \tilde{j}_y h_z);$$
$$\text{where } x, y, z \in \{2, 3, 4\} \text{ and } x \neq y \neq z.$$

Proof. Firstly, we discuss the combination that $e_2e_3e_4f_2f_3f_4 \neq 0$, the remaining three of e_p, f_p, g_p $(p = 2, 3, 4)$ are equal to zeros. The combination $f_2f_3f_4g_2g_3g_4$ can be similar as in the discussion with $e_2e_3e_4f_2f_3f_4$. If $e_2e_3e_4f_2f_3f_4 \neq 0$, $g_2 = g_3 = g_4 = 0$, according to (9), we have $e_2i_3 = e_3i_2$, $e_4i_2 = e_2i_4$, $e_4i_3 = e_3i_4$. The proof of this combination is almost identical as the combination that $e_2e_3e_4 \neq 0$ in Theorem 4.4, the major change is that i_2 (or i_3 or i_4) can be turned into zero by f_2 (or f_3 or f_4) if i_2 (or i_3 or i_4) is nonzero. Thus we conclude that if $e_2e_3e_4f_2f_3f_4 \neq 0$, the remaining three of e_p, f_p, g_p $(p = 2, 3, 4)$ are equal to zeros, under conditions i_x, i_y, i_z being zero or not at the same time, $\tilde{h}_y \tilde{j}_z \neq \tilde{j}_y \tilde{h}_z$, where $x, y, y \in \{2, 3, 4\}$ and $x \neq y \neq z$, we can turn $\mathcal{G}_j$ into canonical form.

Next, we discuss the combination that $e_2f_2g_2e_3f_3g_3 \neq 0$, the remaining three of e_p, f_p, g_p $(p = 2, 3, 4)$are equal to zeros. The combinations $e_2f_2g_2e_3f_3g_3$, $e_2f_2g_2e_4f_4g_4$, $e_3f_3g_3e_4f_4g_4$ can be similar as in the discussion with $e_2f_2g_2e_3f_3g_3$. If $e_2f_2g_2e_3f_3g_3 \neq 0$, $e_4 = f_4 = g_4 = 0$, according to (9), we have $e_3i_2 - e_2i_3 + h_3g_2 - g_3h_2 = 0$,$e_2i_4 = h_4g_2$ and $e_3i_4 = h_4g_3$. Because $e_2e_3g_2g_3 \neq 0$ and $e_3/e_2 = g_3/g_2 = \alpha$, then we obtain $e_2\tilde{i}_3 = g_2\tilde{h}_3$, where $\tilde{i}_3 = i_3 - \alpha i_2$, $\tilde{h}_3 = h_3 - \alpha h_2$. On the other hand, because $e_2e_3g_2g_3 \neq 0$, so $\tilde{h}_3$ and $\tilde{i}_3$ are zero or nonzero at the same time, h_4 and i_4 are zero or nonzero at the same time. This yields four possibilities. These four situations can be discussed as the situation that $e_2f_2g_2 \neq 0$, the remaining six of e_p, f_p, g_p $(p = 2, 3, 4)$ are equal to zeros.

Consequently, if $e_xf_xg_xe_yf_yg_y \neq 0$, the remaining three of e_p, f_p, g_p $(p = 2, 3, 4)$ are equal to zeros, under conditions $\tilde{h}_y = \tilde{i}_y = 0$, $h_zi_z \neq 0$, $\tilde{j}_y \neq 0$, or $\tilde{h}_y\tilde{i}_y \neq 0$, $h_z = i_z = 0$, $j_z \neq 0$, or $\tilde{h}_y\tilde{i}_yh_zi_z \neq 0$, $\tilde{h}_yj_z \neq \tilde{j}_yh_z$, where $x = 2, y = 3, z = 4$, or $x = 2, y = 4, z = 3$, we can turn $\mathcal{G}_j$ into canonical form. □

Theorem 4.7. *If e_p, f_p, g_p $(p = 2,3,4)$ are all nonzero, $\mathcal{G}_j$ can be turned into canonical form under the following conditions:*

$$e_x e_y e_z f_x f_y f_z g_x g_y g_z \neq 0,\ \tilde{h}_y = \tilde{i}_y = 0,\ \tilde{h}_z = \tilde{i}_z \neq 0,\ \tilde{j}_y \neq 0;$$
$$(e_x e_y e_z f_x f_y f_z g_x g_y g_z \neq 0,\ \tilde{h}_y \tilde{i}_y \tilde{h}_z \tilde{i}_z \neq 0,\ \tilde{j}_y \tilde{h}_z \neq \tilde{h}_y \tilde{j}_z);$$
$$where\ x, y, z \in \{2,3,4\}\ and\ x \neq y \neq z.$$

Proof. According to the second conclusion of Property 2.1, if e_p, f_p, g_p ($p = 2,3,4$) are all nonzero, the vectors (e_p, f_p, g_p) $(p = 2,3,4)$ being proportional. Then we can turn $e_3, e_4, f_3, f_4, g_3, g_4$ into zeros due to the vectors (e_p, f_p, g_p) $(p = 2,3,4)$ are proportional. Then we obtain the following for the last three slices of $\mathcal{G}_j$:

$$\left[\begin{array}{cccc|cccc|cccc} 0 & e_2 & h_2 & j_2 & 0 & 0 & \tilde{h}_3 & \tilde{j}_3 & 0 & 0 & \tilde{h}_4 & \tilde{j}_4 \\ 0 & 0 & f_2 & i_2 & 0 & 0 & 0 & \tilde{i}_3 & 0 & 0 & 0 & \tilde{i}_4 \\ 0 & 0 & 0 & g_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where $\tilde{h}_3 = h_3 - \alpha h_2$, $\tilde{j}_3 = j_3 - \alpha j_2$, $\tilde{h}_4 = h_4 - \beta h_2$, $\tilde{j}_4 = j_4 - \beta j_2$ and $\alpha = e_3/e_2$, $\beta = e_4/e_2$.

Now we show that equality $e_2\tilde{i}_3 = g_2\tilde{h}_3$ and $e_2\tilde{i}_4 = g_2\tilde{h}_4$ holds. Firstly, we write $e_3 = \alpha e_2, g_3 = \alpha g_2, e_4 = \beta e_2, g_4 = \beta g_2$. Next, by substituting them into the first two equations of (9), we obtain $e_2\tilde{i}_3 = g_2\tilde{h}_3$ and $e_2\tilde{i}_4 = g_2\tilde{h}_4$. Because $e_2 g_2 \neq 0$, then $\tilde{i}_3$ and $\tilde{h}_3$ are zero or nonzero at the same time, $\tilde{i}_4$ and $\tilde{h}_4$ are zero or nonzero at the same time. This yields four possibilities. These four situation can be discussed just as the situation that $e_2 e_3 e_4 f_2 f_3 f_4 \neq 0$, the remaining three of $e_p, f_p, g_p (p = 2,3,4)$ are equal to zeros.

Consequently, if $e_x f_x g_x e_y f_y, g_y e_z f_z g_z$ are nonzero, under conditions $\tilde{h}_y = \tilde{i}_y = 0, \tilde{h}_z = \tilde{i}_z \neq 0, \tilde{j}_y \neq 0$ or $\tilde{h}_y \tilde{i}_y \tilde{h}_z \tilde{i}_z \neq 0, \tilde{j}_y \tilde{h}_z \neq \tilde{h}_y \tilde{j}_z$, we can turn $\mathcal{G}_j$ into canonical form. □

5 Concluding remarks

In this paper, we have studied under what conditions we can turn $\mathcal{G}_j$ into canonical form (6) if some of the upper triangular entries of the last three slices of $\mathcal{G}_j$ are zeros. In addition, we have shown how to turn $\mathcal{G}_j$ into canonical form under these conditions. In the future, it will be interesting to explore the connection between our three order generalization of the Jordan canonical form and eigenvectors for three order arrays.

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