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Nontrivial periodic solutions to delay difference equations via Morse theory

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Abstract: In this paper some sufficient conditions are obtained to guarantee the existence of nontrivial $4T+2$ periodic solutions of asymptotically linear delay difference equations. The approach used is based on Morse theory.

Keywords: Nontrivial periodic solution, Delay difference equation, Morse theory

MSC: 39A11

1 Introduction

In the present paper we are concerned with the existence of periodic solutions to the system of delay difference equations

$$\Delta x(t) = -f(x(t-T)), \tag{1}$$

where $x \in \mathbf{R}^n$, $\Delta x(t) = x(t+1) - x(t)$, $f \in C(\mathbf{R}^n, \mathbf{R}^n)$ and T is a given positive integer.

In general, (1) may be regarded as a discrete analog of the following differential equation

$$\frac{dx}{dt} = -f(x(t-r)). \tag{2}$$

So far, there have been various approaches developed to the existence of the periodic solutions for delay differential equations since the first study [7] in 1962. As to (2), when $n = 1$, [8] introduced the Yorke-Kaplan's technique in 1974 to study the existence problem of periodic solutions of

$$\frac{dx}{dt} = -f(x(t-1)). \tag{3}$$

They obtained that (3) had 4 periodic solutions under assumptions
(i) $f \in C(\mathbf{R}, \mathbf{R})$ is odd;
(ii) $xf(x) > 0$.

In 2005, by the critical point theory and pseudo-index, Guo and Yu [6] obtained multiplicity results for $4r$ periodic solutions of (2) when $x \in \mathbf{R}^n, f \in C(\mathbf{R}^n, \mathbf{R})$. To our best knowledge, it is the first time that the existence

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of periodic solutions to systems of delay differential equations is dealt with by using variational method. In addition to that, there are many excellent works dealing with (2) by variational method, for example [13–15] and references therein.

It is known that the discrete analogs of differential equations represent the discrete counterpart of corresponding differential equations, and are usually studied in connection with numerical analysis. They occur widely in numerous settings and forms, both in mathematics itself and in its applications to computing, statistics, electrical circuit analysis, biology, dynamical systems, economics and other fields, monograph [1] gives some examples. As to (3), the discrete analog is

$$\Delta x(t) = -f(x(t-1)). \tag{4}$$

Usually, we try to look for 4 periodic nontrivial solutions which satisfy $x(t-2) = -x(t)$ of (4) under assumptions (i) and (ii). However, the answer is negative because (4) has no nontrivial 4 periodic solution at all. In [5], authors give an example

$$\Delta x(t) = -x^3(t-1) \tag{5}$$

and prove that (5) has no nontrivial 4 periodic solution. By the example, we find that there may be many differences between solutions of differential equations and solutions of corresponding difference equations. Given another more classical example, solutions of classical logistic model are simple, whereas its discrete analog difference model has chaotic solutions.

Guo [4, 5] who studied delay difference equations by critical point theory [6, 16–20]. Critical point

On the other hand, since [12] studied (1) when $n = 1$ for the first time, there have been few authors besides Guo [4, 5] who studied delay difference equations by critical point theory [6, 16–20]. Critical point theory is a powerful tool to establish sufficient conditions on the existence of periodic solutions of difference equations. Based on above reasons, our purpose in this paper is to consider the existence of periodic solutions to problem (1). By using Morse theory, we get some existence results on the system (1). To the best of our knowledge, it is the first time that the existence of periodic solutions to systems of delay difference equations is dealt with using Morse theory.

We denote by $\mathbf{R}$, $\mathbf{Z}$ the sets of real numbers and integers, respectively. $\mathbf{R}^n$ is the real space with dimension n, and $[a, b]$ stands for the discrete interval $\{a, a+1, \cdots, b\}$ if $a \le b$ and $a, b \in \mathbf{Z}$.

Throughout this paper we assume that the following (f_1)-(f_3) are satisfied.

(f_1) $f \in C^1(\mathbf{R}^n, \mathbf{R}^n)$ is odd, i.e., for any $x \in \mathbf{R}^n$, $f(-x) = -f(x)$.

(f_2) There exists a continuously differentiable function F, such that the gradient of F is f, i.e., for any $x \in \mathbf{R}^n$, $\nabla_x F(x) = f(x)$ and $F(0) = 0$.

(f_3) There exist real symmetric $n \times n$ matrices A and B such that

$$f(x) = Ax + o(|x|) \qquad \text{as} \quad |x| \longrightarrow \infty, \tag{6}$$

$$f(x) = Bx + o(|x|) \qquad \text{as} \quad |x| \longrightarrow 0, \tag{7}$$

that is, (1) is asymptotically linear at both infinity and origin.

Denote $G_\infty(x) = F(x) - \frac{1}{2}(Ax, x)$ and $G_0(x) = F(x) - \frac{1}{2}(Bx, x)$ respectively, we need further assumptions, which will be employed to determine the critical groups at infinity and at origin respectively.

(f_∞^+) $(G'_\infty(x), x) \ge c_1|x|^{s+1}$, $|G'_\infty(x)| \le c_2|x|^s$ for $x \in \mathbf{R}^n$ with $|x| \ge R$, where constants R, c_1, $c_2 > 0$ and $0 < s < 1$.

(f_∞^-) $(G'_\infty(x), x) \le 0$, $|(G'_\infty(x), x)| \ge c_1|x|^{s+1}$ and $|G'_\infty(x)| \le c_2|x|^s$ for $x \in \mathbf{R}^n$ with $|x| \ge R$, where constants R, c_1, $c_2 > 0$ and $0 < s < 1$.

(f_0^+) $G_0(x) \ge 0$ for $x \in \mathbf{R}^n$ with $|x| \le \varrho$, where $\varrho > 0$ is a constant.

(f_0^-) $G_0(x) \le 0$ for $x \in \mathbf{R}^n$ with $|x| \le \varrho$, where $\varrho > 0$ is a constant.

Remark 1.1. *It is easy to see that* (f_∞^+) *and* (f_∞^-) *imply* (6) *in* (f_3).

Similarly to the argument in [6], for given $n \times n$ real symmetric matrices A, B and integer $k \in [0, T]$, we set

$$N_k = \{\text{the number of negative eigenvalues of} \quad A + 2 \cdot (-1)^k \cdot \sin \frac{(2k+1)\pi}{4T+2} \cdot I\},$$

$$\bar{N}_k = \{\text{the number of nonpositive eigenvalues of} \quad A + 2 \cdot (-1)^k \cdot \sin \frac{(2k+1)\pi}{4T+2} \cdot I\},$$

and

$$\nu(A, B) = \sum_{k=0}^{T} [N_k(A) - N_k(B)],$$

$$\nu_1(A, B) = \sum_{k=0}^{T} [\bar{N}_k(A) - N_k(B)],$$

$$\nu_2(A, B) = \sum_{k=0}^{T} [\bar{N}_k(A) - \bar{N}_k(B)],$$

where I is the $n \times n$ identity matrix.

Now let us state our main results.

Theorem 1.2. *Suppose (f_1)-(f_3) hold and that f is C^1-differentiable near the origin $\boldsymbol{O} \in \mathbf{R}^n$. If $\nu_1(B, B) = 0$, then* (1) *has a nontrivial $4T + 2$ periodic solution x which satisfies $x(t + 2T + 1) = -x(t)$ provided one of the following conditions holds:*
(i) (f_∞^+) *and* $\nu(A, B) \neq 0$*;*
(ii) (f_∞^-) *and* $\nu_1(A, B) \neq 0$.

Theorem 1.3. *Suppose (f_1)-(f_3) hold and that f is C^1-differentiable near the origin $\boldsymbol{O} \in \mathbf{R}^n$. If $\nu_1(B, B) > 0$, then* (1) *has a nontrivial $4T + 2$ periodic solution x which satisfies $x(t + 2T + 1) = -x(t)$ provided one of the following conditions holds:*
(i) (f_∞^+), (f_0^+) *and* $\nu(A, B) \neq 0$*;*
(ii) (f_∞^+), (f_0^-) *and* $\nu_1(B, A) \neq 0$*;*
(iii) (f_∞^-), (f_0^+) *and* $\nu_1(A, B) \neq 0$*;*
(iv) (f_∞^-), (f_0^-) *and* $\nu_2(A, B) \neq 0$.

This paper is divided into four parts. In Section 2, we establish the variational framework associated with (1) and transfer the problem on the existence of periodic solutions of (1) into the existence of critical points of the corresponding functional defined on a suitable Hilbert space. In Section 3, we summarize some basic knowledge on Morse theory which will be used to prove our main results. Also some preliminary results are obtained in this section. The detailed proofs of main results are presented in Section 4.

2 Variational structure

In this section we establish a variational structure which enables us to reduce the existence of $4T+2$ nontrivial periodic solutions of (1) to the existence of critical points of corresponding functional defined on some appropriate function space.

First of all, we recall some notations and preliminary results. Let

$$S = \{x = \{x(t)\}_{t \in \mathbf{Z}} | x = (\cdots, x(-t), \cdots, x(-1), x(0), x(1), \cdots, x(t), \cdots), x(t) \in \mathbf{R}^n\},$$

where n is a given positive integer. For some given integer $T > 0$, E is defined as a subspace of S by

$$E = \{x = \{x(t)\} \in S | x(t + 2T + 1) = -x(t), \quad t \in \mathbf{Z}\}$$

and equipped with the inner product as

$$< x, y >= \sum_{t=1}^{2T+1} (x(t), y(t)), \qquad \forall x, y \in E,$$

then the induced norm is

$$\|x\| = \left(\sum_{t=1}^{2T+1} |x(t)|^2 \right)^{1/2}, \qquad \forall x \in E,$$

where $(\cdot, \cdot)$ and $|\cdot|$ denote the inner product and norm in $\mathbf{R}^n$. It follows that $(E, < \cdot, \cdot >)$ is a Hilbert space, which can be identified with $\mathbf{R}^{(2T+1)n}$.

Define a functional $J : E \to \mathbf{R}$ by

$$J(x) = \sum_{t=1}^{2T+1} x(t+T) \cdot x(t) - \sum_{t=1}^{2T+1} F(x(t)) \qquad \forall x \in E, \tag{8}$$

then $J \in C^1(E, \mathbf{R})$ and if $x \in E$ is a critical point of J, i.e. $J'(x) = 0$, if and only if

$$\frac{\partial J(x)}{\partial x_l(t)} = 0$$

holds for all $t \in [1, 2T+1]$, $l \in [1, n]$. By the same way of [5], we have $x \in E$ and x is a critical point of J when it is a periodic solution of

$$x(t+T) + x(t-T) - f(x(t)) = 0. \tag{9}$$

Together with $x \in E$, $x(t-T) = -x(t+T+1)$, (9) changes into

$$\Delta x(t+T) = -f(x(t)).$$

Since f is odd, we can show that the critical points of J in E are the $4T+2$ periodic solutions of (1). For details, the reader is referred to [12].

We define an operator $L : E \to E$ as

$$(Lx)(t) = x(t+T) + x(t-T), \quad t = 1, 2, \cdots. \tag{10}$$

It is easy to check that L is a bounded linear operator on E. For any $x, y \in E$, by the periodicity of x, y, we have

$$\begin{aligned} < Lx, y >= & \sum_{t=1}^{2T+1} (x(t+T) + x(t-T), y(t)) \\ = & \sum_{t=1}^{2T+1} (x(t+T), y(t)) + \sum_{t=1}^{2T+1} (x(t-T), y(t)) \\ = & \sum_{t=1}^{2T+1} (x(t), y(t-T)) + \sum_{t=1}^{2T+1} (x(t), y(t+T)) \\ = & \sum_{t=1}^{2T+1} (x(t), y(t+T) + y(t-T)) \\ = & < x, Ly > . \end{aligned}$$

It follows that L is self-adjoint.

Define a map

$$\Phi(x) = -\sum_{t=1}^{2T+1} F(x(t)), \qquad \forall x \in E, \tag{11}$$

then $\Phi \in C^1(E, \mathbf{R})$ and J can be rewritten as

$$J(x) = \frac{1}{2} < Lx, x > + \Phi(x), \qquad \forall x \in E. \tag{12}$$

Consider the eigenvalue problem of

$$\begin{cases} x(t+T)+x(t-T)=\lambda x(t), \\ x(t+2T+1)=-x(t). \end{cases} \tag{13}$$

By direct computation, we get

$$\lambda_k = 2\cdot(-1)^k \cos\frac{T-k}{2T+1}\pi, \quad k=0,1,\cdots,T,$$

are eigenvalues of (13). It is obvious that $0 \notin \sigma(L)$, where $\sigma(L)$ is the spectrum of L. Furthermore, when $k=T$, $\lambda_T = 2\cdot(-1)^T$ is an n-multiple eigenvalue of (13) and the corresponding eigenvector is

$$\eta_T = (-1,1,-1,1,\cdots,-1,1)^T;$$

when $0 \le k \le T-1$, $\lambda_k = 2\cdot(-1)^k \cos\frac{T-k}{2T+1}\pi$ is a 2n-multiple eigenvalue of (13) and the corresponding eigenvectors are

$$\eta_k^{(c)} = (\cos\frac{1}{2T+1}t\pi, \cos\frac{3}{2T+1}t\pi, \cdots, \cos\frac{2k+1}{2T+1}t\pi),$$
$$\eta_k^{(s)} = (\sin\frac{1}{2T+1}t\pi, \sin\frac{3}{2T+1}t\pi, \cdots, \sin\frac{2k+1}{2T+1}t\pi).$$

Then for any $x \in E$, x can be expressed as

$$x(t) = a_T\eta_T + \sum_{k=0}^{T-1}(a_k\eta_k^{(c)} + b_k\eta_k^{(s)}) = \sum_{k=0}^{T}(a_k\cos\frac{2k+1}{2T+1}t\pi + b_k\sin\frac{2k+1}{2T+1}t\pi), \tag{14}$$

where a_k, b_k $(0 \le k \ge T)$ are constant vectors.

For later use, we need the following lemma.

Lemma 2.1. *For any $x(j)>0$, $y(j)>0$, $j\in[1,k]$, $k\in\mathbf{Z}$,*

$$\sum_{j=1}^{k} x(j)y(j) \le \left(\sum_{j=1}^{k} x^r(j)\right)^{\frac{1}{r}} \cdot \left(\sum_{j=1}^{k} y^s(j)\right)^{\frac{1}{s}},$$

where $r>1$, $s>1$ and $\frac{1}{r}+\frac{1}{s}=1$.

For any $r>1$, by Lemma 2.1, we can define another norm on E as

$$\|x\|_r = \left(\sum_{t=1}^{2T+1}|x(t)|^r\right)^{1/r}, \qquad \forall x\in E.$$

Obviously, $\|x\| = \|x\|_2$ and there exist constants $c_4 \ge c_3 > 0$ such that

$$c_3\|x\|_r \le \|x\| \le c_4\|x\|_r, \qquad \forall x \in E. \tag{15}$$

3 Some preparatory results

In order to obtain critical points of functional J via Morse theory, we will state some basic facts and some preparatory results which will be used in proofs of our main results.

First, let us recall the definition of Palais-Smale condition.

Let X be a real Banach space, $I \in C(X,\mathbf{R})$. I is a continuously Fréchet differentiable functional defined on X. I is said to satisfy Palais-Smale condition (P.S. for short), if any sequence $\{x(t)\} \subset X$ for which $\{I(x)\}$ is bounded and $I'(x) \to 0$ $(t\to\infty)$ possesses a convergent subsequence in X.

Write $\kappa = \{x \in E | J'(x) = 0\}$. As in [2] and [9], we will work on the following framework under which the qth critical group of J at infinity $C_q(J,\infty)$ can be described precisely, here $q \in \mathbf{Z}$.

(A_∞) $J(x) = \frac{1}{2} < Lx, x > +\Phi(x)$, where $L : E \to E$ is a self-adjoint operator such that 0 is isolated in the spectrum of L. The map $\Phi \in C^1(E, \mathbf{R})$ satisfies $\Phi'(x) = o(\|x\|)$ as $\|x\| \to \infty$. Φ and Φ' map bounded sets into bounded sets. $J(\kappa)$ is bounded from below and J satisfies $(PS)_c$ for $c \ll 0$.

Let (A_∞) hold. Set $V = \mathrm{Ker}(L)$ and $W = V^\perp$. One can split W as $W^+ \oplus W^-$ such that $L|_{W^+}$ is positive definite and $L|_{W^-}$ is negative definite. Denote by $\mu = \dim W^-$, $\nu = \dim V$, the Morse index and the nullity of J at infinity, respectively.

In order to compute the critical group of J at infinity we need the following angle condition at infinity which was built by Bratsch and Li [2]. Here the given angle condition have been made some improvement on [2]. We refer to [3] and [11].

Proposition 3.1. *Let J satisfy (A_∞). Then:*
(i) $C_q(J, \infty) \cong \delta_{q,\mu}\mathbf{Z}$ provided J satisfies the angle condition at infinity:
(AC^+_∞) There exist $M > 0$, $\alpha \in (0, 1)$ such that $< J'(x), v >\geq 0$ for any

$$x = v + w \in E = V \oplus W, \quad \text{with} \quad \|x\| \geq M, \quad \|w\| \leq \alpha\|x\|.$$

(ii) $C_q(J, \infty) \cong \delta_{q,\mu+\nu}\mathbf{Z}$ provided J satisfies the angle condition at infinity:
(AC^-_∞) There exist $M > 0$, $\alpha \in (0, 1)$ such that $< J'(x), v >\leq 0$ for any

$$x = v + w \in E = V \oplus W, \quad \text{with} \quad \|x\| \geq M, \quad \|w\| \leq \alpha\|x\|.$$

When Hilbert space $E = E_0^+ \oplus E_0^-$. One can split E_0^- as $W_0^- \oplus W_0^0$ such that $L|_{W_0^0}$ is zero and $L|_{W_0^-}$ is negative definite. Denote by $\mu_0 = \dim W_0^-$, $\nu_0 = \dim W_0^0$, the Morse index and the nullity of J at 0 respectively. Su [10] gives the following proposition which can be used to compute the critical group of J at origin.

Proposition 3.2. *Let $J \in C^2(E, \mathbf{R})$ satisfy P.S. and $k = \dim E_0^-$. If J has a local linking at 0 corresponding to the spilt $E = E_0^+ \oplus E_0^-$, i.e. there exists $\rho > 0$ small enough such that*

$$J(x) \leq J(0), \quad x \in E^-; \qquad J(x) > J(0), \quad x \in E^+, \quad 0 < \|x\| \leq \rho.$$

Then

$$C_q(J, 0) \cong \delta_{q,k}\mathbf{Z}, \qquad k = \mu_0 \quad \text{or} \quad k = \mu_0 + \nu_0.$$

Recall that, in our setting,

$$J(x) = \frac{1}{2} < Lx, x > +\Phi(x), \qquad \forall x \in E.$$

Denote

$$G_\infty(x) = F(x) - \frac{1}{2}(Ax, x) \qquad \text{and} \qquad \Phi_\infty(x) = -\sum_{t=1}^{2T+1} G_\infty(x(t)),$$

$$G_0(x) = F(x) - \frac{1}{2}(Bx, x) \qquad \text{and} \qquad \Phi_0(x) = -\sum_{t=1}^{2T+1} G_0(x(t)).$$

Let L_A, L_B be bounded linear operators from E to E defined by the following forms

$$(L_A x)(t) = x(t + T) + x(t - T) - Ax(t), \tag{16}$$

$$(L_B x)(t) = x(t + T) + x(t - T) - Bx(t), \tag{17}$$

then J can be reformulated by

$$J(x) = \frac{1}{2} < L_A x, x > +\Phi_\infty(x), \tag{18}$$

or

$$J(x) = \frac{1}{2} < L_B x, x > +\Phi_0(x). \tag{19}$$

For an $n \times n$ symmetric matrix $D \in \mathbf{R}^{n\times n}$, we define linear operator $D : E \to E$ by extending the bilinear forms

$$< Dx, y >= \sum_{t=1}^{2T+1} (Dx(t), y(t)), \qquad x, y \in E.$$

Clearly, D is a bounded linear self-adjoint operator. Moreover, we can easily verify that D is compact on E because E is a finite dimensional Hilbert space. Now, we can draw a conclusion that $L_A = L - A$ and $L_B = L - B$ are bounded linear self-adjoint operators. Furthermore, $\phi_\infty = \phi - A$ and $\phi_0 = \phi - B$ are compact, where $\phi_* = \Phi'_*$ and $* = 0, \infty$.

Lemma 3.3. *Assume that f satisfies* (f_1)*-*(f_3)*. Then* $\Phi_\infty(0) = 0$ *and*

$$\lim_{\|x\|\to+\infty} \frac{\phi_\infty(x)}{\|x\|} = 0. \tag{20}$$

Proof. $\Phi_\infty(0) = 0$ follows by the definition of Φ_∞ and (f_1), (f_2). By (f_3), (i), for any $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$|f(x) - Ax| < \varepsilon|x| + C. \tag{21}$$

Note that

$$< \phi_\infty(x), h >=< (\phi - A)(x), h >= \sum_{t=1}^{2T+1} (f(x(t)) - Ax(t), h(t)), \quad \forall x, h \in E.$$

By Lemma 2.1, we get

$$\begin{aligned} | < \phi_\infty(x), h > | &\le \sum_{t=1}^{2T+1} |f(x(t)) - Ax(t)| \cdot |h(t)| \\ &\le \sum_{t=1}^{2T+1} [\varepsilon \cdot |x(t)| \cdot |h(t)| + C|h(t)|] \\ &\le \varepsilon\|x\| \cdot \|h\| + \sqrt{2T+1}C\|h\|, \end{aligned}$$

this yields

$$\|\phi_\infty(x)\| \le \varepsilon\|x\| + \sqrt{2T+1}C,$$

so

$$\lim_{\|x\|\to+\infty} \sup \frac{\phi_\infty(x)}{\|x\|} \le \varepsilon.$$

By the arbitrariness of ε, we show that $\lim_{\|x\|\to+\infty} \frac{\phi_\infty(x)}{\|x\|} = 0$. □

Lemma 3.4. *Let D be the self-adjoint operator defined by an n×n symmetric matrix D and* $m^+(L-D)$*,* $m^0(L-D)$ *and* $m^-(L-D)$ *denote the dimension of the subspaces of E where* $L - D$ *is positive definite, zero and negative definite respectively. Then*

$$m^-(L-D) = \sum_{k=0}^{T} N_k(D),$$

$$m^0(L-D) = \sum_{k=0}^{T} [\bar{N}_k(D) - N_k(D)],$$

$$m^+(L-D) = (2T+1)n - m^-(L-D) - m^0(L-D).$$

Proof. Consider the operator $L_D = L - D$, which is defined by

$$(L-D)x(t) = x(t+T) + x(t-T) - Dx(t), \qquad \forall x \in E.$$

Together with (14) and $x(t-T) = -x(t-T+2T+1) = -x(t+T+1)$, we get

$$L_D x(t) = \; x(t+T) + x(t-T) - Dx(t)$$

$$
\begin{aligned}
&= x(t+T) - x(t+T+1) - Dx(t) \\
&= \sum_{k=0}^{T}[a_k \cos\frac{2k+1}{2T+1}(t+T)\pi + b_k \sin\frac{2k+1}{2T+1}(t+T)\pi - a_k \cos\frac{2k+1}{2T+1}(t+T+1)\pi \\
&\quad -b_k \sin\frac{2k+1}{2T+1}(t+T+1)\pi - Da_k \cos\frac{2k+1}{2T+1}t\pi - Db_k \sin\frac{2k+1}{2T+1}t\pi] \\
&= \sum_{k=0}^{T}[(2\cdot(-1)^k \cdot \sin\frac{(2k+1)\pi}{4T+2}\cdot I - D)a_k \cos\frac{2k+1}{2T+1}t\pi \\
&\quad +(2\cdot(-1)^k \cdot \sin\frac{(2k+1)\pi}{4T+2}\cdot I - D)a_k \sin\frac{2k+1}{2T+1}t\pi].
\end{aligned}
$$

Consider the eigenvalue problem of operator L_D. Let

$$L_D x(t) = \lambda x(t),$$

where λ is a constant. Then for any $k \in [0, T]$, we have

$$[2\cdot(-1)^k \cdot \sin\frac{(2k+1)\pi}{4T+2}\cdot I - D]a_k = \lambda a_k$$

and

$$[2\cdot(-1)^k \cdot \sin\frac{(2k+1)\pi}{4T+2}\cdot I - D]b_k = \lambda b_k.$$

This implies that λ is an eigenvalue of operator L_D if and only if it is an eigenvalue of $2\cdot(-1)^k\cdot\sin\frac{(2k+1)\pi}{4T+2}\cdot I - D$ for some $k \in [0, T]$. It follows that the conclusions hold. □

4 Proofs of main results

With above preparations, we shall prove our main results in this section. In order to give proofs of our theorems, we need following lemmas.

Lemma 4.1. *Assume that f satisfies (f_1)-(f_3) and $(f_\infty^\pm)$. Then $J(x)$ satisfies P.S. condition.*

Proof. Denote the smallest eigenvalue of L_A is $\lambda_{\min}$. Since $L_A = L - A$ is compact perturbation of L and $0 \notin \sigma(L)$, we find $0 \notin \sigma(L_A)$, that is, $\lambda_{\min} > 0$. Then for any $x \in E$, we have

$$| < L_A x, x > | \geq \lambda_{\min}\|x\|. \tag{22}$$

Let $\{x^{(k)}\} \subset E$ be a PS sequence in E, i.e., there is a constant $M_1 > 0$ such that $|J(x^{(k)})| \leq M_1$ holds for any $k \in \mathbf{N}$ and $J'(x^{(k)}) \to 0$ as $k \to \infty$. Since E is a finite dimensional Hilbert space, here we only need to prove $\{x^{(k)}\}$ is bounded in E.

Since $J'(x^{(k)}) \to 0$ as $k \to \infty$, without generality, we can let $\|J'(x^{(k)})\| \leq 1$ when k is large enough. Write $x^{(k)} = u^{(k)} + v^{(k)}$, where $u^{(k)} \in Y$, $v^{(k)} \in Z$. Here Y, Z are subspaces of E where L_A is positive and negative definite respectively. For sufficiently large k, making use of (21) and (22), we have

$$
\begin{aligned}
\|x^{(k)}\| &= \|u^{(k)} - v^{(k)}\| \geq |J'(x^{(k)}), u^{(k)} - v^{(k)}| \\
&= | < L_A x^{(k)} - \phi_\infty(x^{(k)}), u^{(k)} - v^{(k)} > | \\
&\geq | < L_A x^{(k)}, u^{(k)} - v^{(k)} > | - \sum_{t=1}^{2T+1} |\Big((f-A)x^{(k)}(t), u^{(k)}(t) - v^{(k)}(t)\Big)| \\
&\geq \lambda_{\min}\|x^{(k)}\|\cdot\|u^{(k)} - v^{(k)}\| - \sum_{t=1}^{2T+1} |\Big(\varepsilon|x^{(k)}(t)| + C, u^{(k)}(t) - v^{(k)}(t)\Big)| \\
&\geq \lambda_{\min}\|x^{(k)}\|^2 - \varepsilon\|x^{(k)}\|\cdot\sum_{t=1}^{2T+1}|u^{(k)}(t) - v^{(k)}(t)| - C\sum_{t=1}^{2T+1}|u^{(k)}(t) - v^{(k)}(t)|
\end{aligned}
$$

$$\geq \lambda_{\min}\|x^{(k)}\|^2 - \sqrt{2T+1}\varepsilon\|x^{(k)}\|^2 - \sqrt{2T+1}C\|x^{(k)}\|. \tag{23}$$

For $\lambda_{\min} > 0$, we can choose a sufficiently small $\varepsilon > 0$ such that $\lambda_{\min} > \sqrt{2T+1}\varepsilon$. Then from (23) we get $\{x^{(k)}\}$ is bounded. □

In order to prove our main results by Proposition 3.1, we are in the position to give the verification of these angle conditions at infinity.

Lemma 4.2. *Let f satisfy (f_1)-(f_3) and the functional $J(x)$ be defined by* (12), *we have the following conclusion: if (f_∞^+) (or (f_∞^-)) holds, then J satisfies the angle condition (SAC_∞^-) (or (SAC_∞^+)) at infinity; i.e., there exist $M > 0$, $\alpha \in (0,1)$ such that*

$$< J'(x), \frac{v}{\|v\|} >< 0, \qquad (or \quad < J'(x), \frac{v}{\|v\|} >> 0)$$

for any $x = v + w \in E = V \oplus W$ with $\|x\| \geq M$, $\|w\| \leq \alpha\|x\|$, where $V = Ker(L_A)$ and $W = V^\perp$.

Proof. Set

$$\Omega(M,\varepsilon) = \{x = v + w \in E = V \oplus W \| \|x\| \geq M, \|w\| \leq \varepsilon\|x\|\}, \tag{24}$$

where $M > 0$ and $\varepsilon \in (0,1)$ will be chosen below. For any $x \in \Omega(M,\varepsilon)$, we have

$$\|v\| \geq \sqrt{1-\varepsilon^2}\|x\|, \ \|w\| \leq \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\|v\|. \tag{25}$$

We only prove the case that the functional $J(x)$ satisfies the angle condition (SAC_∞^-) under condition (f_∞^+), the other case is similar and here the proof is omitted.

$$\begin{aligned}
\sum_{t=1}^{2T+1}(G'_\infty(x(t)), v(t)) &= \sum_{t=1}^{2T+1}(G'_\infty(x(t)), x(t) - w(t)) \\
&= \sum_{t=1}^{2T+1}(G'_\infty(x(t)), x(t)) - \sum_{t=1}^{2T+1}(G'_\infty(x(t)), w(t)) \\
&\geq \sum_{t=1}^{2T+1} c_1|x|^{1+s} - \sum_{t=1}^{2T+1}|G'_\infty(x(t)| \cdot |w(t)| \\
&\geq c_1\|x\|_{1+s}^{1+s} - c_2\sum_{t=1}^{2T+1}|x(t)|^s \cdot |w(t)| \\
&\geq c_1\|x\|_{1+s}^{1+s} - c_2(2T+1)^{\frac{1-s}{2}} \cdot \|x\|^s \cdot \|w\| \\
&\geq c_1\|x\|_{1+s}^{1+s} - c_2(2T+1)^{\frac{1-s}{2}} \cdot \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\|v\| \cdot \|x\|^s.
\end{aligned}$$

From (15), for any $x \in E$, there exist constants $0 < r_1 \leq r_2$ such that

$$r_1\|x\|_{1+s}^{1+s} \leq \|x\|^{1+s} \leq r_2\|x\|_{1+s}^{1+s},$$

then

$$\sum_{t=1}^{2T+1}(G'_\infty(x(t)), v(t)) \geq \frac{c_1}{r_2}\|x\|^{1+s} - c_2 \cdot (2T+1)^{\frac{1-s}{2}} \cdot \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\|v\| \cdot \|x\|^s.$$

Let $\|v\| \leq 1$, we have

$$\sum_{t=1}^{2T+1}(G'_\infty(x(t)), v(t)) \geq \frac{c_1}{r_2}\|x\|^{1+s} \cdot \|v\| - c_2 \cdot (2T+1)^{\frac{1-s}{2}} \cdot \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\|v\| \cdot \|x\|^s,$$

it follows

$$\sum_{t=1}^{2T+1}\left(G'_\infty(x(t)), \frac{v(t)}{\|v\|}\right) \geq \left(\frac{c_1}{r_2}\|x\| - c_2 \cdot (2T+1)^{\frac{1-s}{2}} \cdot \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right) \cdot \|x\|^s.$$

Denote $\beta = \frac{c_2\cdot(2T+1)^{\frac{1-s}{2}}\cdot\varepsilon\cdot r_2}{c_1\cdot\sqrt{1-\varepsilon^2}}$, then $\sum_{t=1}^{2T+1}\left(G'_\infty(x(t)), \frac{v(t)}{\|v\|}\right) > 0$ is true when $\|v\| \le 1$ and $\|x\| > \beta$ hold. From (25), we get $\|x\| \le \frac{\|v\|}{\sqrt{1-\varepsilon^2}} \le \frac{1}{\sqrt{1-\varepsilon^2}}$, then we can choose suitable c_1, c_2 and r_2 such that $\varepsilon = \frac{c_1}{c_2(2T+1)^{\frac{1-s}{2}} r_2} \in (0,1)$ and $M \in [\frac{c_2 r_2 \varepsilon (2T+1)^{\frac{1-s}{2}}}{c_1\sqrt{1-\varepsilon^2}}, \frac{1}{\sqrt{1-\varepsilon^2}}]$, it is clear that $M > 0$. Now fixing $M > 0$, $0 < \varepsilon < 1$, we get

$$\sum_{t=1}^{2T+1}\left(G'_\infty(x(t)), \frac{v(t)}{\|v\|}\right) > 0 \tag{26}$$

for any $x = v + w \in \Omega(M, \varepsilon)$ with $\|x\| \ge M$ and $\|w\| \le \varepsilon\|x\|$. Since for any $x \in E$,

$$\begin{aligned} J(x) &= \frac{1}{2} < Lx, x > - \sum_{t=1}^{2T+1} F(x(t)) \\ &= \frac{1}{2} < (L+A)x, x > - \sum_{t=1}^{2T+1} G_\infty(x(t)), \end{aligned}$$

we have

$$< J'(x), \frac{v}{\|v\|} >= - \sum_{t=1}^{2T+1}\left(G'_\infty(x(t)), \frac{v(t)}{\|v\|}\right).$$

By the above argument we get easily that J satisfies the angle condition (SAC_∞^-) at infinity if we take $\alpha = \varepsilon \in (0,1)$. □

Denote $W_0^0 = \operatorname{Ker}(L_B)$, $W_0 = (W_0^0)^\perp$, then $W_0 = W_0^+ \oplus W_0^-$, and $W_0^\pm$ is an invariant subspace according to operator L_B, where L_B is positive definite and negative definite, respectively. Therefore, E can be expressed as

$$E = W_0^0 \oplus W_0^+ \oplus W_0^-. \tag{27}$$

What's more, there exists a constant $\delta > 0$ such that

$$\begin{cases} < L_B x, x > \ge \delta\|x\|^2, & x \in W_0^+, \\ < L_B x, x > \le -\delta\|x\|^2, & x \in W_0^-, \\ < L_B x, x >= 0, & x \in W_0^0. \end{cases} \tag{28}$$

To compute the critical group of J at origin, we are in position to prove that J has a local linking at origin.

Lemma 4.3. *Let f satisfy (f_1)-(f_3) and $(f_0^\pm)$, then J has a local linking at 0 corresponding to the spilt $E = W_0^+ \oplus E_0^-$, where $E_0^- = W_0^- \oplus W_0^0$ (according to condition (f_0^+)) or $E_0^- = W_0^-$ (according to condition (f_0^-)).*

Proof. By (27), given $x \in E$, we can write $x = u + v + w$, where $u \in W_0^+$, $v \in W_0^-$ and $w \in W_0^0$.

First, we prove it under the condition (f_0^+).

Making use of (f_3) (ii) and (28), there exists a constant $\rho \in (0, \varrho]$ such that

$$|G_0(x)| \le \frac{\delta}{3}|x|^2, \qquad |x| \le \rho. \tag{29}$$

On one side, by (19), (28) and (29), we have

$$J(x) \ge \frac{\delta}{2}\|x\|^2 - \frac{\delta}{3}\|x\|^2 > 0. \tag{30}$$

On the other side, since $|x| \le \rho \le \varrho$, when $x \in W_0^- \oplus W_0^0$ and $\|x\| \le \rho$, write $x = v + w$ where $v \in W_0^-$ and $w \in W_0^0$, we get

$$J(x) \le -\frac{1}{2}\delta\|v\|^2 - \sum_{t=1}^{2T+1} G_0(x(t)) \le 0. \tag{31}$$

Together with (30), (31) and $J(0) = 0$, we complete the proof that J has a local linking at 0 under the condition (f_0^+).

Following, we consider the second case with the condition (f_0^-). Similarly, we have

$$J(x) \le -\frac{\delta}{2}\|x\|^2 + \frac{\delta}{3}\|x\|^2 \le 0, \quad \text{as} \quad x \in W_0^-, \quad \|x\| \le \rho. \tag{32}$$

and

$$J(x) \ge \frac{1}{2}\delta\|u\|^2 - \sum_{t=1}^{2T+1} G_0(x(t)), \quad \text{as} \quad x \in W_0^+ \oplus W_0^0, \quad 0 < \|x\| \le \rho. \tag{33}$$

If $u \ne 0$, then (33) means $J(x) > 0$. If $u = 0$, i.e. $x \in W_0^0 \setminus \{0\}$ and $\|x\| \le \rho$, then

$$J(x) = -\sum_{t=1}^{2T+1} G_0(x(t)) \ge 0. \tag{34}$$

In fact we can prove that $J(x) > 0$. Since if $J(x) = 0$ in (34) is true, (1) has infinite solutions. That is, J has a local linking at 0 under the condition (f_0^-). This completes the proof. □

Proof of Theorem 1.2. We will only present the proof for case (i) and the proof for case (ii) is similar.

By Lemma 3.3, we know that Φ_∞ is C^1, $\Phi_\infty(0) = 0$ and

$$\|\Phi'_\infty(x)\| = \|\phi_\infty(x)\| = o(\|x\|), \qquad \text{as} \quad \|x\| \to \infty, \quad x \in E.$$

Combining with Lemma 4.1, we have that J satisfies (A_∞). It follows by Lemma 3.4, Proposition 3.1 and Lemma 4.2 that

$$C_q(J, \infty) \cong \delta_{q,\mu}\mathbf{Z}, \tag{35}$$

where $\mu = m^-(L - A)$.

Since the injection of E into E, with its norm $\|\cdot\|_\infty$, is continuous and f is C^1 differentiable near $\mathbf{0} \in \mathbf{R}^n$, we know that J is C^2 differentiable near the origin $0 \in E$. Further, we have

$$J''(0) = L - B.$$

Since $\nu_1(B, B) = 0$ implies that for every $k \in [0, T]$, $\bar{N}_k(B) - N_k(B) = 0$, we see that 0 in E is a non-degenerate critical point of J. Thus, for every $k \in [0, T]$, we have

$$C_q(J, 0) \cong \delta_{q,\mu^0}\mathbf{Z}, \tag{36}$$

where $\mu^0 = m^-(L - B)$. By the condition $\nu(A, B) \ne 0$, we see that

$$m^-(L - A) \ne m^-(L - B).$$

Thus $\mu \ne \mu^0$. It follows from (35) and (36) that J has different critical groups at infinity and at origin respectively, which implies that J has at least one nontrivial critical point $x \ne 0$, i.e., (1) has a nontrivial $4T + 2$ periodic solution $x(t)$ which satisfies $x(t + 2T + 1) = -x(t)$. □

Proof of Theorem 1.3. Similarly to the proof of Theorem 1.2, we can get (35). By Proposition 3.2, Lemmas 4.1 and 4.1, we have

$$C_q(J, 0) \cong \delta_{q,\mu^0}\mathbf{Z}.$$

Note that $\nu(A, B) \ne 0$ implies $\mu \ne \mu^0$ for every $k \in [0, T]$, we get

$$C_q(J, 0) \ncong C_q(J, \infty),$$

which implies that J has at least one nontrivial critical point $x \ne 0$. The proof is complete. □

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References

[1] Agarwal R.P., Difference Equations and Inequalities: Theory, Methods and Applications, Marcel Dekker, New York, 1992.

[2] Bartsch T., Li S.J., Critical point theory for asyptotically quadratic functionals and applications to problem with resonance, Nonlinear Anal., 1997, 28, 419-441.

[3] Chang K.C., Morse theory on Banach space and its applications to partial differential equations, Chin. Ann. Math. Ser. B, 1983, 4, 381-399.

[4] Guo L.F., Guo Z.M., Periodic solutions to first order superlinear delay difference equation, J. Guangzhou University, 2014, 13, 19-23.

[5] Guo Z.M., Guo L.F., The existence of periodic solutions to higher-order dimensional sublinear delay difference equation, J. Guangzhou University, 2014, 13, 7-12.

[6] Guo Z.M., Yu J.S., Multiplicity results for periodic solutions to delay differential equations via critical point theory, J. Differential Equations, 2005, 218, 15-35.

[7] Jones G.S., The existence of periodic solutions of $f'(x) = -af(x-1)[1+f(x)]$, J. Math. Anal. Appl., 1962, 5, 435-450.

[8] Kaplan J.L., Yorke J.A., Ordinary differential equations which yield periodic solution of delay equations, J. Math. Anal. Appl., 1974, 48, 317-324.

[9] Su J.B., Nontrivial periodic solutions for the asymptotically linear Hamiltonian systems with resonance at infinity, J. Differential Equations, 1998, 145, 252-273.

[10] Su J.B., Multiplicity results for asymptotically linear elliptic problems at resonance, J. Math. Anal. Appl., 2003, 278, 397-408.

[11] Wang Z.Q., Multiple solutions for indefinite functionals and applications to asymptotically linear problems, Acta Math. Sinica, 1989, 5, 101-113.

[12] Xing Q.P., Wang Q.R., Guo Z.M., Existence of periodic solutions to nonlinear diffference equations with delay, Acta Anal. Func. Appl., 2012, 14, 61-70.

[13] Yu J.S., Xiao H.F., Multiple periodic solutions with minimal period 4 of the delay differential equation $\dot{x}(t) = -f(t, x(t-1))$, J. Differential Equations, 2013, 254, 2158-2172.

[14] Yu J.S., A note on periodic solutions of the the delay differential equation $\dot{x}(t) = -f(t, x(t-1))$, Proc. Amer. Math. Soc., 2013, 14, 1281-1288.

[15] Zhang X.S., Meng Q., Nontrivial periodic solutions for delay differential systems via Morse theory, Nonlinear Anal., 2011, 74, 1960-1968.

[16] Zhou Z., Ma D.F., Multiplicity results of breathers for the discrete nonlinear Schrödinger equations with unbounded potentials, Sci. China Math., 2015, 58(4), 781-790.

[17] Zhou Z., Su M.T., Boundary value problems for $2n$-order ϕ_c-Laplacian difference equations containing both advance and retardation, Appl. Math. Lett., 2015, 41, 7-11.

[18] Zhou Z., Yu J.S., Homoclinic solutions in periodic nonlinear difference equations with superlinear nonlinearity, Acta Math. Sin. (Engl. Ser.), 2013, 29(9), 1809-1822.

[19] Zhou Z., Yu J.S., On the existence of homoclinic solutions of a class of discrete nonlinear periodic systems, J. Differential Equations, 2010, 249(5), 1199-1212.

[20] Zhou Z., Yu J.S., Chen Y.M., Homoclinic solutions in periodic difference equations with saturable nonlinearity, Sci. China Math., 2011, 54(1), 83-93.