Open Math. 2018; 16: 862–884 DE GRUYTER

Open Mathematics

Research Article

Xiaoyao Jia*, Xiaoquan Ding, and Juanjuan Gao

Random attractors for stochastic retarded reaction-diffusion equations with multiplicative white noise on unbounded domains

https://doi.org/10.1515/math-2018-0076
Received December 10, 2017; accepted June 12, 2018.

Abstract: In this paper we investigate the stochastic retarded reaction-diffusion equations with multiplicative white noise on unbounded domain $\mathbb{R}^n$ ($n \geq 2$). We first transform the retarded reaction-diffusion equations into the deterministic reaction-diffusion equations with random parameter by Ornstein-Uhlenbeck process. Next, we show the original equations generate the random dynamical systems, and prove the existence of random attractors by conjugation relation between two random dynamical systems. In this process, we use the cut-off technique to obtain the pullback asymptotic compactness.

Keywords: Stochastic reaction-diffusion system, Random attractor, Time-delay, Multiplicative white noise

MSC: 35B40, 35B41, 35K45, 35K57

1 Introduction

In this paper we investigate a class of stochastic partial differential equations, which are widely used in quantum field theory, statistical mechanics and financial mathematics. It is known that there are many dynamical systems, depending on both current and historical states. They are referred to as time-decay dynamical systems and can be described by retarded partial differential equations. Hence, it is very important to study the properties of retarded partial differential equations.

In this paper, we consider the asymptotic behavior of the solutions to the following stochastic retarded reaction-diffusion equation with multiplicative noise in the whole space $\mathbb{R}^n$ ($n \geq 2$):

$$du + (\lambda u - \Delta u)dt = \Big(F(x, u(t, x)) + G(x, u(t - h, x)) + g(x) \Big)\, dt + \epsilon u \circ dw. \tag{1}$$

Here λ is a positive constant; $h > 0$ is the delay time of the system; F and G are given functions satisfying certain conditions which will be given in Section 2; g is a given function defined on $\mathbb{R}^n$; w is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

***Corresponding Author: Xiaoyao Jia:** Mathematics and Statistics School, Henan University of Science and Technology, No.263 Kai-Yuan Road, Luo-Long District, Luoyang, Henan Province, 471023, China, E-mail: jiaxiaoyao@haust.edu.cn
Xiaoquan Ding: Mathematics and Statistics School, Henan University of Science and Technology, No.263 Kai-Yuan Road, Luo-Long District, Luoyang, Henan Province, 471023, China, E-mail: xqding@haust.edu.cn
Juanjuan Gao: Mathematics and Statistics School, Henan University of Science and Technology, No.263 Kai-Yuan Road, Luo-Long District, Luoyang, Henan Province, 471023, China, E-mail: gaojj@haust.edu.cn

the Borel σ-algebra $\mathcal{F}$ on Ω is generated by the compact open topology [2] and $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$; ϵ is a positive parameter; $\circ$ denotes the Stratonovich sense in the stochastic term. We identify $\omega(t)$ with $w_t(\omega)$, i.e. $w_t(\omega) = w(t,\omega) = \omega(t)$, $t \in \mathbb{R}$.

One of the most important problems for stochastic differential equations is to study the asymptotic behavior of the solutions. The asymptotic behavior of random dynamical systems can be described by its random attractors. The existence of attractors for partial differential equations has been studied by many authors, see [3, 12, 16, 17, 20–27, 29, 31, 32, 34] and the references therein. However, to the best of our knowledge, the studies of attractors for retarded partial differential equations are very few, especially for the stochastic retarded partial differential equations. The attractors for the deterministic retarded partial differential equations were investigated in [6, 7, 9, 33, 35], and random attractors for stochastic retarded partial differential equations were studied in [13, 14, 30, 36]. In [13], the authors studied the existence of random attractor for stochastic retarded reaction-diffusion equations with additive noise, while in [14, 30, 36], the authors studied the random attractors for stochastic retarded lattice dynamical systems. In this paper, we will study the existence of the random attractors for stochastic retarded reaction-diffusion equations with multiplicative white noise on unbounded domain $\mathbb{R}^n$. In the bounded domain case, the compactness of random absorbing set can be obtained by Sobolev embeddings theory. However, in the unbounded domain case, we can not get compactness in this way, because Sobolev embeddings are not compact. Hence, the cut-off technique will be used to obtain the compactness. This technique has been used by many authors (see [13, 21, 26, 27]).

The remainder of the paper is organized as follows. In section 2, we recall an important theorem which will be used to obtain the existence of random attractor, and transform the stochastic retarded partial differential equations to random ones by Ornstein-Uhlenbeck process. We will show that for each $\omega \in \Omega$, the random equation has a unique solution. In Section 3, we get some useful estimates for the solution of the random equation. At last, we get the existence of the random attractor.

2 Preliminaries and Random Dynamical Systems

In this section we introduce some notations and recall some basic knowledge about random attractors for random dynamical systems (RDS). The reader can refer to [2, 3, 10, 11, 17] for more details.

First, we introduce some notations which will be used in the following. Let $||\cdot||$ and $(\cdot,\cdot)$ be the norm and inner product in $L^2(\mathbb{R}^n)$. For fixed $h > 0$, let $\mathfrak{S} = C([-h,0], L^2(\mathbb{R}^n))$, and let the norm of $\mathfrak{S}$ be $||f||_{\mathfrak{S}} = \sup_{t\in[-h,0]} ||f(t)||$. Then, it is easy to see that $\mathfrak{S}$ is a Banach space. Let $(X, ||\cdot||_X)$ be a Banach space with Borel σ–algebra $\mathcal{B}(X)$, and let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. For $a \le b$, $t \in [a,b]$ and continuous function $u \in C([a-h,b], L^2(\mathbb{R}^d))$, define $u^t(s) = u(t+s)$ for $s \in [-h,0]$. From the definition, one can see that, for $t \in [a,b]$, $u^t \in \mathfrak{S}$.

Let $(X, ||\cdot||_X)$ be a Banach space with Borel σ–algebra $\mathcal{B}(X)$, and let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space.

Definition 2.1. *$(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system, if $\theta : \mathbb{R}\times\Omega \mapsto \Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F},\mathcal{F})$–measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s$ for all $s,t \in \mathbb{R}$ and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.*

Definition 2.2. *A stochastic process ϕ is called a continuous random dynamical system (RDS) over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$, if ϕ is $(\mathcal{B}([0,+\infty))\times\mathcal{F}\times\mathcal{B}(X),\mathcal{B}(X))$–neasurable, and for all $\omega \in \Omega$,*

(1) the mapping $\phi(t,\omega,\cdot) : X \mapsto X$, $x \mapsto \phi(t,\omega,x)$ is continuous for every $t \ge 0$;

(2) $\phi(0,\omega,\cdot)$ is the identity on X;

(3) $\phi(s+t,\omega,\cdot) = \phi(t,\theta_s\omega,\cdot)\circ\phi(s,\omega,\cdot)$ for all $s,t > 0$.

Definition 2.3. *(1) A random set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called bounded if there exists $x_0 \in X$ and a random variable $r(\omega) > 0$, such that*

$$B(\omega) \subset \{x \in X : \ ||x - x_0||_X \le r(\omega),\ x_0 \in X\}, \quad \text{for all} \quad \omega \in \Omega.$$

(2) A random set $\{B(\omega)\}_{\omega\in\Omega}$ is called a compact random set if $B(\omega)$ is compact for all $\omega \in \Omega$.
(3) A random set $\{B(\omega)\}_{\omega\in\Omega} \subset X$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$, if for $\mathbb{P} - a.e.\ \omega \in \Omega$,

$$\lim_{t\to+\infty} e^{-\gamma t} \sup_{x\in B(\theta_{-t}\omega)} ||x||_X = 0 \ \text{for all} \ \gamma > 0. \tag{2}$$

(4) A random variable $r(\omega)$ is said to be tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$, if for $\mathbb{P} - a.e.\ \omega \in \Omega$,

$$\lim_{t\to+\infty} e^{-\gamma t}|r(\theta_{-t}\omega)| = 0 \ \text{for all} \ \gamma > 0. \tag{3}$$

In the following, we assume that ϕ is a continuous RDS on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$, and $\mathscr{D}$ is a collection of random subsets of X.

Definition 2.4. *Let $\{K(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a random absorbing set for ϕ in $\mathscr{D}$ if for every $\{B(\omega)\} \in \mathscr{D}$ and $\mathbb{P} - a.e.\ \omega \in \Omega$, there exists $T(B, \omega) > 0$, such that*

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \ \text{for all} \ \ t \geq T(B, \omega). \tag{4}$$

Definition 2.5. *A random set $\{A(\omega)\}_{\omega\in\Omega}$ of X is called a $\mathscr{D}$-random attractor (or $\mathscr{D}$-pullback attractor) for ϕ, if the following conditions are satisfied:*
(1) $\{A(\omega)\}_{\omega\in\Omega}$ is compact, and $\omega \mapsto d(x, A(\omega))$ is measurable for every $x \in X$;
(2) $\{A(\omega)\}_{\omega\in\Omega}$ is invariant, that is,

$$\phi(t, \omega, A(\omega)) = A(\theta_t\omega) \ \forall t \geq 0. \tag{5}$$

(3) $\{A(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathscr{D}$, that is, for every $\{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D},\ \omega \in \Omega$,

$$\lim_{t\to+\infty} dist(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), A(\omega)) = 0, \tag{6}$$

where $dist(Y, Z) = \sup_{y\in Y} \inf_{z\in Z} ||y - z||_X$ is the Hausdorff semi-metrix ($Y \subseteq X$, $Z \subseteq X$).

Definition 2.6. *ϕ is said to be $\mathscr{D}$-pullback asymptotically compact in X, if for all $B \in \mathscr{D}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, $\left\{\phi(t_n, \theta_{-t_n}\omega, x_n)\right\}_{i=1}^{\infty}$ has a convergent subsequence in X, as $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$.*

At the end of this section, we refer to [3, 17] for the existence of random attractor for continuous RDS.

Proposition 2.7. *Let $\{K(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ be a random absorbing set for the continuous RDS ϕ in $\mathscr{D}$ and ϕ is $\mathscr{D}$-pullback asymptotically compact in X. Then ϕ has a unique $\mathscr{D}$-random attractor $\{A(\omega)\}_{\omega\in\Omega}$ which is given by*

$$A(\omega) = \cap_{\tau\geq 0}\overline{\cup_{t\geq\tau}\phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}. \tag{7}$$

In the rest of this paper we will assume that $\mathscr{D}$ is the collection of all tempered random subsets of $\mathfrak{S}$ and we will prove that the stochastic retarded reaction-diffusion equation has a $\mathscr{D}$-pullback random attractor.

In the remaining part of this section we show that there is a continuous random dynamical system generated by the following stochastic retarded reaction-diffusion equation on $\mathbb{R}^n$ with the multiplicative noise:

$$du + (\lambda u - \Delta u)dt = \Big(F(x, u(t, x)) + G(x, u(t - h, x)) + g(x) \Big)\, dt + \epsilon u \circ dw, \quad x \in \mathbb{R}^n,\ t > 0, \tag{8}$$

with the initial condition

$$u(t, x) = u_0(t, x), \quad x \in \mathbb{R}^n,\ t \in [-h, 0]. \tag{9}$$

Here g is a given function in $L^2(\mathbb{R}^n)$, and F, G are continuous functions satisfying the following conditions:

(A1) $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that for all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$F(x, s)s \leq -\alpha_1|s|^p + \beta_1(x); \tag{10}$$

$$|F(x,s)| \leq \alpha_2|s|^{p-1} + \beta_2(x); \tag{11}$$

$$\frac{\partial}{\partial s}F(x,s) \leq \alpha_3; \tag{12}$$

$$|\frac{\partial}{\partial x}F(x,s)| \leq \beta_3(x). \tag{13}$$

Here $\alpha_1,\ \alpha_2,\ \alpha_3$ are positive constants, $p > 2$, $\beta_1(x),\ \beta_2(x),\ \beta_3(x)$ are nonnegative functions on $\mathbb{R}^n$, such that $\beta_1(x) \in L^1(\mathbb{R}^n)$, and $\beta_2(x), \beta_3(x) \in L^2(\mathbb{R}^n)$.

(A2) $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that for all $x \in \mathbb{R}^n$ and $s_1, s_2 \in \mathbb{R}$,

$$|G(x,s_1) - G(x,s_2)| \leq \alpha_4|s_1 - s_2|; \tag{14}$$

$$|G(x,s)| \leq \beta_4(x)|s| + \beta_5(x). \tag{15}$$

Here α_4 is a positive constant, $\beta_4(x), \beta_5(x)$ are nonnegative functions on $\mathbb{R}^n$ such that $\beta_4(x) \in L^{\frac{2p}{p-2}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\beta_5(x) \in L^2(\mathbb{R}^n)$.

Example 2.8. *For $x \in \mathbb{R}^n$, $s, s_1, s_2 \in \mathbb{R}$ and $p > 2$, let*

$$F(x,s) = -\alpha_1|s|^{p-1}sgn(s); \qquad G(x,s) = \alpha_4 e^{-x^2}s.$$

It is easy to check that the functions F and G in Example 2.8 satisfy Condition (A1) and (A2).

In what follows we consider the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ which is defined in Section 1. Let

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}. \tag{16}$$

Then $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is an ergodic metric dynamical system. Since the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is canonical, one has

$$w(t,\omega) = \omega(t), \quad w(t, \theta_s\omega) = w(t+s,\omega) - w(s,\omega). \tag{17}$$

To study the random attractor for problem (8)-(9), we first transform that system into a deterministic system with random parameter. Let

$$z(\theta_t\omega) \equiv -\mu\int_{-\infty}^{0} e^{\mu s}(\theta_t\omega)(s)ds, \ \ t \in \mathbb{R}.$$

Then, one has that $z(\theta_t\omega)$ is the Ornstein-Uhlenbeck process and solves the following equation (see [15] for details):

$$dz + \mu z dt = dw(t). \tag{18}$$

Moreover, the random variable $z(\theta_t\omega)$ is tempered, and $z(\theta_t\omega)$ is $\mathbb{P}$-a.e. continuous. It follows from Proposition 4.3.3 [2] that there exists a tempered function $r(\omega) > 0$ such that

$$|z(\omega)| + |z(\omega)|^2 \leq r(\omega), \tag{19}$$

where $r(\omega)$ satisfies, for $\alpha > 0$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$r(\theta_t\omega) \leq e^{\frac{\alpha}{2}|t|}r(\omega), \quad t \in \mathbb{R}. \tag{20}$$

Then it follows that for $\alpha > 0$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$|z(\theta_t\omega)| + |z(\theta_t\omega)|^2 \leq e^{\frac{\alpha}{2}|t|}r(\omega), \quad t \in \mathbb{R}, \tag{21}$$

$$|z(\theta_{t-h}\omega)| + |z(\theta_{t-h}\omega)|^2 \leq e^{\frac{\alpha}{2}|t-h|}r(\omega) \leq e^{\frac{\alpha}{2}h}e^{\frac{\alpha}{2}|t|}r(\omega), \quad t \in \mathbb{R}. \tag{22}$$

By [9], $z(\theta_t\omega)$ has the following properties:

$$\lim_{t\to\pm\infty}\frac{z(\theta_t\omega)}{t} = 0, \tag{23}$$

$$\lim_{t\to\pm\infty}\frac{1}{|t|}\int_{-t}^{0}|z(\theta_s\omega)|ds = E|z|, \tag{24}$$

$$E|z| \le \sqrt{E|z|^2} = \frac{1}{\sqrt{2\mu}}. \tag{25}$$

It follows from (23) and (25) that there exists a positive constant $\mu > 0$ small enough, such that for $t > 0$ large enough,

$$2\epsilon\mu\int_{-t}^{0}|z(\theta_s\omega)|ds \le \frac{\lambda}{4}t; \qquad 2\epsilon|z(\theta_t\omega)| \le \frac{\lambda}{8}t. \tag{26}$$

In this paper we consider the weak solutions of (8)-(9) and (30)-(31).

Definition 2.9. *For any $u_0 \in \mathfrak{S}$, let $u : [0,\infty)\times\Omega \to L^2(\mathbb{R}^n)$. Suppose that $u(\cdot,\omega,u_0) : [0,\infty) \to L^2(\mathbb{R}^n)$ is continuous and $\frac{\partial u}{\partial t}$ is measurable. Then we say that u is a weak solution of (8)-(9), if $u(t,\omega,u_0) = u_0(t,\omega)$, $t \in [-h,0]$, $\omega \in \Omega$, and $u \in L^2([0,T],H^1(\mathbb{R}^n)) \cap L^p([0,T],L^p(\mathbb{R}^n))$, for any $T > 0$, and for any $\phi \in C_0^\infty(\mathbb{R}^n)$*

$$\begin{aligned}&\big(u(T),\phi\big) - \big(u_0(0),\phi\big) + \int_0^T \big(u(t),\lambda\phi - \Delta\phi\big)\,dt\\ &= \int_0^T \big(F(\cdot,u(t)) + G(\cdot,u(t-h)) + g,\phi\big)\,dt + \epsilon\int_0^T (\phi, u\circ dw),\end{aligned} \tag{27}$$

$\mathbb{P}$-a.e. $\omega \in \Omega$.

Suppose that u is a weak solution of (8)-(9). Let $v(t) = e^{-\epsilon z(\theta_t\omega)}u(t)$, $v_0(t) = e^{-\epsilon z(\theta_t\omega)}u_0(t)$ and $\psi = e^{\epsilon z(\theta_t\omega)}\phi$. Then one has that

$$\begin{aligned}\big(u(T),\phi\big) - \big(u_0(0),\phi\big) &= \int_0^T\left(\frac{\partial u}{\partial t},\phi\right)dt = \int_0^T\left(\frac{\partial}{\partial t}\left(e^{\epsilon z(\theta_t\omega)}v(t)\right),\phi\right)dt\\ &= \int_0^T\left(\frac{\partial}{\partial t}v(t),\psi\right)dt + \epsilon\int_0^T(\psi, v\circ dz)\\ &= \big(v(T),\psi\big) - \big(v_0(0),\psi\big) + \epsilon\int_0^T(\psi,v\circ dw) - \mu\epsilon\int_0^T\big(\psi, z(\theta_t\omega)v\big)\,dt.\end{aligned} \tag{28}$$

In the last step of (28) we use (18). Hence, it follows from (28) and the definition of weak solution that

$$\begin{aligned}&\big(v(T),\psi\big) - \big(v_0(0),\psi\big) + \int_0^T\big(v(t),\lambda\psi - \Delta\psi\big)\,dt\\ &= \int_0^T e^{-\epsilon z(\theta_t\omega)}\left(F(\cdot,e^{\epsilon z(\theta_t\omega)}v(t)) + G(\cdot,e^{\epsilon z(\theta_{t-h}\omega)}v(t-h)) + g,\psi\right)dt + \mu\epsilon\int_0^T\big(\psi,z(\theta_t\omega)v\big)\,dt.\end{aligned} \tag{29}$$

Then function v satisfying (29) is said to be the weak solution of the following equation:

$$\begin{aligned}\frac{\partial v}{\partial t} + \lambda v - \Delta v =& e^{-\epsilon z(\theta_t\omega)}F(x,e^{\epsilon z(\theta_t\omega)}v(t,x)) + e^{-\epsilon z(\theta_t\omega)}G(x,e^{\epsilon z(\theta_{t-h}\omega)}v(t-h,x))\\ &+ e^{-\epsilon z(\theta_t\omega)}g(x) + \epsilon\mu z(\theta_t\omega)v,\end{aligned} \tag{30}$$

with the initial condition

$$v_0(t,x) = e^{-\epsilon z(\theta_t\omega)}u_0(t,x), \quad x \in \mathbb{R}^n,\ t \in [-h,0]. \tag{31}$$

Equation (30)-(31) can be seen as a deterministic partial differential equation with random coefficients.

We use a similar method as in [4] to obtain the existence of weak solution to equation (30)-(31). First, using the Galerkin method as in [28], we can get that, under the condition (A1) and (A2), for any bounded domain $O \subset \mathbb{R}^n$, (30)-(31) has a unique weak solution $v(\cdot,\omega,v_0) \in C\left([0,\infty), L^2(O)\right) \cap L^2_{loc}\left([0,\infty), H^1_0(O)\right) \cap L^p_{loc}\left([0,\infty), L^p(O)\right)$, for $\mathbb{P}$-a.e. $\omega \in \Omega$. We can take the domain to be a family of balls, and the radius of balls tend to ∞. Then, one can get that $v \in L^2(\mathbb{R}^n)$ is the unique weak solution of (30)-(31) on $\mathbb{R}^n$. Then, $u(t) = e^{\epsilon z(\theta_t\omega)}v(t)$ is a unique weak solution to (8)-(9). Similar as Theorem 12 [13], we can show that (30)-(31) generates a continuous random dynamical system $(\Phi(t))_{t\ge 0}$ over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$, with

$$\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) = v^t(\cdot,\omega,v_0), \quad \omega\in\Omega, \quad v_0 \in \mathfrak{S}, \tag{32}$$

and (8)-(9) generates a continuous random dynamical system $(\Psi(t))_{t\ge 0}$ over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$, with

$$\Psi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) = u^t(\cdot,\omega,u_0), \quad \omega\in\Omega, \quad u_0 \in \mathfrak{S}. \tag{33}$$

Notice that two dynamical systems are conjugate to each other. Therefore, in the following sections, we only consider the existence of random attractor of $(\Phi(t))_{t\ge 0}$.

3 Uniform estimates of solutions

In this section we prove the existence of the random attractor for random dynamical system $(\Phi(t))_{t\ge 0}$. We first give some useful estimates on the mild solutions of equation (30)-(31).

Lemma 3.1. *Random dynamical system Φ has a random absorbing set $\{K(\omega)\}_{\omega\in\Omega}$ in $\mathcal{D}$, that is, for any $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\mathbb{P}$-a.e.$\omega \in \Omega$, there is $T_B(\omega) > 0$, such that $\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega)$ for all $t \ge T_B(\omega)$.*

Proof. Taking the inner product of (30) with v, we get that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||v||^2 + \lambda||v||^2 + ||\nabla v||^2 &= e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} F(x, e^{\epsilon z(\theta_t\omega)}v(t,x))v dx \\ &+ e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} G(x, e^{\epsilon z(\theta_{t-h}\omega)}v(t-h,x))v dx + e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} g(x)v dx + \epsilon\mu z(\theta_t\omega)||v||^2.\end{aligned} \tag{34}$$

We now estimate each term on the right hand side of (34). For the first term, by condition (A1) and Young inequality we have that

$$\begin{aligned}e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} F(x, e^{\epsilon z(\theta_t\omega)}v(t,x))v dx &\le e^{-2\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} -\alpha_1|e^{\epsilon z(\theta_t\omega)}v|^p + \beta_1(x)dx \\ &= -\alpha_1 e^{(p-2)\epsilon z(\theta_t\omega)}||v||^p_{L^p} + e^{-2\epsilon z(\theta_t\omega)}||\beta_1||_{L^1}.\end{aligned} \tag{35}$$

Note that by Young inequality, we have that for all $a, b, c > 0$, $\alpha, \beta, \gamma > 1$, and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$,

$$abc \le \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta} + \frac{c^\gamma}{\gamma}. \tag{36}$$

For the second term on the right hand side of (34), by condition (A2) and (36), we obtain that

$$\begin{aligned}&e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} G(x, e^{\epsilon z(\theta_{t-h}\omega)}v(t-h,x))v dx \\ &\le e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} \beta_4(x)\left|e^{\epsilon z(\theta_{t-h}\omega)}v(t-h,x)v\right| dx + e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} \beta_5(x)|v|dx\end{aligned}$$

$$= \int_{\mathbb{R}^n} \left(e^{\frac{p-2}{p}\epsilon z(\theta_t\omega)}|v|\right) \left|v(t-h,x)\right| \left(e^{\frac{2-2p}{p}\epsilon z(\theta_t\omega)+\epsilon z(\theta_{t-h}\omega)}\beta_4(x)\right) dx + e^{-\epsilon z(\theta_t\omega)} \int_{\mathbb{R}^n} \beta_5(x)|v|dx$$
$$\le \frac{\alpha_1}{2} e^{(p-2)\epsilon z(\theta_t\omega)}||v||_{L^p}^p + \frac{\lambda}{4} e^{-\frac{\lambda}{2}h}||v(t-h,x)||^2 + c_1 e^{\frac{4-4p}{p-2}\epsilon z(\theta_t\omega)+\frac{2p}{p-2}\epsilon z(\theta_{t-h}\omega)}||\beta_4||^{\frac{2p}{p-2}}$$
$$+\frac{\lambda}{4}||v||^2 + \frac{1}{\lambda} e^{-2\epsilon z(\theta_t\omega)}||\beta_5||^2, \tag{37}$$

where c_1 is a positive constant depending on α, λ and h. For the third term on the right hand side of (34), by Young inequality,

$$e^{-\epsilon z(\theta_t\omega)} \int_{\mathbb{R}^n} g(x)v dx \le \frac{1}{\lambda} e^{-2\epsilon z(\theta_t\omega)}||g||^2 + \frac{\lambda}{4}||v||^2. \tag{38}$$

Then it follows from (34) - (38) that, for all $t \ge 0$

$$\frac{d}{dt}||v||^2 + 2||\nabla v||^2 + \alpha_1 e^{(p-2)\epsilon z(\theta_t\omega)}||v||_{L^p}^p$$
$$\le \left(2\epsilon\mu|z(\theta_t\omega)| - \lambda\right)||v||^2 + \frac{\lambda}{2} e^{-\frac{\lambda}{2}h}||v(t-h,x)||^2 + c_2 e^{-2\epsilon z(\theta_t\omega)} + c_3 e^{\frac{4-4p}{p-2}\epsilon z(\theta_t\omega)+\frac{2p}{p-2}\epsilon z(\theta_{t-h}\omega)}$$
$$= \left(2\epsilon\mu|z(\theta_t\omega)| - \frac{\lambda}{2}\right)||v||^2 + \frac{\lambda}{2}\left(e^{-\frac{\lambda}{2}h}||v(t-h,x)||^2 - ||v||^2\right) + c_2 e^{-2\epsilon z(\theta_t\omega)} + c_3 e^{\frac{4-4p}{p-2}\epsilon z(\theta_t\omega)+\frac{2p}{p-2}\epsilon z(\theta_{t-h}\omega)}, \tag{39}$$

with $c_2 = 2||\beta_1||_{L^1} + \frac{2}{\lambda}||\beta_5||^2 + \frac{2}{\lambda}||g||^2$, $c_3 = 2c_1||\beta_4||^{\frac{2p}{p-2}}$. Applying Gronwall inequality, we obtain that, for all $t \ge 0$,

$$||v(t)||^2 + 2\int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau}||\nabla v||^2 ds + \alpha_1 \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau+(p-2)\epsilon z(\theta_s\omega)}||v||_{L^p}^p ds$$
$$\le e^{-\frac{\lambda}{2}t+2\epsilon\mu\int_0^t |z(\theta_\tau\omega)|d\tau}||v_0||^2 + \frac{\lambda}{2}\int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau}\left(e^{-\frac{\lambda}{2}h}||v(s-h)||^2 - ||v||^2\right) ds$$
$$+c_2\int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau-2\epsilon z(\theta_t\omega)} ds + c_3\int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)} ds. \tag{40}$$

Notice that

$$\int_0^t e^{\frac{\lambda}{2}s-2\epsilon\mu\int_0^s |z(\theta_\tau\omega)|d\tau} e^{-\frac{\lambda}{2}h}||v(s-h)||^2 ds$$
$$= \int_{-h}^{t-h} e^{\frac{\lambda}{2}(s+h)-2\epsilon\mu\int_0^{s+h} |z(\theta_\tau\omega)|d\tau} e^{-\frac{\lambda}{2}h}||v(s)||^2 ds$$
$$= \int_{-h}^{0} e^{\frac{\lambda}{2}s-2\epsilon\mu\int_0^{s+h} |z(\theta_\tau\omega)|d\tau}||v(s)||^2 ds + \int_0^{t-h} e^{\frac{\lambda}{2}s-2\epsilon\mu\int_0^{s+h} |z(\theta_\tau\omega)|d\tau}||v(s)||^2 ds$$
$$\le h||v_0||_{\mathfrak{S}} + \int_0^{t-h} e^{\frac{\lambda}{2}s-2\epsilon\mu\int_0^{s} |z(\theta_\tau\omega)|d\tau}||v(s)||^2 ds$$
$$\le h||v_0||_{\mathfrak{S}} + \int_0^{t} e^{\frac{\lambda}{2}s-2\epsilon\mu\int_0^{s} |z(\theta_\tau\omega)|d\tau}||v(s)||^2 ds. \tag{41}$$

Hence, it follows from (41) that

$$\frac{\lambda}{2}\int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_t\omega)|d\tau}\left(e^{-\frac{\lambda}{2}h}||v(s-h)||^2 - ||v||^2\right) ds$$

$$
\begin{aligned}
&= \frac{\lambda}{2} e^{-\frac{\lambda}{2}t+2\epsilon\mu\int_0^t |z(\theta_t\omega)|d\tau} \int_0^t e^{\frac{\lambda}{2}s-2\epsilon\mu\int_0^s |z(\theta_t\omega)|d\tau} e^{-\frac{\lambda}{2}h} ||v(s-h)||^2 ds \\
&\qquad - \frac{\lambda}{2} e^{-\frac{\lambda}{2}t+2\epsilon\mu\int_0^t |z(\theta_t\omega)|d\tau} \int_0^t e^{\frac{\lambda}{2}s-2\epsilon\mu\int_0^s |z(\theta_t\omega)|d\tau} ||v||^2 ds \\
&\le \frac{\lambda h}{2} e^{-\frac{\lambda}{2}t+2\epsilon\mu\int_0^t |z(\theta_t\omega)|d\tau} ||v_0||_{\mathfrak{S}}.
\end{aligned}
\tag{42}
$$

Then, (40) and (42) imply that

$$
\begin{aligned}
&||v(t)||^2 + 2\int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_t\omega)|d\tau} ||\nabla v||^2 ds + \alpha_1 \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_t\omega)|d\tau+(p-2)\epsilon z(\theta_s\omega)} ||v||_{L^p}^p ds \\
&\le (1+\frac{\lambda h}{2}) e^{-\frac{\lambda}{2}t+2\epsilon\mu\int_0^t |z(\theta_t\omega)|d\tau} ||v_0||_{\mathfrak{S}}^2 + c_2 \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_t\omega)|d\tau-2\epsilon z(\theta_t\omega)} ds \\
&\qquad + c_3 \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_t\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)} ds.
\end{aligned}
\tag{43}
$$

For fixed $\sigma \in [-h, 0]$, we have that, for $t \ge -\sigma$,

$$
\begin{aligned}
||v(t+\sigma)||^2 &\le (1+\frac{\lambda h}{2})\, e^{-\frac{\lambda}{2}(t+\sigma)+2\epsilon\mu\int_0^{t+\sigma} |z(\theta_\tau\omega)|d\tau} ||v_0||_{\mathfrak{S}}^2 \\
&\quad + c_2 \int_0^{t+\sigma} e^{\frac{\lambda}{2}(s-t-\sigma)+2\epsilon\mu\int_s^{t+\sigma} |z(\theta_\tau\omega)|d\tau-2\epsilon z(\theta_s\omega)} ds \\
&\quad + c_3 \int_0^{t+\sigma} e^{\frac{\lambda}{2}(s-t-\sigma)+2\epsilon\mu\int_s^{t+\sigma} |z(\theta_\tau\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)} ds \\
&\le (1+\frac{\lambda h}{2})\, e^{\frac{\lambda}{2}(h-t)+2\epsilon\mu\int_0^t |z(\theta_\tau\omega)|d\tau} ||v_0||_{\mathfrak{S}}^2 \\
&\quad + c_2 e^{\frac{\lambda h}{2}} \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau-2\epsilon z(\theta_s\omega)} ds \\
&\quad + c_3 e^{\frac{\lambda h}{2}} \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)} ds.
\end{aligned}
\tag{44}
$$

and for $t \in [0, -\sigma]$,

$$
\begin{aligned}
||v(t+\sigma)||^2 &\le ||v_0||_{\mathfrak{S}}^2 \\
&\le (1+\frac{\lambda h}{2})\, e^{\frac{\lambda}{2}(h-t)+2\epsilon\mu\int_0^t |z(\theta_\tau\omega)|d\tau} ||v_0||_{\mathfrak{S}}^2 \\
&\quad + c_2 e^{\frac{\lambda h}{2}} \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau-2\epsilon z(\theta_s\omega)} ds \\
&\quad + c_3 e^{\frac{\lambda h}{2}} \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)} ds.
\end{aligned}
\tag{45}
$$

Due to (44) and (45), we find that for all $t \ge 0$

$$
||v^t||_{\mathfrak{S}}^2 \le (1+\frac{\lambda h}{2})\, e^{\frac{\lambda}{2}(h-t)+2\epsilon\mu\int_0^t |z(\theta_\tau\omega)|d\tau} ||v_0||_{\mathfrak{S}}^2
$$

$$+c_2 e^{\frac{\lambda h}{2}} \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau-2\epsilon z(\theta_s\omega)} ds$$

$$+c_3 e^{\frac{\lambda h}{2}} \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)} ds. \tag{46}$$

Replacing ω by $\theta_{-t}\omega$ in (46), we get that for all $t \geq 0$

$$\begin{aligned}||v^t(\theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2_{\mathfrak{S}} \leq (1+\frac{\lambda h}{2})\, e^{\frac{\lambda}{2}(h-t)+2\epsilon\mu\int_0^t |z(\theta_{\tau-t}\omega)|d\tau}||v_0(\theta_{-t}\omega)||^2_{\mathfrak{S}}\\ +c_2 e^{\frac{\lambda h}{2}} \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau-2\epsilon z(\theta_{s-t}\omega)} ds\\ +c_3 e^{\frac{\lambda h}{2}} \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_{s-t}\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-t-h}\omega)} ds.\end{aligned} \tag{47}$$

By (26), one has that for any $v_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$,

$$\begin{aligned}&\lim_{t\to+\infty}(1+\frac{\lambda h}{2})\, e^{\frac{\lambda}{2}(h-t)+2\epsilon\mu\int_0^t |z(\theta_{\tau-t}\omega)|d\tau}||v_0||^2_{\mathfrak{S}}\\ =&\lim_{t\to+\infty}(1+\frac{\lambda h}{2})\, e^{\frac{\lambda}{2}(h-t)+2\epsilon\mu\int_{-t}^0 |z(\theta_\tau\omega)|d\tau}||v_0||^2_{\mathfrak{S}}\\ \leq&\lim_{t\to+\infty}(1+\frac{\lambda h}{2})\, e^{\frac{\lambda}{2}(h-t)+\frac{\lambda}{4}t}||v_0||^2_{\mathfrak{S}} = 0.\end{aligned} \tag{48}$$

It follows from (26) that for any $v_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$,

$$\begin{aligned}c_2 e^{\frac{\lambda h}{2}} \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau-2\epsilon z(\theta_{s-t}\omega)} ds = c_2 e^{\frac{\lambda h}{2}} \int_{-t}^0 e^{\frac{\lambda}{2}s+2\epsilon\mu\int_s^0 |z(\theta_\tau\omega)|d\tau-2\epsilon z(\theta_s\omega)} ds\\ \leq c_2 e^{\frac{\lambda h}{2}} \int_{-\infty}^0 e^{\frac{\lambda}{4}s-2\epsilon z(\theta_t\omega)} ds \leq c_2 e^{\frac{\lambda h}{2}} \int_{-\infty}^0 e^{\frac{\lambda}{2}s-\frac{\lambda}{4}s-\frac{\lambda}{8}s} ds < +\infty.\end{aligned} \tag{49}$$

Similarly, we can get that

$$\begin{aligned}&c_3 e^{\frac{\lambda h}{2}} \int_0^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_{s-t}\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-t-h}\omega)} ds\\ =&c_3 e^{\frac{\lambda h}{2}} \int_{-t}^0 e^{\frac{\lambda}{2}s+2\epsilon\mu\int_s^0 |z(\theta_\tau\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)} ds\\ \leq&c_3 e^{\frac{\lambda h}{2}} \int_{-\infty}^0 e^{\frac{\lambda}{4}s+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)} ds < +\infty.\end{aligned} \tag{50}$$

Set

$$\rho_1^2(\omega) = 1 + c_2 e^{\frac{\lambda h}{2}} \int_{-\infty}^0 e^{\frac{\lambda}{4}s-2\epsilon z(\theta_s\omega)} ds + c_3 e^{\frac{\lambda h}{2}} \int_{-\infty}^0 e^{\frac{\lambda}{4}s+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)} ds. \tag{51}$$

Then, there exists a $T_B(\omega) > 0$, such that for all $t > T_B(\omega)$,

$$||v^t(\theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2_{\mathfrak{S}} \leq \rho_1^2(\omega). \tag{52}$$

To show $\rho_1^2(\omega)$ is tempered, we need only to prove that for any $\gamma > 0$ small enough, the following holds

$$\lim_{t\to+\infty} e^{-\gamma t}\rho_1^2(\theta_{-t}\omega) = 0. \tag{53}$$

Using (26) again, one has that for $\gamma < \lambda$ small enough,

$$\begin{aligned}
&\lim_{t\to+\infty} e^{-\gamma t}\int_{-\infty}^{0} e^{\frac{\lambda}{4}s-2\epsilon z(\theta_{s-t}\omega)}ds \le \lim_{t\to+\infty} e^{-\gamma t}\int_{-\infty}^{0} e^{\frac{\gamma}{4}s-2\epsilon z(\theta_{s-t}\omega)}ds\\
&= \lim_{t\to+\infty} e^{-\gamma t}\int_{-\infty}^{t} e^{\frac{\gamma}{4}(s+t)-2\epsilon z(\theta_{s}\omega)}ds = \lim_{t\to+\infty} e^{-\frac{3}{4}\gamma t}\left(\int_{-\infty}^{0} e^{\frac{\gamma}{4}s-2\epsilon z(\theta_{s}\omega)}ds + \int_{0}^{t} e^{\frac{\gamma}{4}s-2\epsilon z(\theta_{s}\omega)}ds\right)\\
&\le \lim_{t\to+\infty} e^{-\frac{3}{4}\gamma t}\left(\int_{-\infty}^{0} e^{\frac{\gamma}{4}s-\frac{\gamma}{8}s}ds + \int_{0}^{t} e^{\frac{\gamma}{4}s+\frac{\gamma}{8}s}ds\right) = 0.
\end{aligned} \tag{54}$$

Similarly, we can get that

$$\lim_{t\to+\infty} e^{-\gamma t}\int_{-\infty}^{0} e^{\frac{\lambda}{4}s+\frac{4-4p}{p-2}\epsilon z(\theta_{s-t}\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-t-h}\omega)}ds = 0. \tag{55}$$

In view of (54) and (55), we find that $\rho_1^2(\omega)$ is tempered. This ends the proof. □

In Lemma 3.1, we show that $\{K(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for Φ. To prove Φ has a random attractor, we need to show that Φ is $\mathscr{D}$-pullback asymptotically compact. Therefore, we should obtain some estimates for ∇v.

Lemma 3.2. *There exists a tempered random variable $\rho_2(\omega) > 0$ such that for any $\{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ and $v_0(\omega) \in B(\omega)$, there exists a $T_B(\omega) > 0$ such that the solution v of (30)-(31) satisfies, for $\mathbb{P}$-a.e. $\omega \in \Omega$, for all $t \ge T_B(\Omega)$,*

$$\int_{t}^{t+1} ||\nabla v(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2 ds \le \rho_2(\omega). \tag{56}$$

Proof. By (43), we have that for all $t \ge 0$

$$\begin{aligned}
&2\int_{0}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_t\omega)|d\tau}||\nabla v||^2 ds \le (1+\frac{\lambda h}{2})e^{-\frac{\lambda}{2}t+2\epsilon\mu\int_0^t |z(\theta_t\omega)|d\tau}||v_0||^2_{\mathfrak{S}} +\\
&c_2\int_{0}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_t\omega)|d\tau-2\epsilon z(\theta_t\omega)}ds + c_3\int_{0}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_t\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)}ds.
\end{aligned} \tag{57}$$

Let $T_B(\omega)$ be the positive constant defined in Lemma 4.1. Replacing ω by $\theta_{-t}\omega$ in (57), by (48) - (50), one has that for all $t \ge T_B(\omega)$,

$$\begin{aligned}
&2\int_{0}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau}||\nabla v(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2 ds\\
&\le (1+\frac{\lambda h}{2})e^{-\frac{\lambda}{2}t+2\epsilon\mu\int_0^t |z(\theta_{\tau-t}\omega)|d\tau}||v_0(\theta_{-t}\omega)||^2_{\mathfrak{S}} + c_2\int_{0}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau-2\epsilon z(\theta_{s-t}\omega)}ds\\
&+c_3\int_{0}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_{s-t}\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-t-h}\omega)}ds
\end{aligned}$$

$$\leq \rho_1^2(\omega). \tag{58}$$

In another aspect, for $s \in [t, t+1]$,

$$\begin{aligned}
&\int_0^{t+1} e^{\frac{\lambda}{2}(s-t-1)+2\epsilon\mu\int_s^{t+1}|z(\theta_{\tau-t-1}\omega)|d\tau}||\nabla v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))||^2 ds \\
&\geq \int_t^{t+1} e^{\frac{\lambda}{2}(s-t-1)+2\epsilon\mu\int_s^{t+1}|z(\theta_{\tau-t-1}\omega)|d\tau}||\nabla v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))||^2 ds \\
&\geq e^{-\frac{\lambda}{2}-2\epsilon\mu\max\limits_{-1\leq\tau\leq 0}|z(\theta_\tau\omega)|}\int_t^{t+1}||\nabla v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))||^2 ds.
\end{aligned} \tag{59}$$

It follows from (58) and (59) that

$$\int_t^{t+1}||\nabla v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))||^2 ds \leq e^{\frac{\lambda}{2}+2\epsilon\mu\max\limits_{-1\leq\tau\leq 0}|z(\theta_\tau\omega)|}\rho_1^2(\omega). \tag{60}$$

Replacing ω by $\theta_1\omega$ in the last inequality we obtain that

$$\int_t^{t+1}||\nabla v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2 ds \leq e^{\frac{\lambda}{2}+2\epsilon\mu\max\limits_{-1\leq\tau\leq 0}|z(\theta_\tau\omega)|}\rho_1^2(\omega) \equiv \rho_2(\omega). \tag{61}$$

This ends the proof. □

Lemma 3.3. *There exists a tempered random variable $\rho_3(\omega) > 0$ such that for any $\{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ and $v_0(\omega) \in B(\omega)$, there exists a $T_B(\omega) > 0$ such that the solution v of (30)-(31) satisfies, for $\mathbb{P}$-a.e. $\omega \in \Omega$, for all $t \geq T_B(\omega) + h + 1$, and $\sigma_1, \sigma_2 \in [-h, 0]$,*

$$\left|\left|\nabla v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\right|\right|^2 \leq \rho_3(\omega), \tag{62}$$

$$\left|\int_{t+\sigma_1}^{t+\sigma_2}||\Delta v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2 ds\right| \leq \rho_4(\omega). \tag{63}$$

Proof. Taking the inner product of (30) with $-\Delta v$, we get that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla v||^2 + \lambda||\nabla v||^2 + ||\Delta v||^2 = -e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} F(x, e^{\epsilon z(\theta_t\omega)}v(t,x))\Delta v dx - \\
&e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} G(x, e^{\epsilon z(\theta_{t-h}\omega)}v(t-h,x))\Delta v dx - e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} g(x)\Delta v dx + \epsilon\mu z(\theta_t\omega)||\nabla v||^2.
\end{aligned} \tag{64}$$

We now estimate each term on the right-hand side of (64). For the first term, by Young inequality and condition (A1), we have

$$\begin{aligned}
&-e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} F(x, e^{\epsilon z(\theta_t\omega)}v(t,x))\Delta v dx \\
&= e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}\frac{\partial F}{\partial x}(x, e^{\epsilon z(\theta_t\omega)}v)\cdot\nabla v dx + \int_{\mathbb{R}^n}\frac{\partial F}{\partial s}(x, e^{\epsilon z(\theta_t\omega)}v)|\nabla v|^2 dx \\
&\leq e^{-\epsilon z(\theta_t\omega)}||\beta_3||\,||\nabla v|| + \alpha_3||\nabla v||^2 \\
&\leq (1+\alpha_3)||\nabla v||^2 + \frac{1}{4}e^{-2\epsilon z(\theta_t\omega)}||\beta_3||^2.
\end{aligned} \tag{65}$$

For the second term, it follows from condition (A2) and Young inequality that

$$\begin{aligned}
&\left|-e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} G(x, e^{\epsilon z(\theta_{t-h}\omega)}v(t-h,x))\Delta v dx\right| \\
&\leq e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}\left(e^{\epsilon z(\theta_{t-h}\omega)}\beta_4(x)|v(t-h,x)| + \beta_5(x)\right)\Delta v dx \\
&\leq \frac{1}{8}||\Delta v||^2 + 2e^{-2\epsilon z(\theta_t\omega)+2\epsilon z(\theta_{t-h}\omega)}\int_{\mathbb{R}^n}\beta_4^2(x)|v(t-h,x)|^2 dx + \frac{1}{8}||\Delta v||^2 + 2e^{-2\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}\beta_5^2(x)dx \\
&\leq \frac{1}{4}||\Delta v||^2 + 2e^{-2\epsilon z(\theta_t\omega)+2\epsilon z(\theta_{t-h}\omega)}||\beta_4||^2_{L^\infty}||v(t-h)||^2 + 2e^{-2\epsilon z(\theta_t\omega)}||\beta_5||^2. \qquad (66)
\end{aligned}$$

The third term is bounded by

$$\left|-e^{-\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n} g(x)\Delta v dx\right| \leq \frac{1}{4}||\Delta v||^2 + e^{-2\epsilon z(\theta_t\omega)}||g||^2. \qquad (67)$$

By (64) - (67), we find that for all $t \geq 0$,

$$\frac{d}{dt}||\nabla v||^2 + ||\Delta v||^2 \leq 2(1+\alpha_3)||\nabla v||^2 + c_5 e^{-2\epsilon z(\theta_t\omega)} + 4e^{-2\epsilon z(\theta_t\omega)+2\epsilon z(\theta_{t-h}\omega)}||\beta_4||^2_{L^\infty}||v(t-h)||^2, \qquad (68)$$

with $c_5 = 4||\beta_5||^2 + 2||g||^2 + \frac{1}{2}||\beta_3||^2$. Let $T_B(\omega)$ be the positive constant defined in Lemma 3.1, take $t \geq T_B(\omega)$ and $s \in [t, t+1]$. Integrating (68) over $[s, t+1]$, we get that

$$\begin{aligned}
||\nabla v(t+1,\omega,v_0(\omega))||^2 &\leq ||\nabla v(s,\omega,v_0(\omega))||^2 + 2(1+\alpha_3)\int_s^{t+1}||\nabla v(\tau,\omega,v_0(\omega))||^2 d\tau \\
&+c_5\int_s^{t+1} e^{-2\epsilon z(\theta_\tau\omega)}d\tau + 4||\beta_4||^2_{L^\infty}\int_s^{t+1} e^{-2\epsilon z(\theta_\tau\omega)+2\epsilon z(\theta_{\tau-h}\omega)}||v(\tau-h,\omega,v_0(\omega))||^2 d\tau. \qquad (69)
\end{aligned}$$

Integrating the above inequality with respect to s over $[t, t+1]$, we get that

$$\begin{aligned}
||\nabla v(t+1,\omega,v_0(\omega))||^2 &\leq 2(2+\alpha_3)\int_t^{t+1}||\nabla v(\tau,\omega,v_0(\omega))||^2 d\tau \\
&+c_5\int_t^{t+1} e^{-2\epsilon z(\theta_\tau\omega)}d\tau + 4||\beta_4||^2_{L^\infty}\int_t^{t+1} e^{-2\epsilon z(\theta_\tau\omega)+2\epsilon z(\theta_{\tau-h}\omega)}||v(\tau-h,\omega,v_0(\omega))||^2 d\tau. \qquad (70)
\end{aligned}$$

Replacing ω by $\theta_{-t-1}\omega$ in (70), we obtain that

$$\begin{aligned}
||\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))||^2 &\leq 2(2+\alpha_3)\int_t^{t+1}||\nabla v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))||^2 d\tau \\
&+c_5\int_t^{t+1} e^{-2\epsilon z(\theta_{\tau-t-1}\omega)}d\tau + 4||\beta_4||^2_{L^\infty}\int_t^{t+1} e^{-2\epsilon z(\theta_{\tau-t-1}\omega)+2\epsilon z(\theta_{\tau-t-h-1}\omega)}||v(\tau-h,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))||^2 d\tau. \qquad (71)
\end{aligned}$$

It follows from (56) that for all $t \geq T_B(\omega) + h$,

$$2(2+\alpha_3)\int_t^{t+1}||\nabla v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))||^2 d\tau \leq 2(2+\alpha_3)\rho_2(\theta_{-1}\omega), \qquad (72)$$

$$\int_t^{t+1} e^{-2\epsilon z(\theta_{\tau-t-1}\omega)}d\tau \leq e^{2\epsilon \max_{-1\leq\tau\leq 0}|z(\theta_\tau\omega)|}, \qquad (73)$$

and

$$\begin{aligned}
&\int_{t}^{t+1} e^{-2\epsilon z(\theta_{\tau-t-1}\omega)+2\epsilon z(\theta_{\tau-t-h-1}\omega)}||\nabla v(\tau-h,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))||^2 d\tau\\
&\le e^{2\epsilon \max\limits_{-1\le\tau\le0}\left(|z(\theta_\tau\omega)|+|z(\theta_{\tau-h}\omega)|\right)} \int_{t-h}^{t-h+1} ||\nabla v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))||^2 d\tau\\
&= e^{2\epsilon \max\limits_{-1\le\tau\le0}\left(|z(\theta_\tau\omega)|+|z(\theta_{\tau-h}\omega)|\right)} \int_{t-h}^{t-h+1} ||\nabla v(\tau,\theta_{-t+h-1}(\theta_{-h}\omega),v_0(\theta_{-t+h-1}(\theta_{-h}\omega))||^2 d\tau\\
&\le e^{2\epsilon \max\limits_{-1\le\tau\le0}\left(|z(\theta_\tau\omega)|+|z(\theta_{\tau-h}\omega)|\right)}\rho_2(\theta_{-h}\omega).
\end{aligned} \tag{74}$$

It follows from (71) - (74) that, for all $t \ge T_B(\omega)+h$

$$\begin{aligned}
||\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))||^2 &\le 2(2+\alpha_3)\rho_2(\theta_{-1}\omega)+c_5 e^{2\epsilon \max\limits_{-1\le\tau\le0}|z(\theta_\tau\omega)|}\\
&+4||\beta_4||^2_{L^\infty} e^{2\epsilon \max\limits_{-1\le\tau\le0}\left(|z(\theta_\tau\omega)|+|z(\theta_{\tau-h}\omega)|\right)}\rho_2(\theta_{-h}\omega)\equiv\rho_3(\omega).
\end{aligned} \tag{75}$$

Then we have that for all $t \ge T_B(\omega)+h+1$,

$$||\nabla v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 \le \rho_3(\omega). \tag{76}$$

Let $t \ge h$, $-h \le \sigma_1 \le \sigma_2 \le 0$. Integrating (68) over $[t+\sigma_1, t+\sigma_2]$, we obtain that

$$\begin{aligned}
&||\nabla v(t+\sigma_2,\omega,v_0(\omega))||^2 + \int_{t+\sigma_1}^{t+\sigma_2} ||\Delta v(\tau,\omega,v_0(\omega))||^2 d\tau\\
&\le ||\nabla v(t+\sigma_1,\omega,v_0(\omega))||^2 + 2(1+\alpha_3)\int_{t+\sigma_1}^{t+\sigma_2} ||\nabla v(\tau,\omega,v_0(\omega))||^2 d\tau\\
&+c_5\int_{t+\sigma_1}^{t+\sigma_2} e^{-2\epsilon z(\theta_\tau\omega)}d\tau + 4||\beta_4||^2_{L^\infty}\int_{t+\sigma_1}^{t+\sigma_2} e^{-2\epsilon z(\theta_\tau\omega)+2\epsilon z(\theta_{\tau-h}\omega)}||v(\tau-h,\omega,v_0(\omega))||^2 d\tau.
\end{aligned} \tag{77}$$

Replacing ω by $\theta_{-t}\omega$ in the last inequality, we have that

$$\begin{aligned}
&||\nabla v(t+\sigma_2,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 + \int_{t+\sigma_1}^{t+\sigma_2} ||\Delta v(\tau,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 d\tau\\
&\le ||\nabla v(t+\sigma_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 + 2(1+\alpha_3)\int_{t+\sigma_1}^{t+\sigma_2} ||\nabla v(\tau,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 d\tau\\
&+c_5\int_{t+\sigma_1}^{t+\sigma_2} e^{-2\epsilon z(\theta_{\tau-t}\omega)}d\tau + 4||\beta_4||^2_{L^\infty}\int_{t+\sigma_1}^{t+\sigma_2} e^{-2\epsilon z(\theta_{\tau-t}\omega)+2\epsilon z(\theta_{\tau-t-h}\omega)}||v(\tau-h,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 d\tau.
\end{aligned} \tag{78}$$

Using (76), we get that

$$\int_{t+\sigma_1}^{t+\sigma_2} ||\nabla v(\tau,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 d\tau = \int_{t+\sigma_1}^{t+\sigma_2} ||\nabla v(\tau,\theta_{-\tau}(\theta_{\tau-t}\omega),v_0(\theta_{-\tau}(\theta_{\tau-t}\omega)))||^2 d\tau \le h\max_{-h\le\tau\le0}\rho_3(\theta_\tau\omega)). \tag{79}$$

By (52), we can get that

$$4||\beta_4||^2_{L^\infty}\int_{t+\sigma_1}^{t+\sigma_2} e^{-2\epsilon z(\theta_{\tau-t}\omega)+2\epsilon z(\theta_{\tau-t-h}\omega)}||v(\tau-h,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2 d\tau$$

$$\leq 4\|\beta_4\|_{L^\infty}^2 \max_{-2h\leq\tau\leq -h} \rho_1^2(\theta_\tau\omega) \int_{t+\sigma_1}^{t+\sigma_2} e^{-2\epsilon z(\theta_{\tau-t}\omega)+2\epsilon z(\theta_{\tau-t-h}\omega)} d\tau$$

$$= 4h\|\beta_4\|_{L^\infty}^2 \max_{-2h\leq\tau\leq -h} \rho_1^2(\theta_\tau\omega) e^{2\epsilon \max\limits_{-h\leq\tau\leq 0}\left(|z(\theta_\tau\omega)|+|z(\theta_{\tau-h}\omega)|\right)}. \tag{80}$$

Notice that

$$\int_{t+\sigma_1}^{t+\sigma_2} e^{-2\epsilon z(\theta_{\tau-t}\omega)} d\tau = \int_{\sigma_1}^{\sigma_2} e^{-2\epsilon z(\theta_\tau\omega)} d\tau \leq h e^{2\epsilon \max\limits_{-h\leq\tau\leq 0} |z(\theta_\tau\omega)|}. \tag{81}$$

It follows from (76), (78)-(81) that

$$\int_{t+\sigma_1}^{t+\sigma_2} \|\Delta v(\tau, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 d\tau \leq (1+2h(1+\alpha_3)) \max_{-h\leq\tau\leq 0} \rho_3(\theta_\tau\omega) + c_5 h e^{2\epsilon \max\limits_{-h\leq\tau\leq 0} |z(\theta_\tau\omega)|} +$$

$$4h\|\beta_4\|_{L^\infty}^2 \max_{-2h\leq\tau\leq -h} \rho_1^2(\theta_\tau\omega) e^{2\epsilon \max\limits_{-h\leq\tau\leq 0}\left(|z(\theta_\tau\omega)|+|z(\theta_{\tau-h}\omega)|\right)} \equiv \rho_4(\omega). \tag{82}$$

This ends the proof. □

Lemma 3.4. *Let $B(\omega) \in \mathscr{D}$ and $v_0(\omega) \in B(\omega)$. Then for any $\epsilon_1 > 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists a $T^\star = T^\star(B, \omega, \epsilon_1) > 0$, and $R^\star = R^\star(\omega, \epsilon_1) > 0$ such that the solution (30)-(31), satisfies for all $t \geq T^\star$,*

$$\sup_{s\in[-h,0]} \int_{|x|\geq R^\star} |v^t(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))| dx \leq \epsilon_1. \tag{83}$$

Proof. Let ρ be a smooth function defined on $\mathbb{R}^+$ such that $0 \leq \rho(s) \leq 1$ for all $s \in \mathbb{R}^+$, and

$$\rho(s) = \begin{cases} 0, & 0 \leq s \leq 1, \\ 1, & s \geq 2. \end{cases} \tag{84}$$

Then there exists a positive constant c_0 such that $|\rho'(s)| \leq c_0$ for all $s \in \mathbb{R}^+$. Taking the inner product of (30) with $\rho\left(\frac{|x|^2}{k^2}\right) v$, we get that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|v|^2 dx + \lambda \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|v|^2 dx - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) v\Delta v dx$$

$$= e^{-\epsilon z(\theta_t\omega)} \int_{\mathbb{R}^n} F(x, e^{\epsilon z(\theta_t\omega)} v)\rho\left(\frac{|x|^2}{k^2}\right) v dx + e^{-\epsilon z(\theta_t\omega)} \int_{\mathbb{R}^n} G(x, e^{\epsilon z(\theta_{t-h}\omega)} v(t-h))\rho\left(\frac{|x|^2}{k^2}\right) v dx$$

$$+ e^{-\epsilon z(\theta_t\omega)} \int_{\mathbb{R}^n} g(x)\rho\left(\frac{|x|^2}{k^2}\right) v dx + \epsilon\mu z(\theta_t\omega) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|v|^2 dx. \tag{85}$$

We now estimate each term in (85). For the third term on the left hand-side of (85), we have that

$$-\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) v\Delta v dx = \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2 dx + \int_{\mathbb{R}^n} \rho'\left(\frac{|x|^2}{k^2}\right) v \frac{2x}{k^2}\cdot\nabla v dx$$

$$= \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2 dx + \int_{k\leq|x|\leq\sqrt{2}k} \rho'\left(\frac{|x|^2}{k^2}\right) v \frac{2x}{k^2}\cdot\nabla v dx, \tag{86}$$

and

$$\left|\int_{k\leq|x|\leq\sqrt{2}k} \rho'\left(\frac{|x|^2}{k^2}\right) v \frac{2x}{k^2}\cdot\nabla v dx\right|$$

$$\leq \frac{2\sqrt{2}}{k} \int\limits_{k \leq |x| \leq \sqrt{2}k} |\rho'\left(\frac{|x|^2}{k^2}\right)| \, |v| \, |\nabla v| dx$$

$$\leq \frac{4c_0}{k} \int\limits_{\mathbb{R}^n} |v| \, |\nabla v| dx \leq \frac{2c_0}{k} \left(||v||^2 + ||\nabla v||^2\right). \tag{87}$$

By (86) and (87), one has that

$$-\int\limits_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) v\Delta v dx \geq \int\limits_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\nabla v|^2 dx - \frac{2c_0}{k}\left(||v||^2 + ||\nabla v||^2\right). \tag{88}$$

For the first term on the right-hand side of (85) by condition (A1), we have

$$e^{-\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} F(x, e^{\epsilon z(\theta_t\omega)} v) \rho\left(\frac{|x|^2}{k^2}\right) v dx$$

$$= e^{-2\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} F(x, e^{\epsilon z(\theta_t\omega)} v) \rho\left(\frac{|x|^2}{k^2}\right) e^{\epsilon z(\theta_t\omega)} v dx$$

$$\leq -\alpha_1 e^{(p-2)\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^p dx + e^{-2\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \beta_1(x) dx. \tag{89}$$

For the second term on the right-hand side of (85) we have that

$$\left| e^{-\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} G(x, e^{\epsilon z(\theta_{t-h}\omega)} v(t-h)) \rho\left(\frac{|x|^2}{k^2}\right) v dx \right|$$

$$\leq e^{-\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} \beta_4(x) \left| e^{\epsilon z(\theta_{t-h}\omega)} v(t-h) \right| \rho\left(\frac{|x|^2}{k^2}\right) |v| dx + e^{-\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} \beta_5(x) \rho\left(\frac{|x|^2}{k^2}\right) |v| dx. \tag{90}$$

By (36), the first term on the right-hand side of (90) is bounded by

$$e^{-\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} \beta_4(x) \left| e^{\epsilon z(\theta_{t-h}\omega)} v(t-h) \right| \rho\left(\frac{|x|^2}{k^2}\right) |v| dx$$

$$= \int\limits_{\mathbb{R}^n} \left(e^{-\frac{p-2}{p}\epsilon z(\theta_t\omega)} |v| \rho^{\frac{1}{p}}\left(\frac{|x|^2}{k^2}\right)\right) \left(\rho^{\frac{1}{2}}\left(\frac{|x|^2}{k^2}\right) |v(t-h)|\right) \left(\rho^{\frac{p-2}{2p}}\left(\frac{|x|^2}{k^2}\right) e^{\frac{2-2p}{p}\epsilon z(\theta_t\omega) + \epsilon z(\theta_{t-h}\omega)} \beta_4(x)\right) dx$$

$$\leq \alpha_1 \int\limits_{\mathbb{R}^n} e^{(p-2)\epsilon z(\theta_t\omega)} \rho\left(\frac{|x|^2}{k^2}\right) |v|^p dx + \frac{\lambda}{4} e^{-\frac{\lambda}{2}h} \int\limits_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v(t-h)|^2 dx$$

$$+ c_6 \int\limits_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) e^{\frac{4-4p}{p-2}\epsilon z(\theta_t\omega) + \frac{2p}{p-2}\epsilon z(\theta_{t-h}\omega)} \left(\beta_4(x)\right)^{\frac{2p}{p-2}} dx, \tag{91}$$

and the second term on the right-hand side of (90) is bounded by

$$e^{-\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} \beta_5(x) \rho\left(\frac{|x|^2}{k^2}\right) |v| dx \leq \frac{\lambda}{4} \int\limits_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + \frac{1}{\lambda} e^{-2\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \beta_5^2(x) dx. \tag{92}$$

For the third term on the right-hand side of (85) we have that

$$e^{-\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} g(x) \rho\left(\frac{|x|^2}{k^2}\right) v dx$$

$$\leq \frac{\lambda}{4} \int\limits_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + \frac{1}{\lambda} e^{-2\epsilon z(\theta_t\omega)} \int\limits_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) g^2(x) dx. \tag{93}$$

It follows from (85) - (93) that

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2dx \le \left(2\epsilon\mu|z(\theta_t\omega)|-\frac{\lambda}{2}\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2dx$$
$$+\frac{\lambda}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(e^{-\frac{\lambda}{2}h}|v(t-h)|^2-|v|^2\right)dx+\frac{4c_0}{k}\left(||v||^2+||\nabla v||^2\right)$$
$$+2e^{-2\epsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left[\beta_1(x)+\frac{1}{\lambda}\beta_5^2(x)+\frac{1}{\lambda}g^2(x)\right]dx$$
$$+2c_6\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)e^{\frac{4-4p}{p-2}\epsilon z(\theta_t\omega)+\frac{2p}{p-2}\epsilon z(\theta_{t-h}\omega)}\beta_4^{\frac{2p}{p-2}}(x)dx. \tag{94}$$

Let $T_B(\omega)$ be the positive constant defined in Lemma 3.1. Set $T_1(B,\omega)\ge T_B(\omega)+h+1$. Applying Gronwall inequality to (94), we obtain that for $t\ge T_1$,

$$\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2dx \le e^{\frac{\lambda}{2}(T_1-t)+2\epsilon\mu\int_{T_1}^t|z(\theta_\tau\omega)|d\tau}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v(T_1)|^2dx$$
$$+\frac{\lambda}{2}\int_{T_1}^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t|z(\theta_\tau\omega)|d\tau}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(e^{-\frac{\lambda}{2}h}|v(t-h)|^2-|v|^2\right)dx\,ds$$
$$+\frac{4c_0}{k}\int_{T_1}^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t|z(\theta_\tau\omega)|d\tau}\left(||v||^2+||\nabla v||^2\right)ds$$
$$+2\int_{T_1}^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t|z(\theta_\tau\omega)|d\tau-2\epsilon z(\theta_s\omega)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left[\beta_1(x)+\frac{1}{\lambda}\beta_5^2(x)+\frac{1}{\lambda}g^2(x)\right]dx\,ds$$
$$+2c_6\int_{T_1}^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t|z(\theta_\tau\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\beta_4^{\frac{2p}{p-2}}(x)dx\,ds. \tag{95}$$

If we take $t\ge T_1+h$, then, by (95), we can get that for all $\sigma\in[-h,0]$,

$$\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v(t+\sigma)|^2dx \le e^{\frac{\lambda}{2}(T_1-t-\sigma)+2\epsilon\mu\int_{T_1}^{t+\sigma}|z(\theta_\tau\omega)|d\tau}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v(T_1)|^2dx$$
$$+\frac{\lambda}{2}\int_{T_1}^{t+\sigma} e^{\frac{\lambda}{2}(s-t-\sigma)+2\epsilon\mu\int_s^{t+\sigma}|z(\theta_\tau\omega)|d\tau}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(e^{-\frac{\lambda}{2}h}|v(s-h)|^2-|v|^2\right)dx\,ds$$
$$+\frac{4c_0}{k}\int_{T_1}^{t+\sigma} e^{\frac{\lambda}{2}(s-t-\sigma)+2\epsilon\mu\int_s^{t+\sigma}|z(\theta_\tau\omega)|d\tau}\left(||v||^2+||\nabla v||^2\right)ds$$
$$+2\int_{T_1}^{t+\sigma} e^{\frac{\lambda}{2}(s-t-\sigma)+2\epsilon\mu\int_s^{t+\sigma}|z(\theta_\tau\omega)|d\tau-2\epsilon z(\theta_s\omega)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left[\beta_1(x)+\frac{1}{\lambda}\beta_5^2(x)+\frac{1}{\lambda}g^2(x)\right]dx\,ds$$
$$+2c_6\int_{T_1}^{t+\sigma} e^{\frac{\lambda}{2}(s-t-\sigma)+2\epsilon\mu\int_s^{t+\sigma}|z(\theta_\tau\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\beta_4^{\frac{2p}{p-2}}(x)dx\,ds. \tag{96}$$

It follows that for all $t\ge T_1+h$,

$$\sup_{-h\le\sigma\le0}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v(t+\sigma)|^2dx \le e^{\frac{\lambda}{2}(T_1+h-t)+2\epsilon\mu\int_{T_1}^t|z(\theta_\tau\omega)|d\tau}||v(T_1)||_{\mathfrak{S}}^2$$

$$+\frac{\lambda}{2}e^{\frac{\lambda h}{2}}\int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(e^{-\frac{\lambda}{2}h}|v(s-h)|^2-|v|^2\right)dx\,ds$$

$$+\frac{4c_0}{k}e^{\frac{\lambda h}{2}}\int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau}\left(||v||^2+||\nabla v||^2\right)ds$$

$$+2e^{\frac{\lambda h}{2}}\int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau-2\epsilon z(\theta_s\omega)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left[\beta_1(x)+\frac{1}{\lambda}\beta_5^2(x)+\frac{1}{\lambda}g^2(x)\right]dx\,ds$$

$$+2c_1e^{\frac{\lambda h}{2}}\int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_\tau\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_s\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-h}\omega)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\beta_4^{\frac{2p}{p-2}}(x)dx\,ds. \tag{97}$$

Replacing ω by $\theta_{-t}\omega$, we obtain from (97) that for all $t \geq T_1(B,\omega)$,

$$\sup_{-h\le\sigma\le 0}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v^t(\sigma,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx \le e^{\frac{\lambda h}{2}}e^{\frac{\lambda}{2}(T_1-t)+2\epsilon\mu\int_{T_1}^t |z(\theta_{\tau-t}\omega)|d\tau}||v^{T_1}(\theta_{-t}(\omega),v_0(\theta_{-t}(\omega)))||_{\mathfrak{S}}^2$$

$$+\frac{\lambda}{2}e^{\frac{\lambda h}{2}}\int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)e^{-\frac{\lambda}{2}h}|v(s-h,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2-|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\,ds$$

$$+\frac{4c_0}{k}e^{\frac{\lambda h}{2}}\int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau}\left(||v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2+||\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2\right)ds$$

$$+2e^{\frac{\lambda h}{2}}\int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau-2\epsilon z(\theta_{s-t}\omega)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left[\beta_1(x)+\frac{1}{\lambda}\beta_5^2(x)+\frac{1}{\lambda}g^2(x)\right]dx\,ds$$

$$+2c_1e^{\frac{\lambda h}{2}}\int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_{s-t}\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-t-h}\omega)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\beta_4^{\frac{2p}{p-2}}(x)dx\,ds. \tag{98}$$

Now we estimate each term on the right hand side of (98). We first replace t by T_1 and replace ω by $\theta_{-t}\omega$ in (46), we have that

$$||v(T_1,\theta_{-t}\omega,\theta_{-t}v_0(\omega))||_{\mathfrak{S}}^2 \le (1+\frac{\lambda h}{2})\,e^{\frac{\lambda}{2}(h-T_1)+2\epsilon\mu\int_0^{T_1}|z(\theta_{\tau-t}\omega)|d\tau}||v_0(\theta_{-t}\omega)||_{\mathfrak{S}}^2$$

$$+c_2e^{\frac{\lambda}{2}h}\int_0^{T_1} e^{\frac{\lambda}{2}(s-T_1)+2\epsilon\mu\int_s^{T_1}|z(\theta_{\tau-t}\omega)|d\tau-2\epsilon z(\theta_{s-t}\omega)}ds$$

$$+c_3e^{\frac{\lambda}{2}h}\int_0^{T_1} e^{\frac{\lambda}{2}(s-T_1)+2\epsilon\mu\int_s^{T_1}|z(\theta_{\tau-t}\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_{s-t}\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-t-h}\omega)}ds. \tag{99}$$

Therefore, for the first term on the right hand side of (98),

$$e^{\frac{\lambda h}{2}}e^{\frac{\lambda}{2}(T_1-t)+2\epsilon\mu\int_{T_1}^t |z(\theta_{\tau-t}\omega)|d\tau}||v^{T_1}(\theta_{-t}(\omega),v_0(\theta_{-t}(\omega)))||_{\mathfrak{S}}^2$$

$$\le (1+\frac{\lambda h}{2})e^{\lambda h}\,e^{-\frac{\lambda}{2}t+2\epsilon\mu\int_0^t|z(\theta_{\tau-t}\omega)|d\tau}||v_0(\theta_{-t}\omega)||_{\mathfrak{S}}^2$$

$$+c_2e^{\lambda h}\int_0^{T_1} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t|z(\theta_{\tau-t}\omega)|d\tau-2\epsilon z(\theta_{s-t}\omega)}ds$$

$$+c_3e^{\lambda h}\int_0^{T_1} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t|z(\theta_{\tau-t}\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_{s-t}\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-t-h}\omega)}ds. \tag{100}$$

We now estimate the terms in (100) as follows. For the first term, by (26), we find that for $t \geq T_1$ and $v_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$,

$$\begin{aligned}
&\lim_{t\to+\infty}(1+\frac{\lambda h}{2})e^{\lambda h}\ e^{-\frac{\lambda}{2}t+2\epsilon\mu\int_0^t|z(\theta_{\tau-t}\omega)|d\tau}||v_0(\theta_{-t}\omega)||^2_{\mathfrak{S}}\\
=&\lim_{t\to+\infty}(1+\frac{\lambda h}{2})e^{\lambda h}\ e^{-\frac{\lambda}{2}t+2\epsilon\mu\int_{-t}^0|z(\theta_{\tau}\omega)|d\tau}||v_0(\theta_{-t}\omega)||^2_{\mathfrak{S}}\\
\leq&\lim_{t\to+\infty}(1+\frac{\lambda h}{2})e^{\lambda h}\ e^{-\frac{\lambda}{2}t+\frac{\lambda}{4}t}||v_0(\theta_{-t}\omega)||^2_{\mathfrak{S}}=0.
\end{aligned}\tag{101}$$

For the second term, by (26)

$$\begin{aligned}
&\lim_{t\to+\infty}c_2e^{\lambda h}\int_0^{T_1}e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t|z(\theta_{\tau-t}\omega)|d\tau-2\epsilon z(\theta_{s-t}\omega)}ds\\
=&\lim_{t\to+\infty}c_2e^{\lambda h}\int_0^{T_1}e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_{s-t}^0|z(\theta_{\tau}\omega)|d\tau-2\epsilon z(\theta_{s-t}\omega)}ds\\
\leq&\lim_{t\to+\infty}c_2e^{\lambda h}\int_0^{T_1}e^{\frac{\lambda}{2}(s-t)-\frac{\lambda}{4}(s-t)-\frac{\lambda}{8}(s-t)}ds=0.
\end{aligned}\tag{102}$$

Similarly, we can get the following estimate for the third term on the right-hand side of (100):

$$\lim_{t\to+\infty}c_3e^{\lambda h}\int_0^{T_1}e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t|z(\theta_{\tau-t}\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_{s-t}\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-t-h}\omega)}ds=0.\tag{103}$$

It follows from (100) - (103) that

$$\lim_{t\to+\infty}e^{\frac{\lambda h}{2}}e^{\frac{\lambda}{2}(T_1-t)+2\epsilon\mu\int_{T_1}^t|z(\theta_{\tau-t}\omega)|d\tau}||v^{T_1}(\theta_{-t}(\omega),v_0(\theta_{-t}(\omega)))||^2_{\mathfrak{S}}=0,\tag{104}$$

which implies that for any $\epsilon_1>0$, there is a $T_2=T_2(B,\omega,\epsilon)>T_1$ such that

$$e^{\frac{\lambda h}{2}}e^{\frac{\lambda}{2}(T_1-t)+2\epsilon\mu\int_{T_1}^t|z(\theta_{\tau-t}\omega)|d\tau}||v^{T_1}(\theta_{-t}(\omega),v_0(\theta_{-t}(\omega)))||^2_{\mathfrak{S}}\leq\epsilon_1.\tag{105}$$

Next, we estimate the second term on the right hand side of (98). Similarly as (42), we can obtain that

$$\begin{aligned}
&\frac{\lambda}{2}e^{\frac{\lambda h}{2}}\int_{T_1}^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t|z(\theta_{\tau-t}\omega)|d\tau}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)e^{-\frac{\lambda}{2}h}|v(s-h,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2-|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\,ds\\
\leq&\frac{\lambda}{2}e^{\frac{\lambda h}{2}}e^{\frac{\lambda}{2}(T_1-t)+2\epsilon\mu\int_0^t|z(\theta_{\tau-t}\omega)|d\tau}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\sup_{-h\leq\sigma\leq0}|v^{T_1}(\sigma,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\,ds\\
\leq&\frac{\lambda}{2}e^{\frac{\lambda h}{2}}e^{\frac{\lambda}{2}(T_1-t)+2\epsilon\mu\int_0^t|z(\theta_{\tau-t}\omega)|d\tau}||v^{T_1}(\theta_{-t}\omega,v_0(\theta_{-t}\omega))||_{\mathfrak{S}}.
\end{aligned}\tag{106}$$

By (104) and (106), we know that for any $\epsilon_1>0$, there exists a $T_3=T_3(B,\omega,\epsilon)>T_1$ such that

$$\begin{aligned}
&\frac{\lambda}{2}e^{\frac{\lambda h}{2}}\int_{T_1}^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t|z(\theta_{\tau-t}\omega)|d\tau}\times\\
&\times\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)e^{-\frac{\lambda}{2}h}|v(s-h,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2-|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\,ds\leq\epsilon_1.
\end{aligned}\tag{107}$$

For the third term on the right hand side of (98), by Lemma 3.1 and Lemma 3.3, we obtain that

$$\int_{T_1}^t e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t|z(\theta_{\tau-t}\omega)|d\tau}\left(||v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2+||\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))||^2\right)ds$$

$$\leq \int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau} \left(\rho_1^2(\theta_{s-t}\omega)+\rho_3(\theta_{s-t}\omega)\right) ds$$
$$= \int_{T_1-t}^{0} e^{\frac{\lambda}{2}s+2\epsilon\mu\int_s^0 |z(\theta_{\tau}\omega)|d\tau} \left(\rho_1^2(\theta_{s}\omega)+\rho_3(\theta_{s}\omega)\right) ds$$
$$\leq \int_{-\infty}^{0} e^{\frac{\lambda}{2}s-\frac{\lambda}{4}s} \left(\rho_1^2(\theta_{s}\omega)+\rho_3(\theta_{s}\omega)\right) ds < \infty. \tag{108}$$

This implies that there exists a $T_4 = T_4(B, \omega, \epsilon) > T_1$ and $R_2 = R_2(\omega, \epsilon_1)$ such that for all $t \geq T_4$ and $k \geq R_2$,

$$\frac{4c_0}{k} e^{\frac{\lambda h}{2}} \int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau} \left(||v(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2 + ||\nabla v(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2\right) ds \leq \epsilon_1. \tag{109}$$

Note that $\beta_1 \in L^1(\mathbb{R}^n)$, $\beta_5 \in L^2(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n)$. Thus, there is $R_3 = R_3(\omega, \epsilon_1)$ such that for all $k \geq R_3$

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left[\beta_1(x)+\frac{1}{\lambda}\beta_5^2(x)+\frac{1}{\lambda}g^2(x)\right] dx \leq \int_{|x|\geq k} \left[\beta_1(x)+\frac{1}{\lambda}\beta_5^2(x)+\frac{1}{\lambda}g^2(x)\right] dx \leq \epsilon_1. \tag{110}$$

Then, for the fourth term on the right hand side of (98) we have that

$$2e^{\frac{\lambda h}{2}} \int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau-2\epsilon z(\theta_{s-t}\omega)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left[\beta_1(x)+\frac{1}{\lambda}\beta_5^2(x)+\frac{1}{\lambda}g^2(x)\right] dx\, ds$$
$$\leq 2\epsilon_1 e^{\frac{\lambda h}{2}} \int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau-2\epsilon z(\theta_{s-t}\omega)} ds$$
$$= 2\epsilon_1 e^{\frac{\lambda h}{2}} \int_{T_1-t}^{0} e^{\frac{\lambda}{2}s+2\epsilon\mu\int_s^0 |z(\theta_{\tau}\omega)|d\tau-2\epsilon z(\theta_{s}\omega)} ds$$
$$\leq 2\epsilon_1 e^{\frac{\lambda h}{2}} \int_{-\infty}^{0} e^{\frac{\lambda}{2}s-\frac{\lambda}{4}s-\frac{\lambda}{8}s} ds = \frac{16}{\lambda} e^{\frac{\lambda h}{2}} \epsilon_1. \tag{111}$$

Notice that $\beta_4 \in L^{\frac{2p}{p-2}}(\mathbb{R}^n)$. Thus, there is $R_4 = R_4(\epsilon_1, \omega)$ such that for all $k \geq R_4$,

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \beta_4^{\frac{2p}{p-2}}(x) dx \leq \int_{|x|\geq k} \beta_4^{\frac{2p}{p-2}}(x) dx \leq \epsilon_1. \tag{112}$$

Then, similarly as (111), there exists a constant $c > 0$, such that

$$2c_1 e^{\frac{\lambda h}{2}} \int_{T_1}^{t} e^{\frac{\lambda}{2}(s-t)+2\epsilon\mu\int_s^t |z(\theta_{\tau-t}\omega)|d\tau+\frac{4-4p}{p-2}\epsilon z(\theta_{s-t}\omega)+\frac{2p}{p-2}\epsilon z(\theta_{s-t-h}\omega)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \beta_4^{\frac{2p}{p-2}}(x) dx\, ds \leq c\epsilon_1. \tag{113}$$

Let $T = \max\{T_1, T_2, T_3 . T_4\}$ and $R = \max\{R_1, R_2, R_3\}$. Then, it follows from (98), (105), (107), (109), (111) and (113) that

$$\sup_{-h\leq\sigma\leq 0} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v^t(\sigma, \theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 dx \leq (c + \frac{16}{\lambda} e^{\frac{\lambda h}{2}} + 3)\epsilon_1, \tag{114}$$

This ends the proof. □

Lemma 3.5. *Let $B(\omega) \in \mathscr{D}$ and $v_0(\omega) \in B(\omega)$. Then there are tempered random variables $\rho_5(\omega), \rho_6(\omega)$, such that the solution of (30)-(31) satisfies for all $t \geq T_B(\omega) + 2h + 1$, and $\sigma, \sigma_1, \sigma_2 \in [-h, 0]$,*

$$||v^t(\sigma, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2_{H^1(\mathbb{R}^n)} \leq \rho_5(\omega); \tag{115}$$

$$||v^t(\sigma_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)) - v^t(\sigma_2, \theta_{-t}\omega, v_0(\theta_{-t}\omega))|| \leq \rho_6(\omega)|\sigma_1 - \sigma_2|^{1/2}. \tag{116}$$

Proof. By Lemma 3.1 and Lemma 3.3, we get that for all $t \geq T_B(\omega) + h + 1$, and $\sigma \in [-h, 0]$,

$$\begin{aligned} &||v^t(\sigma, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2_{H^1(\mathbb{R}^n)} \\ &= ||v(t+\sigma, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2 + ||\nabla v(t+\sigma, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2 \\ &\leq \max_{-h\leq\sigma\leq 0} \left(\rho_1^2(\theta_\sigma\omega) + \rho_3(\theta_\sigma\omega)\right) \equiv \rho_5(\omega). \end{aligned} \tag{117}$$

For $\sigma_1, \sigma_2 \in [-h, 0]$ (assuming $\sigma_1 \leq \sigma_2$ for simplicity), one has that,

$$\begin{aligned} &||v^t(\sigma_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)) - v^t(\sigma_2, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2 \\ &= ||v(t+\sigma_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)) - v(t+\sigma_2, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2 \\ &= ||\int_{t+\sigma_1}^{t+\sigma_2} \frac{d}{dt} v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2 ds \\ &\leq \lambda \int_{t+\sigma_1}^{t+\sigma_2} ||v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||ds + \int_{t+\sigma_1}^{t+\sigma_2} ||\Delta v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||ds \\ &\quad + \int_{t+\sigma_1}^{t+\sigma_2} ||e^{-\epsilon z(\theta_{s-t}\omega)} F(x, e^{\epsilon z(\theta_{s-t}\omega)} v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)))||ds \\ &\quad + \int_{t+\sigma_1}^{t+\sigma_2} ||e^{-\epsilon z(\theta_{s-t}\omega)} G(x, e^{\epsilon z(\theta_{s-t-h}\omega)} v(s-h, \theta_{-t}\omega, v_0(\theta_{-t}\omega)))||ds \\ &\quad + \int_{t+\sigma_1}^{t+\sigma_2} ||e^{-\epsilon z(\theta_{s-t}\omega)} g||ds + \epsilon\mu \int_{t+\sigma_1}^{t+\sigma_2} ||z(\theta_{s-t}\omega) v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||ds. \end{aligned} \tag{118}$$

Now, we estimate each term on the right hand side of (118). By Lemma 3.1, and Lemma 3.3 we find that for all $t \geq T_B(\omega) + 2h + 1$,

$$\int_{t+\sigma_1}^{t+\sigma_2} ||v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||ds = \int_{t+\sigma_1}^{t+\sigma_2} ||v(s, \theta_{-s}(\theta_{s-t}\omega), v_0(\theta_{-s}(\theta_{s-t}\omega)))||ds \leq \max_{-h\leq\tau\leq 0} \rho_1(\theta_\tau\omega)|\sigma_1 - \sigma_2|; \tag{119}$$

$$\int_{t+\sigma_1}^{t+\sigma_2} ||z(\theta_{s-t}\omega) v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||ds \leq \max_{-h\leq\tau\leq 0} z(\theta_\tau\omega)\rho_1(\theta_\tau\omega)|\sigma_1 - \sigma_2|; \tag{120}$$

$$\int_{t+\sigma_1}^{t+\sigma_2} ||\Delta v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||ds \leq \left(\int_{t+\sigma_1}^{t+\sigma_2} ||\Delta v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||^2 ds\right)^{1/2} |\sigma_1 - \sigma_2|^{1/2} \leq \rho_4(\omega)|\sigma_1 - \sigma_2|^{1/2}; \tag{121}$$

and

$$\int_{t+\sigma_1}^{t+\sigma_2} ||e^{-\epsilon z(\theta_{s-t}\omega)} g||ds \leq \max_{-h\leq\tau\leq 0} e^{\epsilon|z(\theta_\tau\omega)|} ||g||\, |\sigma_1 - \sigma_2|. \tag{122}$$

Using conditions (A1) and (A2), we obtain

$$
\begin{aligned}
&\int_{t+\sigma_1}^{t+\sigma_2} ||e^{-\epsilon z(\theta_{s-t}\omega)} F(x, e^{\epsilon z(\theta_{s-t}\omega)} v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)))|| ds \\
&\le \int_{t+\sigma_1}^{t+\sigma_2} \alpha_2 e^{(p-2)\epsilon z(\theta_{s-t}\omega)} ||v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||^{p-1} + e^{-\epsilon z(\theta_{s-t}\omega)} ||\beta_2|| ds \\
&\le \left(\alpha_2 \max_{-h\le\tau\le 0} e^{(p+2)\epsilon|z(\theta_\tau\omega)|} \rho_1^{p-1}(\theta_\tau\omega) + \max_{-h\le\tau\le 0} e^{\epsilon|z(\theta_\tau\omega)|} ||\beta_2||\right) |\sigma_1 - \sigma_2|,
\end{aligned}
\tag{123}
$$

and

$$
\begin{aligned}
&\int_{t+\sigma_1}^{t+\sigma_2} ||e^{-\epsilon z(\theta_{s-t}\omega)} G(x, e^{\epsilon z(\theta_{s-t-h}\omega)} v(s-h, \theta_{-t}\omega, v_0(\theta_{-t}\omega)))|| ds \\
&\le \left(\max_{-h\le\tau\le 0} e^{\epsilon|z(\theta_\tau\omega)| + \epsilon|z(\theta_{\tau-h}\omega)|} \rho_1(\theta_{\tau-h}\omega) + ||\beta_5||\right) |\sigma_1 - \sigma_2|.
\end{aligned}
\tag{124}
$$

The lemma follows from (119) to (124). This ends the proof. □

Next, we use Ascoli theorem to show that Φ is $\mathscr{D}$ pullback asymptotically compact.

Lemma 3.6. *The random dynamical system Φ is $\mathscr{D}$ pullback asymptotically compact in $\mathfrak{S}$; that is, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the sequence $\{\Phi(t_n, \theta_{-t_n}, v_0(\theta_{-t_n}))\}_{n=1}^{\infty}$ has a convergent subsequence in $\mathfrak{S}$ provided $t_n \to \infty$ and $v_0(\theta_{-t_n}) \in B(\theta_{t_n}\omega)$.*

Proof. Let Q_k be the set of $\{x \in \mathbb{R}^n : |x| < k\}$. Since $t_n \to \infty$, there exists $N = N(B, \omega)$ such that $t_n > T_B(\omega) + 2h + 1$, for all $n > N$. Using Lemma 3.5, we obtain that for $n > N$ and $\sigma \in [-h, 0]$,

$$||v^{t_n}(\sigma, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega))||_{H^1(Q_k)} \le \rho_5(\omega). \tag{125}$$

This implies that $\{\Phi(t_n, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega))\}_{n=N}^{\infty}$ is relatively compact in $L^2(Q_k)$, since the embedding from $H^1(Q_k)$ to $L^2(Q_k)$ is compact. From Lemma 3.5, we can also obtain that for all $n > N$, and $\sigma_1, \sigma_2 \in [-h, 0]$,

$$||v^{t_n}(\sigma_1, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega)) - v^{t_n}(\sigma_1, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega))||_{L^2(Q_k)} \le \rho_6(\omega)|\sigma_1 - \sigma_2|^{1/2}. \tag{126}$$

This means that $\{\Phi(t_n, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega))\}_{n=N}^{\infty}$ is equicontinuous. By Ascoli Theorem we have that for fixed $k \in \mathbb{N}$, $\{v^{t_n}(\sigma, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega))\}_{n=1}^{\infty}$ is relatively compact in $C([-h, 0]; L^2(Q_k))$. Therefore, for each $k \in \mathbb{N}$, there exists a subsequence $\{\Phi(t_{n_i}, \theta_{-t_{n_i}}\omega, v_0(\theta_{-t_{n_i}}\omega))\}$ converges to $\eta_k(\cdot, \omega)$ in $C([-h, 0]; L^2(Q_k))$. Notice that, for fixed $\sigma \in [-h, 0]$ and $\omega \in \Omega$, $\eta_{k+1}(\sigma, \omega)$ coincides with $\eta_k(\sigma, \omega)$ on Q_k. Hence, for any $\epsilon > 0$, there exists $N_1 = N_1(\omega, \epsilon) \in \mathbb{N}$, such that for all $i > N_1$,

$$\sup_{\sigma\in[-h,0]} ||v^{t_{n_i}}(\sigma, \theta_{-t_{n_i}}\omega, v_0(\theta_{-t_{n_i}}\omega)) - \eta(\sigma, \omega)||_{L^2(Q_k)} \le \epsilon. \tag{127}$$

By Lemma 3.1, we have that for all $\sigma \in [-h, 0]$, and $i > N$,

$$||v^{t_{n_i}}(\sigma, \theta_{-t_{n_i}}\omega, v_0(\theta_{-t_{n_i}}\omega))||_{L^2(Q_k)} \le \sup_{\sigma\in[-h,0]} ||v^{t_{n_i}}(\sigma, \theta_{-t_{n_i}}\omega, v_0(\theta_{-t_{n_i}}\omega))||_{L^2(\mathbb{R}^n)} \le \rho_1(\omega). \tag{128}$$

Hence, for all $k \in \mathbb{N}$,

$$||\eta(\sigma, \omega)||_{L^2(Q_k)} \le \lim_{i\to\infty} ||v^{t_{n_i}}(\sigma, \theta_{-t_{n_i}}\omega, v_0(\theta_{-t_{n_i}}\omega))||_{L^2(Q_k)} \le \rho_1(\omega). \tag{129}$$

It follows that,

$$||\eta(\sigma, \omega)||_{L^2(\mathbb{R}^n)} \le \sup_{k\in\mathbb{N}} ||\eta(\sigma, \omega)||_{L^2(Q_k)} \le \rho_1(\omega). \tag{130}$$

Therefore, $\eta(\sigma,\omega) \in L^2(\mathbb{R}^n)$. By Lemma 3.4, for every $B \in \mathscr{D}$ and $\epsilon_1 > 0$, there exist $K = K(\omega, \epsilon_1)$ and $N_2 = N_2(\omega, \epsilon_1)$, such that for all $i > N_2$ and $k > K$,

$$\sup_{\sigma\in[-h,0]} ||v^{t_{n_i}}(\sigma, \theta_{-t_{n_i}}, v_0(\theta_{-t_{n_i}}))||_{L^2(|x|\geq k)} \leq \epsilon_1. \tag{131}$$

Notice that for all $l > k > K$ and $\sigma \in [-h, 0]$,

$$\begin{aligned} ||\xi(\sigma,\omega)||_{L^2(k\leq|x|<l)} &= \lim_{i\to\infty} ||v^{t_{n_i}}(\sigma, \theta_{-t_{n_i}}\omega, v_0(\theta_{-t_{n_i}}\omega))||_{L^2(k\leq|x|<l)} \\ &\leq \sup_{\sigma\in[-h,0], i>N_2} ||v^{t_{n_i}}(\sigma, \theta_{-t_{n_i}}\omega, v_0(\theta_{-t_{n_i}}\omega))||_{L^2(|x|\geq k)} \leq \epsilon_1. \end{aligned} \tag{132}$$

It follows that

$$\sup_{\sigma\in[-h,0]} ||\xi(\sigma,\omega)||_{L^2(|x|\geq k)} \leq \sup_{\sigma\in[-h,0], l>k} ||\xi(\sigma,\omega)||_{L^2(k\leq|x|<l)} \leq \epsilon_1. \tag{133}$$

Then, (113), (127) and (131) imply that for all $i > \max N_1, N_2$,

$$\begin{aligned} &\sup_{\sigma\in[-h,0]} ||v^{t_{n_i}}(\sigma, \theta_{-t_{n_i}}\omega, v_0(\theta_{-t_{n_i}}\omega)) - \xi(\sigma,\omega)|| \\ &\leq \sup_{\sigma\in[-h,0]} ||v^{t_{n_i}}(\sigma, \theta_{-t_{n_i}}\omega, v_0(\theta_{-t_{n_i}}\omega)) - \xi(\sigma,\omega)||_{L^2(Q_k)} + 2\sup_{\sigma\in[-h,0]} ||v^{t_{n_i}}(\sigma, \theta_{-t_{n_i}}\omega, v_0(\theta_{-t_{n_i}}\omega))||_{L^2(|x|\geq k)} \\ &\qquad +2\sup_{\sigma\in[-h,0]} ||\xi(\sigma,\omega)||_{L^2(|x|\geq k)} \\ &\leq 5\epsilon_1. \end{aligned} \tag{134}$$

This ends the proof. □

Theorem 3.7. *The random dynamical system Φ has a unique $\mathscr{D}$-random attractor in $\mathfrak{S}$.*

Proof. By Lemma 3.1, we know that Φ has a random absorbing set, and by Lemma 3.6, we obtain that Φ is $\mathscr{D}$-pullback asymptotically compact in $\mathfrak{S}$. Therefore, by Proposition 2.1, Φ has an unique $\mathscr{D}$-pullback attractor. This ends the proof. □

Competing interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgement: This work is partially supported by the National Natural Science Foundation of China under Grants 11301153 and 11271110, the Key Programs for Science and Technology of the Education Department of Henan Province under Grand 12A110007, and the Scientific Research Funds of Henan University of Science and Technology.

References

[1] Antoci F., Prizzi M., Reaction-diffusion equations on unbounded thin domains, *Topol. Methods Nonlinear Anal.*, 2011, 18, 283-302.
[2] Arnold L., Random Dynamical Systems, 1998, *New York:Springer-Verlag*.
[3] Bates P.W., Lisei H., Lu K., Attractors for stochastic lattics dynamical systems, *Stoch. Dyn.*, 2006, 6, 1-21.
[4] Bates P.W., Lu K., Wang B., Random attractors for stochastic reaction-diffusion equations on unbounded domains, *Journal of Differential Equations*, 2009, 246(2), 845-869.
[5] Bensoussan A., Da Prato G., Delfour M. C., et al., Representation and control of infinite dimensional systems, 1992, *Boston: Birkhauser*.
[6] Bouzahir H., Ezzinbi K., Global attractor for a class of partial functional differential equations with infinite delay, *Topics in Func. Differential and Difference Equa.*, 2001, 29, 63-71.

[7] Caraballo T., Marin-Rubio P., Valero J., Autonomous and non-autonomous attractors for differential equations with delays, *J. Diff. Equa.*, 2005, 208(1), 9-41.
[8] Caraballo T., Real J., Attractors for 2D-Navier-Stokes models with delays, *J. Diff. Equa.*, 2004, 205(2), 271-297.
[9] Caraballo T., Kloeden P.E., Schmalfuβ B., Exponentially stable stationary solutions for stochastic evolution equations and their perturbation, *Appl.Math.Optim.*, 2004, 50(9), 183-207.
[10] Chueshov I., Monotone Random Systems Theory and Applications, 2002, *New York:Springer-Verlag*.
[11] Crauel H., Debussche A., Flandoli F., Random attractors, *J. Dyn. Diff. Equs.*, 1997, 9, 307-341.
[12] Cui H., Langa J.A., Li Y., Regularity and structure of pullback attractors for reaction-diffusion type systems without uniqueness, *Nonlinear Analysis: Theory, Methods and Applications*, 2016, 140, 208-235.
[13] Ding X., Jiang J., Random attractors for stochastic retarded reaction-diffusion equations on unbounded domains, *Abstract and Applied Analysis*, 2013, Hindawi Publishing Corporation.
[14] Ding X., Jiang J., Random attractors for stochastic retarded lattice dynamical systems, *Abstract and Applied Analysis*, 2012, Hindawi Publishing Corporation.
[15] Fan X., Attractors for a damped stochastic wave equation of sina-Gordon type with sublinear multiplicative noise, 2006, *Stoc. Anal. Appl.*, 24, 767-793.
[16] Fan X., Chen H., Attractors for the stochastic reaction-diffusion equation driven by linear multiplicative noise with a variable coefficient, *J. Math. Anal. Appl.*, 2013, 398(2), 715-728.
[17] Flandoli F., Schmalfuβ B., Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise, *Stoch. Rep.*, 1996, 59, 21-45.
[18] Garcia-Luengo J., Marin-Rubio P., Reaction-diffusion equations with non-autonomous force in H-1 and delays under measurability conditions on the driving delay term, *Journal of mathematical analysis and applications*, 2014, 417(1), 80-95.
[19] Gawarecki L., Mandrekar V., Stochastic differential equations in infinite dimensions: with applications to stochastic partial differential equations, 2010, *Springer Science and Business Media*.
[20] Gorban N.V., Kapustyan O.V., Kasyanov P.O., Uniform trajectory attractor for non-autonomous reaction-diffusion equations with Caratheodory's nonlinearity, *Nonlinear Analysis*, 2014, 98, 13-26.
[21] Gu A., Xiang H., Upper semicontinuity of random attractors for stochastic three-component reversible Gray-Scott system, *App. Math. Comp.*, 2013, 225, 387-400.
[22] Hale J.K., Lin X., Raugel G., Upper semicontinuity of attractors for approximations of semigroups and PDE's, *Math. Comp.*, 1988, 50(181), 89-123.
[23] Hale J.K., Raugel G., Upper semicontinuity of the attractor for a singularly perturbed hyperbolic equation, *J. Diff. Equa.*, 1988, 73(2), 197-214.
[24] Imkeller P., Schmalfuss B., The conjugacy of stochastic and random differential equations and the existence of global attractors, *J. Dyn. Diff. Equ.*, 2001, 13, 215-249.
[25] Kloeden P.E., Langa J.A., Flattening,sequeezing and the existence of random attractors, *Proc. R. Soc. A*, 2007, 463, 163-181.
[26] Wang Z., Zhou S., Random attractor for stochastic reaction-diffusion equation with multiplicative noise on unbounded domains, *J. Math. Anal. Appl.*, 2011, 384, 160-172.
[27] Wang B., Upper semicontinuity of random attractors for non-compact random dynamical systems, *J. Diff. Equa.*, 2009, 139, 1-18.
[28] Wang X., Lu K., Wang B., Random attractors for delay parabolic equations with additive noise and deterministic nonautonomous forcing, *SIAM Journal on Applied Dynamical Systems*, 2015, 14(2), 1018-1047.
[29] Xiao Y., Sun C., Higher-order asymptotic attraction of pullback attractors for a reaction-diffusion equation in non-cylindrical domains, *Nonlinear Analysis: Theory, Methods and Applications*, 2015, 113, 309-322.
[30] Yan W., Li Y., Ji S., Random attractors for first order stochastic retarded lattice dynamical systems, *J. Math. Phy.*, 2010, 51(3), 032702.
[31] You Y., Global dynamics of an autocatalytic reaction-diffusion system with functional response, *J. App. Anal. Comp.*, 2011, 1(1): 121-142.
[32] You Y., Random attractors and robustness for stochastic reversible reaction-diffusion systems, *J. App. Anal. Comp.*, 2013, 34(1), 301-333.
[33] Zhao C., Zhou S., Attractors of retarded first order lattice systems, *Nonlinearity*, 2013, 20(8): 61-72.
[34] Zhao W., H1-random attractors for stochastic reaction-diffusion equations with additive noise, *Nonlinear Analysis: Theory, Methods and Application*, 2013, 84, 61-72.
[35] Zhou S., Attractors for second order lattice dynamical systems, *J. Diff. Equa.*, 2002, 179(2), 605-624.
[36] Zhou S., Wang Z., Random attractors for stochastic retarded lattice systems, *J. Diff. Equa. and Appl.*, 2013, 19(9), 1523-1543.