Open Mathematics

Research Article

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Algebras of right ample semigroups

https://doi.org/10.1515/math-2018-0075
Received November 1, 2017; accepted May 22, 2018.

Abstract: Strict RA semigroups are common generalizations of ample semigroups and inverse semigroups. The aim of this paper is to study algebras of strict RA semigroups. It is proved that any algebra of strict RA semigroups with finite idempotents has a generalized matrix representation whose degree is equal to the number of non-zero regular $\mathcal{D}$-classes. In particular, it is proved that any algebra of finite right ample semigroups has a generalized upper triangular matrix representation whose degree is equal to the number of non-zero regular $\mathcal{D}$-classes. As its application, we determine when an algebra of strict RA semigroups (right ample monoids) is semiprimitive. Moreover, we prove that an algebra of strict RA semigroups (right ample monoids) is left self-injective iff it is right self-injective, iff it is Frobenius, and iff the semigroup is a finite inverse semigroup.

Keywords: Right ample semigroup, Semigroup algebra, Generalized (upper triangular) matrix representation, Left self-injective algebra, Frobenius algebra

MSC: 20M30, 16G60

1 Introduction

The mathematical structures which encode information about partial symmetries are certain generalizations of groups, called inverse semigroups. Abstractly, inverse semigroups are regular semigroups each of whose elements has exactly one inverse; equivalently, a regular semigroup is inverse if and only if its idempotents commute. Like groups, inverse semigroups first arose in questions concerned with the solutions of equations, but this time in Lie's attempt to find the analogue of Galois theory for differential equations. The symmetries of such equations form what are now termed Lie pseudogroups, and inverse semigroups are, several times removed, the corresponding abstract structures. In addition to their early appearance in differential geometry, inverse semigroups have found a number of other applications in recent years including: C^*-algebras; tilings, quasicrystals and solid-state physics; combinatorial group theory; model theory; and linear logic. Inverse semigroups have been widely investigated. For inverse semigroups, the readers are referred to the monographs of Petrich [1] and Lawson [2]. Because of the important role of inverse semigroups in the theory of semigroups, there are attempts to generalize inverse semigroups. (Left; Right) ample semigroups originally introduced by Fountain in [3] are generalizations of inverse semigroups in the range of (left pp semigroups; right pp semigroups) abundant semigroups.

Inverse semigroup algebras are a class of semigroup algebras which is widely investigated. For example, Crabb and Munn considered the semiprimitivity of combinatorial inverse semigroup algebras (see [4]); algebras of free inverse semigroups (see [5, 6]); nil-ideals of inverse semigroup algebras (see [7]). More

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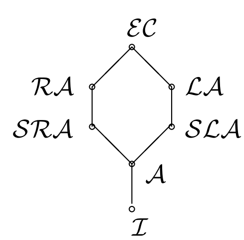
results on inverse semigroup algebras are collected in a survey of Munn [8]. Recently, Steinberg [9, 10] investigated representations of finite inverse semigroups. Inverse semigroups are ample semigroups and any ample semigroup can be viewed as a subsemigroup of some inverse semigroup. Ample semigroups include cancellative monoids and path semigroups of quivers. In [11, 12], Okniński studied algebras of cancellative semigroups. Guo and Chen [13] proved that any algebra of finite ample semigroups has a generalized upper triangular matrix representation. Guo and Shum [14] established the construction of algebras of ample semigroups each of whose $\mathcal{J}^*$-classes contains a finite number of idempotents. For the related results on semigroup algebras, the reader can be referred to the books of Okniński [15], and Jesper and Okniński [16].

Frobenius algebras are algebras with non-degenerate bilinear mappings. They are closely related to the representation theory of groups which appear in many branches of algebras, algebraic geometry and combinatorics. Frobenius algebras and their generalizations, such as quasi-Frobenius algebras and right (left) self-injective algebras, play an important role and become a central topic in algebra. Wenger proved an important result (Wenger Theorem): an algebra of an inverse semigroup is left self-injective iff it is right self-injective; iff it is quasi-Frobenius; iff the inverse semigroup is finite (see [17]). In [18], Guo and Shum determined when an ample semigroup algebra is Frobenius and generalized Wenger Theorem to ample semigroup algebras.

Right (Left) ample semigroups are known as generalizations of ample semigroups and include left (right) cancellative monoids as proper subclass. So, it is natural to probe algebras of right ample semigroups. This is the aim of paper. Indeed, any ample semigroup is both a right ample semigroup and a left ample semigroup. The symmetry property is a “strict" one in the theory of semigroups. It is interesting whether the semigroup algebras have the “similar" properties if we destroy the “symmetry". To be precise, for what “weak symmetry" assumption is the Wenger Theorem valid? By weakening the condition: S is a left ample semigroup, to the condition: Each $\widehat{\mathcal{L}}$-class of S contains an idempotent, we introduce strict RA semigroups (dually, strict LA semigroups), which include ample semigroups and inverse semigroups as its proper subclasses. The following picture illustrates the relationship between strict RA (LA) semigroups and other classes of semigroups:



where $\mathcal{EC}$ is the class of EC-semigroups (that is, semigroups whose idempotents commute), for EC-semigroups, see [19, 20]; $\mathcal{SRA}$ is the class of strict RA semigroups; $\mathcal{SLA}$ is the class of strict LA semigroups; $\mathcal{A}$ is the class of ample semigroups; $\mathcal{I}$ is the class of inverse semigroups, and the symbolic “|" means that the upper class of semigroups includes properly the lower one.

Our ideas in this paper are somewhat similar as in [13, 18] and inspired by the references [21, 22]. In Section 2, we obtain some properties of strict RA semigroups. Section 3 is devoted to the representation theory of generalized matrices for algebras of strict RA semigroups. It is verified that any algebra of a strict RA semigroup with finite idempotents has a generalized matrix representation whose degree is equal to the number of non-zero regular $\mathcal{D}$-classes (Theorem 3.2), extending the main result in [13]. In particular, we prove that any algebra of a finite right ample monoid has a generalized upper triangular matrix representation (Corollary 3.5). As applications of these representation theorems, we determine when an algebra of strict RA finite semigroups is semiprimitive (Theorem 3.10). This extends the related result of Guo and Chen in [13]. In particular, a sufficient and necessary condition for an algebra of right ample finite monoids to be semiprimitive is obtained (Corollary 3.11). In Section 4, we shall determine when an algebra of right ample semigroups is left self-injective. It is shown that an algebra of a strict RA semigroup is left self-injective, iff it is right self-injective, iff it is Frobenius and iff the Strict RA semigroup is a finite inverse semigroup (Theorem 4.11). This result extends the Wenger Theorem to the case for strict RA semigroups. Especially, it is verified that if an algebra of a right ample monoid is left (resp. right) self-injective, then the monoid is finite (Corollary 4.18). So, we give a positive answer to [15, Problem 6, p.328] for the case that S is a strict RA (LA) semigroup (especially, a right ample monoid). It is interesting to find that

– the "distance" between strict RA (LA) semigroups with finite inverse semigroups is the left (right) self-injectivity;
– for an algebra of strict RA (LA) semigroups, left (right) self-injectivity= quasi-Frobenoius = Frobenius.

In this paper we shall use the notions and notations of the monographs [23] and [15]. For right ample semigroups, the reader can be referred to [3].

2 Strict RA semigroups

Let S be a semigroup; we denote by $E(S)$ the set of idempotents of S, by S^1 the semigroup obtained from S by adjoining an identity if S does not have one.

To begin with, we recall some known results on Green's relations. For any $a, b \in S$, define

$$\begin{aligned} a\mathcal{L}b &\iff S^1a = S^1b \text{ i.e. } a = xb, b = ya \text{ for some } x, y \in S^1; \\ a\mathcal{R}b &\iff aS^1 = bS^1 \text{ i.e. } a = bu, b = av \text{ for some } u, v \in S^1; \\ \mathcal{H} &= \mathcal{L} \cap \mathcal{R}; \\ \mathcal{D} &= \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}. \end{aligned}$$

In general, $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence. a is regular in S if there exists $x \in S$ such that $axa = a$. Equivalently, a is regular if and only if $a\mathcal{D}e$ for some $e \in E(S)$. And, S is called regular if each element of S is regular in S. We use D_x to denote the $\mathcal{D}$-class of S containing x, and call a $\mathcal{D}$-class containing a regular element a regular $\mathcal{D}$-class. It is well known that any element of a regular $\mathcal{D}$-class is regular and that the $\mathcal{D}$-class containing the zero element 0 is just the set $\{0\}$.

As generalizations of Green's $\mathcal{L}$- and $\mathcal{R}$-relations, we have $\mathcal{L}^*$- and $\mathcal{R}^*$-relations defined on S by

$$\begin{aligned} &a\mathcal{L}^*b \text{ if } (ax = ay \Leftrightarrow bx = by \text{ for all } x, y \in S^1), \\ &a\mathcal{R}^*b \text{ if } (xa = ya \Leftrightarrow xb = yb \text{ for all } x, y \in S^1). \end{aligned}$$

It is well known that $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence. In general, $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. And, if a, b are regular, then $a\mathcal{L}\ (\mathcal{R})b$ if and only if $a\mathcal{L}^*\ (\mathcal{R}^*)b$.

Definition 2.1. *A semigroup S is* right ample *if*

(A) its idempotents commute;
(B) every element a is $\mathcal{L}^*$-related a (unique) idempotent a^*;
(C) for any $a \in S$ and $e \in E(S)$, $ea = a(ea)^*$.

Left ample semigroups are defined by duality. Moreover, a semigroup is *ample* if it is both left ample and right ample. Obviously, inverse semigroups are (left; right) ample semigroups.

As generalizations of $\mathcal{L}^*$ and $\mathcal{R}^*$, we have $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{R}}$ defined on S by

$$\begin{aligned} &a\widehat{\mathcal{L}}b \text{ if } (a = ae \Leftrightarrow b = be \text{ for all } e \in E(S^1)), \\ &a\widehat{\mathcal{R}}b \text{ if } (a = ea \Leftrightarrow b = eb \text{ for all } e \in E(S^1)). \end{aligned}$$

In general, $\widehat{\mathcal{L}}$ is not a right congruence and $\widehat{\mathcal{R}}$ is not a left congruence. Clearly, $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widehat{\mathcal{L}}$ and $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widehat{\mathcal{R}}$.

Lemma 2.2. *Let $a \in S$ and $e \in E(S)$. If $a\widehat{\mathcal{R}}e$, then $ea = a$. Moreover, if $g, h \in E(S)$ and $g\widehat{\mathcal{R}}h$, then $g\mathcal{R}h$.*

Proof. Since $e^2 = e$, the result follows from $a\widehat{\mathcal{R}}e$. The rest is trivial. □

By Lemma 2.2, it is evident that for regular elements a, b, $a\mathcal{L}\ (\mathcal{R})b$ if and only if $a\widehat{\mathcal{L}}\ (\widehat{\mathcal{R}})b$. Especially, on a regular semigroup $\mathcal{L} = \widehat{\mathcal{L}}$ and $\mathcal{R} = \widehat{\mathcal{R}}$.

Definition 2.3. *A semigroup S is* strict RA *if S is a right ample semigroup in which for any $a \in S$, there exists an idempotent e such that $a\widehat{\mathcal{R}}e$. Strict LA semigroups are defined by duality.*

For a strict RA semigroup, each $\widehat{\mathcal{R}}$-class contains exactly one idempotent; for, if e, f are idempotents and $e\widehat{\mathcal{R}}f$, then by Lemma 2.2, $e\mathcal{R}f$, so that $e = f$ since idempotents of a strict RA semigroup commute. We shall denote by $a^{\#}$ the unique idempotent in the $\widehat{\mathcal{R}}$-class containing a. Obviously, ample semigroups are strict LA (strict RA) semigroups. So, strict RA semigroups (strict LA semigroups) are common generalizations of ample semigroups and inverse semigroups. Indeed, strict RA semigroups include ample semigroups as its proper subclass; for, it is easy to see that left cancellative monoids are strict RA semigroups but not all of left cancellative monoids are ample semigroups.

Example 2.4. Assume that $Q = (V, E)$ is a quiver with vertices $V = \{1, 2, \cdots, r\}$. Denote by $(i_1|i_2|\cdots|i_n)$ the path: $\overset{i_1}{\circ}\to\overset{i_2}{\circ}\to\ \cdots\ \to\overset{i_n}{\circ}$. In particular, we call the path of Q without vertices the empty path of Q, denoted by 0. For any $i \in V$, we appoint to have an empty path e_i. Let $P(Q)$ be the set of all paths of Q. On $P(Q)$, define a multiplication by: for $(i_1|\cdots|i_m), (j_1|\cdots|j_n) \in P(Q)$,

$$(i_1|\cdots|i_m) \circ (j_1|\cdots|j_n) = \begin{cases} (i_1|\cdots|i_m|j_2|\cdots|j_n) & \text{if } i_m = j_1; \\ \\ 0 & \text{otherwise.} \end{cases}$$

Evidently, the path algebra RQ of the quiver Q over R is just the contracted semigroup $R_0[P(Q)]$. By a routine computation, $(P(Q), \circ)$ is a semigroup in which

- 0 is the zero element of $P(Q)$ and $e_1, e_2, \cdots, e_r, 0$ are all idempotents of $P(Q)$;
- $e_{i_1}\mathcal{R}^*(i_1|\cdots|i_m)\mathcal{L}^*e_{i_m}$;
- $e_ie_j = 0$ whenever $i \neq j$.

It is easy to see that $P(Q)$ is a strict RA semigroup having only finite idempotents.

Example 2.5. Let $\mathcal{C}$ be a small category and 0 a symbol. On the set

$$S(\mathcal{C}) := \left(\sqcup_{A,B\in Obj(\mathcal{C})} Hom(A, B)\right) \sqcup \{0\},$$

define: for $\alpha, \beta \in \sqcup_{A,B\in Obj(\mathcal{C})} Hom(A, B)$,

$$\alpha * \beta = \begin{cases} \alpha \circ \beta & \text{if } \alpha \text{ and } \beta \text{ can be composed,} \\ \\ 0 & \text{otherwise} \end{cases}$$

and $0 * \alpha = \alpha * 0 = 0 * 0 = 0$. It is a routine check that $(S(\mathcal{C}), *)$ is a semigroup with zero 0, called the *category semigroup* of $\mathcal{C}$. An arrow $\alpha \in Hom(A, B)$ is an *isomorphism* if there exists $\beta \in Hom(B, A)$ such that $\alpha\beta = 1_A$ and $\beta\alpha = 1_B$. We call $\mathcal{C}$ a *groupoid* if any arrow of $\mathcal{C}$ is an isomorphism (for groupoids, see [22] and their references); and *left* (resp. *right*) *cancellative* if for any arrows α, β, γ of $\mathcal{C}$, $\beta = \gamma$ whenever $\alpha\beta = \alpha\gamma$ (resp. $\gamma\alpha = \gamma\alpha$). And, $\mathcal{C}$ is *cancellative* if $\mathcal{C}$ is both left cancellative and right cancellative. It is not difficult to see that any groupoid is a cancellative category. Leech categories and Clifford categories are left cancellative categories (for these kinds of categories, see [24]).

(A) Assume that $\mathcal{C}$ is a left cancellative category. By computation, in the semigroup $S(\mathcal{C})$, for any $\alpha \in Hom(A, B)$,
  - $1_A\mathcal{L}^*\alpha\widehat{\mathcal{R}}1_B$;
  - $E(S(\mathcal{C})) = \{0\} \sqcup \{1_C : C \in Obj(\mathcal{C})\}$.

  It is not difficult to check that $S(\mathcal{C})$ is a strict RA semigroup. By dual arguments, if $\mathcal{C}$ is cancellative, then $S(\mathcal{C})$ is an ample semigroup.

(B) When $\mathcal{C}$ is a groupoid. In this case, there exists $\beta \in Hom(B, A)$ such that $\alpha\beta = 1_A$. It follows that $\alpha\beta\alpha = \alpha$. This means that α is regular and hence $S(\mathcal{C})$ is a regular semigroup. Again by the foregoing arguments, $S(\mathcal{C})$ is indeed an inverse semigroup.

Lemma 2.6. *Let S be a right ample semigroup and $E(S) \cdot S = S$. If $|E(S)| < \infty$ then S is strict RA.*

Proof. Let $a \in S$ and denote by e_a the minimal idempotent under the partial order ω on $E(S)$ (that is, $e\omega f$ if and only if $e = ef = fe$) such that $e_a a = a$. Obviously, for any $f \in E(S)$, $fe_a = e_a$ can imply that $fa = a$; conversely, if $fa = a$ then $fe_a \cdot a = a$ and $fe_a = e_a$ since $fe_a\omega e_a$. Thus $a\widehat{\mathcal{R}}e_a$, so S is strict RA. □

Corollary 2.7. *Let S be a right ample monoid. If S is finite, then S is strict RA.*

For a right ample semigroup S, define: for $a, b \in S$,

$$a \leq_r b \text{ if and only if } a = be \text{ for some } e \in E(S). \tag{1}$$

In (1), the idempotent e can be chosen as a^*; for, by $a = be$, we have $a = ae$, so that $a^* = a^*e$, thereby $a = aa^* = bea^* = ba^*$. So, $a \leq_r b$ if and only if $a = ba^*$. It is not difficult to check that $\leq_r$ is a partial order on S. Moreover, we have

Lemma 2.8. *If S is a right ample semigroup, then with respect to $\leq_r$, S is an ordered semigroup.*

Proof. The proof is a routine check and we omit the detail. □

For an element a of a right ample semigroup S, we denote

$$O(a) = \{x \in S : x \leq_r a\}.$$

If $x \leq_r a$ then $x = ax^*$, so that $x = aa^*x^* = xa^*$, hence $x^* = x^*a^*$, therefore $x^*\omega a^*$, that is, $x^* \in \omega(a^*)$ where $\omega(a^*) = \{e \in E(S) : e\omega a^*\}$. On the other hand, for $f, h \in \omega(a^*)$, if $af = ah$ then $f = a^*f = a^*h = h$. Consequently, $O(a) = \{af : f \in \omega(a^*)\}$ and $|O(a)| = |\omega(a^*)|$.

Proposition 2.9. *Let S be a strict RA semigroup and $a, b, x \in S$. If $x \leq_r ab$, then there exist uniquely $u, v \in S$ such that*

(O1) $u \leq_r a, v \leq_r b$;
(O2) $u^\# = x^\#, u^* = v^\#$ and $x^* = v^*$;
(O3) $x = uv$.

Proof. Denote $v = (x^\# a)^* bx^*$ and $u = x^\# av^\#$. We have $uv = x^\# av^\# \cdot (x^\# a)^* bx^* = x^\# abx^* = x$. Because S is a right ample semigroup, we have $x^\# a^\# \leq_r a^\#$ and $a^*v^\# \leq_r a^*$, so that $u = x^\# a^\# \cdot a \cdot a^*v^\# \leq_r a^\# \cdot a \cdot a^* = a$; and $(x^\# a)^* b^\# \leq_r b^\#$, $b^*x^* \leq_r b^*$, so that $v = (x^\# a)^* b^\# \cdot b \cdot b^*x^* \leq_r b^\# bb^* = b$.

Note that $v = (x^\# a)^* bx^* = vx^*$, we observe $v^* = v^*x^*$. On the other hand, since $x = uv = uvv^* = xv^*$, we get $x^* = x^*v^*$. Thus $x^* = v^*$ since $E(S)$ commutes.

Indeed, by $u = x^\# av^\#$, we have $u = uv^\#$, so that $u^* = u^*v^\#$. It follows that $u^* \leq_r v^\#$. Consider that $\mathcal{L}^*$ is a right congruence, we have $u^*\mathcal{L}^*u = x^\# av^\# \mathcal{L}^* (x^\# a)^* v^\#$ and $u^* = (x^\# a)^* v^\#$ since each $\mathcal{L}^*$-class of S contains exactly one idempotent. So,

$$v = v^\# v = v^\# (x^\# a)^* bx^* = v^\# (x^\# a)^* v = u^* v,$$

further by definition, $v^\# = u^*v^\#$. It follows that $v^\# \leq_r u^*$. Therefore $u^* = v^\#$.

By the definition of u, $u = x^\# u$ and $x^\# u^\# = u^\#$. But $u^\# u = u$, so $u^\# x = x$. It follows that $u^\# x^\# = x^\#$. Therefore $u^\# = x^\#$ since $E(S)$ commutes.

Finally, we let m, n be elements of S satisfying the properties of u and those of v in $(O1)$, $(O2)$ and $(O3)$, respectively. Then $m^* = n^\#$ and $v^* = x^* = n^*$. By the arguments before Lemma 2.8, $n = bn^* = bv^* = v$ and $m = am^* = an^\#$. By the first equality, $n^\# = v^\#$. Thus $u = x^\# av^\# = x^\# an^\# = x^\# m = m^\# m = m$. This proves the uniqueness of u and v. □

Assume now that T is a strict RA semigroup with zero element 0. Let E be the set of nonzero idempotents of T. For any $a \in T\backslash\{0\}$, $(a)_{ij}$ will denote the $E \times E$ matrix with entry a in the (i, j) position and zeros elsewhere. Also, we still use 0 to denote the $E \times E$ matrix each of whose entries is 0. Set

$$M(T) = \{(a)_{ij} : ia = a = aj\} \cup \{0\}$$

and define an operation on $M(T)$ by: $A, B \in M(T)$,

$$AB = \begin{cases} 0 & \text{if } A = 0 \text{ or } B = 0; \\ 0 & \text{if } A = (a)_{ij}, B = (b)_{kl} \text{ and } j \neq k; \\ (ab)_{il} & \text{if } A = (a)_{ij}, B = (b)_{kl} \text{ and } j = k. \end{cases}$$

It is easy to check that with respect to the above operation, $M(T)$ is a semigroup with zero element 0. Moreover, we can observe

Lemma 2.10. *The zero* 0 *and the elements* $(i)_{ii}$ *(*$i \in E$*) are all idempotents of* $M(T)$*, and orthogonal each other.*

Write $X(T) = \{0\} \cup \{(a)_{ij} : a^{\#} = i, a^* = j\}$. We denote by $PRA(T)$ the subsemigroup of $M(T)$ generated by $X(T)$. It is not difficult to see that

$$PRA(T) = \{(a)_{ij} : \text{there exist } x_1, x_2, \cdots, x_n \in T \text{ such that } a = x_1x_2\cdots x_n \\ x_1^{\#} = i, x_n^* = j, x_k^* = x_{k+1}^{\#} \text{ for } k = 1, 2, \cdots, n-1\}.$$

In what follows, for any $a \in T$, we use $\overline{a}$ to denote $(a)_{ij}$ with $a^{\#} = i$, $a^* = j$.

Proposition 2.11. *In the semigroup* $PRA(T)$*, for any* $(a)_{ij} \in PRA(T)$*, we have*
(A) $a^* = j$.
(B) $(a)_{ij}$ *is regular if and only if* a *is regular in* T*. Moreover, if* a *is regular, then* $(a)_{ij} \in X(T)$.
(C) $(a)_{ij}\mathcal{L}^*(j)_{jj}$.

Proof. (A) By the definition of $PRA(T)$, there exist $x_1, x_2, \cdots, x_n \in T$ such that $a = x_1x_2\cdots x_n$, $x_k^* = x_{k+1}^{\#}$ where $k = 1, 2, \cdots, n-1$, $x_1^{\#} = i$ and $x_n^* = j$. Since $\mathcal{L}^*$ is a left congruence, we have

$$\begin{aligned} a &= x_1x_2\cdots x_n\mathcal{L}^*x_1^*x_2\cdots x_n \\ &= x_2^{\#}x_2\cdots x_n\mathcal{L}^*\cdots\mathcal{L}^*x_{n-1}^*x_n = x_n\mathcal{L}^*x_n^* \\ &= j \end{aligned}$$

and further since each $\mathcal{L}^*$-class of a left ample semigroup contains exactly one idempotent, we get $a^* = j$.

(B) Let $(a)_{ij}$ be regular. Then there exists $(x)_{ji} \in PRA(T)$ such that

$$(a)_{ij}(x)_{ji}(a)_{ij} = (a)_{ij}.$$

It follows that $axa = a$, whence a is regular in T.

Conversely, assume that a is regular in T and let y be an inverse of a in T. Since $x_1x_2\cdots x_nyx_1x_2\cdots x_n = x_1x_2\cdots x_n$ and $x_1\mathcal{L}^*i_1$, we have $x_2\cdots x_nyx_1x_2\cdots x_{n-1}x_n = x_2\cdots x_{n-1}x_n$, and hence

$$\begin{aligned} x_3\cdots x_{n-1}x_n \cdot yx_1 \cdot x_2\cdots x_{n-1} &= x_2^*x_3\cdots x_{n-1}x_n \cdot yx_1 \cdot x_2\cdots x_n \\ &= x_2^*x_3\cdots x_{n-1}x_n \cdot yx_1x_2 \cdot x_3\cdots x_{n-1}x_n \\ &= x_3^{\#}x_3\cdots x_{n-1}x_n \\ &= x_3\cdots x_n. \end{aligned}$$

It follows that $x_3\cdots x_n$ is regular. Continuing this process we have that x_n is regular. By Lemma 2.2, $x_n^{\#}\mathcal{R}x_n$. Now, $x_1x_2\cdots x_nyx_1x_2\cdots x_n = x_1x_2\cdots x_n$ can imply that $x_1x_2\cdots x_nyx_1x_2\cdots x_{n-1}x_n^{\#} = x_1x_2\cdots x_{n-1}x_n^{\#}$, whence $x_1\cdots x_{n-1} = x_1\cdots x_{n-1} \circ x_ny \circ x_1x_2\cdots x_{n-1}$ and further regular in T. By the foregoing proof, x_{n-1} is regular. Applying these arguments to $x_1 \cdots x_{n-2}$, we know that x_{n-2} is regular, therefore x_l is regular for $l = 1, \cdots, n$. But $x_k \mathcal{L}^* x_k^*$, so $x_k \mathcal{L} x_k^*$ and as $\mathcal{L}$ is a right congruence,

$$\begin{aligned} a &= x_1 x_2 \cdots x_n \mathcal{L} x_1^* x_2 \cdots x_n \\ &= x_2 \cdots x_n \mathcal{L} \cdots \mathcal{L} x_{n-1}^* x_n = x_n \mathcal{L} x_n^* \\ &= j \end{aligned}$$

and $a^* = j$ since each $\mathcal{L}^*$-class of a strict RA semigroup contains exactly one idempotent. On the other hand, since $\mathcal{R}$ is a left congruence, we have

$$\begin{aligned} a &= x_1 x_2 \cdots x_n \mathcal{R} x_1 x_2 \cdots x_{n-1} x_n^{\#} \\ &= x_1 x_2 \cdots x_{n-1} \mathcal{R} \cdots \mathcal{R} x_1 x_2^{\#} = x_1 \mathcal{R} x_1^{\#} \\ &= i \end{aligned}$$

and $a^{\#} = i$ since each $\mathcal{R}^*$-class of a left ample semigroup contains exactly one idempotent. We have now proved that $(a)_{ij} \in X(T)$. It follows that

$$ay = a^{\#} = i \text{ and } ya = a^* = j.$$

Furthermore, $(y)_{ji} \in PRA(T)$. Consequently, $(a)_{ij}$ is regular since $(a)_{ij}(y)_{ji}(a)_{ij} = (aya)_{ij} = (a)_{ij}$.

(C) Now let $(x)_{jk}, (y)_{jl} \in PRA(T)$. If $(a)_{ij}(x)_{jk} = (a)_{ij}(y)_{jl}$, then $(ax)_{ik} = (ay)_{il}$, hence $k = l$ and $ax = ay$. By the second equality and applying (A), we have $jx = jy$, and further $(j)_{jj}(x)_{jk} = (jx)_{ik} = (jy)_{il} = (j)_{jj}(y)_{jl}$; if $(a)_{ij}(x)_{jk} = (a)_{ij}$, then $(a)_{ij}(x)_{jk} = (a)_{ij} = (a)_{ij}(j)_{jj}$, and by the foregoing proof, $(j)_{jj}(x)_{jk} = (j)_{jj}(j)_{jj}$. We have now proved that for all $(x)_{jk}, (y)_{jl} \in PRA(T)^1$, we have $(j)_{jj}(x)_{jk} = (j)_{jj}(y)_{jl}$ whenever $(a)_{ij}(x)_{jk} = (a)_{jl}$. This and the equality $(a)_{ij}(j)_{jj} = (a)_{ij}$ derive that $(a)_{ij} \mathcal{L}^* (j)_{jj}$. □

3 Generalized matrix representations

Let $R_1, R_2, \cdots, R_n$ be associative rings (algebras) with identity and let R_{ij} be a left R_i- right R_j-bimodule for $i, j = 1, 2, \cdots, n$ and $i < j$. We call the formal $n \times n$ matrix

$$\begin{pmatrix} a_1 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_n \end{pmatrix}$$

with $a_i \in R_i, a_{ij} \in R_{ij}$ for $i, j = 1, 2, \cdots, n$, a *generalized $n \times n$ matrix*. For $1 \le i, j, k \le n$, there is a (R_i, R_k)-bimodule homomorphism $\phi_{ijk} : R_{ij} \otimes_{R_j} R_{jk} \to R_{ik}$ such that the square

$$\begin{tikzcd} R_{ij} \otimes R_{jk} \otimes R_{kl} \arrow[r, "id_{ij} \otimes \phi_{jkl}"] \arrow[d, "\phi_{ijk} \otimes id_{kl}"'] & R_{ij} \otimes R_{jl} \arrow[d, "\phi_{ijl}"] \\ R_{ik} \otimes R_{kl} \arrow[r, "\phi_{ikl}"'] & R_{il} \end{tikzcd}$$

is commutative. We denote $(a \otimes b)\phi_{ijk}$ by ab. With respect to matrix addition and matrix multiplication, the set

$$\begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ R_{21} & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_n \end{pmatrix}$$

of all generalized matrices is an R-algebra, called a *generalized matrix algebra* of degree n. If $R_{ij} = 0$ for any $1 \le j \le i \le n$, then we call the generalized matrix algebra a *generalized upper matrix algebra* of degree n.

Definition 3.1. *A ring (An algebra) has a* generalized matrix representation of degree n *if there exists a ring (an algebra) isomorphism*

$$\phi : \mathfrak{A} \to \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ R_{21} & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_n \end{pmatrix}.$$

For a semigroup S with zero 0, we denote by $D_{reg}(S)$ the number of nonzero regular $\mathcal{D}$-classes of S. By [23, Proposition 3.2, p.45], $D_{reg}(S) = |\{D_e : e \in E(S)\backslash\{0\}\}| = |E(S)/\mathcal{D}^S| - 1$. We arrive at the main result of this section.

Theorem 3.2 (Generalized Matrix Representation Theorem). *Let S be a strict RA semigroup and R a commutative ring with unity. If $|E(S)| < \infty$ then $R_0[S]$ has a generalized matrix representation of degree $D_{reg}(S)$.*

Proof. Define a map φ by

$$\varphi : S \to R_0[PRA(S)]; \quad s \mapsto \sum_{u \in O(s)} (u)_{u^\#, u^*}$$

and span this map linearly to $R_0[S]$. By the arguments before Proposition 2.9, $|O(s)| = |\omega(s^*)| \leq |E(S)| < \infty$ and hence φ is well defined.

For $s, t \in S$, by Proposition 2.9, there exist uniquely $x \in O(s), y \in O(t)$ such that $u = xy$, $x^\# = u^\#$, $x^* = y^\#$ and $y^* = u^*$, for any $u \in O(st)$. So,

$$\begin{aligned} \varphi(st) &= \sum_{u \in O(st)} (u)_{u^\#, u^*} \\ &= \sum_{u = xy, x^\# = u^\#, x^* = y^\#, y^* = u^*, x \in O(s), y \in O(t)} (xy)_{x^\#, y^*} \\ &= \sum_{u = xy, x^\# = u^\#, x^* = y^\#, y^* = u^*, x \in O(s), y \in O(t)} (x)_{x^\#, x^*} (y)_{y^\#, y^*} \qquad \text{by Proposition 2.9} \\ &= \left(\sum_{x \in O(s)} (x)_{x^\#, x^*} \right) \left(\sum_{y \in O(t)} (y)_{y^\#, y^*} \right) \\ &= \varphi(s)\varphi(t) \end{aligned}$$

and φ is a homomorphism.

Let $x = \sum_{u \in supp(x)} r_u u$, $y = \sum_{v \in supp(y)} r_v v \in R_0[S]\backslash\{0\}$ and $\varphi(x) = \varphi(y)$. Denote

$$\begin{aligned} &\Lambda(x) = \{a^* : a \in supp(x)\}; \\ &m(x, i) = \sum_{u \in supp(x), u^* = i} r_u u, \end{aligned}$$

and let $Max(x)$ be the set of maximal elements of $\Lambda(x)$ under $\leq_r$. Let $Max^*(x) = \{a^* : a \in Max(x)\}$ and $M^1(x) = \sum_{i \in Max^*(x)} m(x, i)$. Define recursively

$$M^k(x) = \sum_{l \in Max^*\left(x - \sum_{j=1}^{k-1} \varphi(M^j(x))\right)} m\left(x - \sum_{j=1}^{k-1} \varphi(M^j(x)), l \right),$$

where $M^0(x) = 0$ and $M^i(0) = 0$ for any positive integer i. By the definition of φ, there must be a positive integer n such that $M^n(x) = 0$. Let $n(x)$ be the smallest integer such that $M^{n(x)}(x) = 0$. Now again by the definition of φ, it is not difficult to see that $x = \sum_{i=1}^{n(x)} M^i(x)$. Because $R_0[S]$ is a free R-module with a basis $S\backslash\{0\}$, $\varphi(x) = \varphi(y)$ can imply that $M^1(x) = M^1(y)$. It follows that $\varphi(x - M^1(x)) = \varphi(y - M^1(y))$. By the foregoing proof, $M^2(x) = M^1(x - M^1(x)) = M^1(y - M^1(y)) = M^2(y)$. Continuing this process, we have $n(x) = n(y)$ and $M^k(x) = M^k(y)$ for $2 \leq k \leq n(x)$. Therefore

$$x = \sum_{i=1}^{n(x)} M^i(x) = \sum_{i=1}^{n(y)} M^i(y) = y,$$

and φ is injective.

Now let $|E(S)| = n+1$. Then we may assume that $E(S)\backslash\{0\} = \{e_1, e_2, \cdots, e_n\}$. For any $(a)_{ij} \in PRA(S)$, we have $ia = a$ and $a^* = j$. It follows that $aj = a$. Note that $i = e_k$ and $j = e_l$ for some $1 \le k, l \le n$. Thus $(e_k)_{ii}(a)_{ij} = (a)_{ij} = (a)_{ij}(e_l)_{jj}$ and

$$\begin{aligned}\left(\sum_{p=1}^{n} \overline{e_p}\right) \cdot (a)_{ij} &= (e_k)_{ii}(a)_{ij} \\ &= (a)_{ij} \\ &= (a)_{ij}(e_l)_{jj} \\ &= (a)_{ij} \cdot \left(\sum_{p=1}^{n} \overline{e_p}\right).\end{aligned}$$

Therefore $1 = \sum_{p=1}^{n} \overline{e_p}$ is the identity of $R_0[PRA(S)]$. So, $R_0[S]$ has an identity.

For convenience, we identify $R_0[S]$ with $T := \varphi(R_0[S])$. If $\overline{e_k}T \cong \overline{e_l}T$, then there exist $x \in \overline{e_k}R_0[T]\overline{e_l}$, $y \in \overline{e_l}R_0[T]\overline{e_k}$ such that $\overline{e_k} = xy$, $yx = \overline{e_l}$. Thus there are $a \in$supp(x), $b \in$supp(y) such that $\overline{e_k} = ab$, $ba = \overline{e_l}$. As $x \in \overline{e_k}R_0[T]\overline{e_l}$, we get $\overline{e_k}a = a$. Now, $\overline{e_k}\mathcal{R}a$ and a is a regular element of $PRA(S)$. By Proposition 2.11, $a = (u)_{pq}$ for some a regular element u of S. But

$$(e_k u e_l)_{e_k,e_l} = (e_k)_{e_k,e_k}(u)_{pq}(e_l)_{e_l,e_l} = \overline{e_k}a\overline{e_l} = a = (u)_{pq},$$

so by the multiplication of $PRA(S)$, $e_k = p$, $e_l = q$ and hence $e_k = u^{\#}$, $e_l = u^*$. Thus $e_k\mathcal{R}u\mathcal{L}e_l$. In other words, $e_k\mathcal{D}e_l$ in the semigroup S. Now we prove that if $\overline{e_k}T \cong \overline{e_l}T$, then $e_k\mathcal{D}e_l$ in the semigroup S. Because the reverse is obvious, it is now verified that $\overline{e_k}T \cong \overline{e_l}T$ if and only if $e_k\mathcal{D}e_l$ in the semigroup S.

Let $\pi = \cup_{i=1}^{r} E_i$ be the partition of $E(S)\backslash\{0\}$ induced by $\mathcal{D}|_{E(S)}$ and let $\{f_1, f_2, \cdots, f_r\}$ be representatives of this partition π. Moreover, we let $n_k = |E(D_{f_k})|$ where D_{f_k} is the $\mathcal{D}$-class of S containing f_k. (Of course, $r = D_{reg}(S)$.) Then by the foregoing proof,

$$T = \oplus_{i=1}^{n} \overline{e_i}T \cong \oplus_{k=1}^{r} n_k \overline{f_k}T$$

and as $\overline{f_k}$'s are mutually orthogonal, T is isomorphic to the generalized matrix algebra

$$\begin{pmatrix} M_{n_1}(\overline{f_1}T\overline{f_1}) & M_{n_1,n_2}(\overline{f_1}T\overline{f_2}) & \cdots & M_{n_1,n_r}(\overline{f_1}T\overline{f_r}) \\ M_{n_2,n_1}(\overline{f_2}T\overline{f_1}) & M_{n_2}(\overline{f_2}T\overline{f_2}) & \cdots & M_{n_2,n_r}(\overline{f_2}T\overline{f_r}) \\ \cdots & \cdots & \cdots & \cdots \\ M_{n_r,n_1}(\overline{f_r}T\overline{f_1}) & M_{n_r,n_2}(\overline{f_r}T\overline{f_2}) & \cdots & M_{n_r}(\overline{f_r}T\overline{f_r}) \end{pmatrix}. \tag{2}$$

By the construction of $PRA(S)$, $\overline{f_k}T\overline{f_l} = R_0[\overline{M_{kl}}])$ where $\mathcal{N}$ is the set of positive integers and

- $M_{ij} = \left\{\prod_{k=1}^{m} x_i : m \in \mathcal{N}, x_1, \cdots, x_k \in S, x_1^{\#} = f_i, x_m^* = f_j, x_k^* = x_{k+1}^* \text{ for } 1 \le k \le m-1\right\}$;
- $\overline{M_{ij}} = \{(a)_{f_i,f_j} : a \in M_{ij}\}$.

The proof is finished. □

Example 3.3. Let S be a semigroup each of whose $\mathcal{L}^*$-classes contains at least one idempotent, and assume that the idempotents of S are in the center of S. By [25], S is a strong semilattice Y of left cancellative monoids M_α with $\alpha \in Y$. A routine check can show that S is a strict RA semigroup in which for any $a \in S$, $a^{\#} = a^* = f_\alpha$, where f_α is the identity of M_α. It is not difficult to see that $PRA(S) = \sum_{\alpha \in Y}\{(a)_{f_\alpha,f_\alpha} : a \in M_\alpha\}$. Now let $f_1, f_2, \cdots, f_n$ be all nonzero idempotents of S and f_i be the identity of the left cancellative monoid M_{α_i} for $i = 1, 2, \cdots, n$. With notations in the proof of Theorem 3.2, $M_{i,j} = \varnothing$ when $i \neq j$. So, by Theorem 3.2, $R_0[S]$ is isomorphic to the generalized matrix algebra $diag(R[M_{\alpha_1}], R[M_{\alpha_2}], \cdots, R[M_{\alpha_n}])$.

As in [26], a ring (an algebra) $\mathfrak{A}$ has a *generalized upper triangular matrix representation of degree n* if there exists a ring (algebra) isomorphism

$$\phi : \mathfrak{A} \to \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix}.$$

Theorem 3.4 (Generalized Triangular Matrix Representation Theorem). *Let S be a strict RA semigroup and R a commutative ring. If*
(A) $|E(S)| < \infty$; and
(B) for any $e \in E(S)$, $M_{e,e} = \{x \in S : ex = x, e = x^*\}$ is a subgroup of S,
then $R_0[S]$ has a generalized upper triangular matrix representation of degree $D_{reg}(S)$.

Proof. Let us turn back to the proof of Theorem 3.2. By hypothesis, M_{f_i,f_i} is a subgroup of S, in other words, any element $x \in M_{f_i,f_i}$ is in a subgroup of S, and so $x\mathcal{H}e$ for some $e \in E(S)$. But $x^* = f_i$, now $e\mathcal{L}f_i$ and further $e = f_i$. Thus $x^\# = f_i = x^*$. It follows that $x \in M_{ii}$ and $M_{f_i,f_i} \subseteq M_{ii}$. On the other hand, it is clear that $M_{ii} \subseteq M_{f_i,f_i}$. Therefore $M_{ii} = M_{f_i,f_i}$ and is a subgroup of S.

We may claim:

Claim A. *For any $1 \leq k, l \leq r$ with $k \neq l$, if $M_{kl} \neq \varnothing$, then $M_{lk} = \varnothing$.*

Indeed, if otherwise, we pick $a \in M_{kl}$, $b \in M_{lk}$, and $(a)_{f_k,f_l}, (b)_{f_l,f_k} \in PRA(S)$. Also, $a^* = f_l$, $f_k a = a$, $f_l b = b$ and $b^* = f_k$, hence $ab\mathcal{L}^*b$, $ba\mathcal{L}^*a$. It follows that $(ab)_{f_k,f_k}, (ba)_{f_l,f_l} \in PRA(S)$. Thus $ab \in M_{kk}$, $ba \in M_{ll}$. But M_{kk}, M_{ll} are both subgroups, so $ab\mathcal{H}f_k$, $ba\mathcal{H}f_l$. It follows that a and b are regular in S, and $a\mathcal{R}f_k\mathcal{L}b$, $a\mathcal{L}f_k\mathcal{R}b$. Thus $f_k\mathcal{D}f_l$. This is contrary to that $\{f_1, f_2, \cdots, f_r\}$ are representatives of the partition of $E(S)\backslash\{0\}$ induced by $\mathcal{D}|_{E(S)}$. So, we prove Claim A.

We next verify:

Claim B. *For any $1 \leq i, j, k \leq r$, if $M_{ij} \neq \varnothing$ and $M_{jk} \neq \varnothing$, then $M_{ik} \neq \varnothing$.*

Pick $x \in M_{ij}, y \in M_{jk}$. So, $x^* = f_j$, $f_j y = y$ and $xy\mathcal{L}^*y\mathcal{L}^*f_k$. It follows that $(x)_{f_i,f_j}(y)_{f_j,f_k} = (xy)_{f_i,f_k}$. Hence $xy \in M_{ik}$ and $M_{ik} \neq \varnothing$. This results in Claim B.

Consider the quiver Q whose vertex set is $V = \{1, 2, \cdots, r\}$ and in which there is an edge from i to j if and only if $M_{ij} \neq \varnothing$. By Claim B, there is a path from i to l if and only if $M_{il} \neq \varnothing$. Again by Claim A, the quiver Q has no cycles. Now by [41, Corollary, p.143], the vertices of Q can be labeled $V = \{1, 2, \cdots, r\}$ in such a way that if there is an edge from i to j then $i < j$. This means that we can relabel $V = \{1, 2, \cdots, r\}$ such that if $M_{ij} \neq \varnothing$ then $i < j$. In this case, the algebra (2) is a generalized upper triangular matrix algebra. The proof is completed. □

Let us turn back to the proof of Theorem 3.2 again. Assume now that S is finite. For $x, y \in M_{ii}$, we have $x^* = f_i = y^*, f_i x = x$ and $f_i y = y$. So, $xy\mathcal{L}^*f_i y = y\mathcal{L}^*f_i$. Thus $xy \in M_{ii}$ and M_{ii} is a subsemigroup of S. For $a \in M_{ii}$, if $ax = ay$, then $x = f_i x = f_i y = y$ and M_{ii} is left cancellative. But M_{ii} is finite, so M_{ii} is a subgroup. Now by Theorem 3.4, the following corollary is immediate.

Corollary 3.5. *Let S be a strict RA semigroup and R a commutative ring. If S is finite, then $R_0[S]$ has a generalized upper triangular matrix representation of degree $D_{reg}(S)$.*

Remark 3.6. Note that ample semigroups are strict RA semigroups. By Corollary 3.5, any algebra of ample finite semigroups has a generalized upper matrix representation. So, Corollary 3.5 extends the triangular matrix representation theorem on algebras of ample finite semigroups [13, Theorem 4.5].

By Corollary 2.7, any right ample finite monoid is strict RA. So, we have the following corollary.

Corollary 3.7. *Let S be a right ample monoid and R a commutative ring. If S is finite, then $R_0[S]$ has a generalized upper triangular matrix representation of degree $D_{reg}(S)$.*

Example 3.8. With notation in Example 2.4, by Theorem 3.2, $R_0[P(Q)]$ is isomorphic to the generalized matrix algebra

$$\begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ R_{21} & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_n \end{pmatrix},$$

where R_i is the subalgebra of $R_0[P(Q)]$ generated by all paths from i to i, and R_{ij} is the free module with the set of all paths from i to j as a base.

Now let Q have no loops. In this case, $M_{e_i,e_i} = \{e_i\}$ is a subgroup of $P(Q)$. By Theorem 3.4, $R_0[P(Q)]$ has a generalized upper triangular matrix representation.

Example 3.9. Let $\mathcal{C}$ be a left cancellative category with $|Obj(\mathcal{C})| < \infty$. By Example 2.5, $S(\mathcal{C})$ is a strict RA semigroup in which $|E(S(\mathcal{C}))| < \infty$. Obviously, the relation

$$\pi = \{(1_A, 1_B) : A, B \in Obj(\mathcal{C}) \text{ and there exist arrows } \alpha, \beta \text{ such that } 1_A = \alpha\beta, 1_B = \beta\alpha\}$$

is an equivalence on the set $Y := E(S(\mathcal{C}))\backslash\{0\}$. Let $Y/\pi = \{Y_1, Y_2, \cdots, Y_r\}$. Moreover, we let $X_i = \{A \in Obj(\mathcal{C}) : 1_B \in Y_i\}$ and $n_i = |X_i|$. If pick $A_i \in X_i$, then by Theorem 3.2, $R_0[S(\mathcal{C})]$ is isomorphic to the generalized matrix algebra

$$\begin{pmatrix} M_{n_1}(R[Hom(A_1,A_1)]) & M_{n_1,n_2}(R[Hom(A_1,A_2)]) & \cdots & M_{n_1,n_r}(R[Hom(A_1,A_r)]) \\ M_{n_2,n_1}(R[Hom(A_2,A_1)]) & M_{n_2}(R[Hom(A_2,A_2)]) & \cdots & M_{n_2,n_r}(R[Hom(A_2,A_r)]) \\ \vdots & \vdots & \ddots & \vdots \\ M_{n_r,n_1}(R[Hom(A_r,A_1)]) & M_{n_r,n_2}(R[Hom(A_r,A_2)]) & \cdots & M_{n_r}(R[Hom(A_r,A_r)]) \end{pmatrix}.$$

Now let $\mathcal{C}$ be a groupoid. In this case, $Hom(A_i, A_j) = \varnothing$ whenever $i \neq j$, and further $R_0[S(\mathcal{C})]$ is isomorphic to the generalized matrix algebra

$$\begin{pmatrix} M_{n_1}(R[Hom(A_1,A_1)]) & 0 & \cdots & 0 \\ 0 & M_{n_2}(R[Hom(A_2,A_2)]) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_r}(R[Hom(A_r,A_r)]) \end{pmatrix}.$$

By definition, the category algebra $R\mathcal{C}$ of $\mathcal{C}$ over R is just the contracted semigroup algebra $R_0[S(\mathcal{C})]$. For category algebras, see [33, 38, 39].

We conclude this section by giving a sufficient and necessary condition for the semigroup algebra of a strict RA finite semigroup to be semiprimitive, which is just [13, Theorem 5.5] when the semigroup is ample.

Theorem 3.10. *Let S be a strict RA semigroup and R a commutative ring. If S is finite, then $R_0[S]$ is semiprimitive if and only if the following conditions are satisfied:*
(A) S is an inverse semigroup;
(B) for every maximum subgroup G of S, $R[G]$ is semiprimitive.

Proof. By [13, Theorem 5.4], it suffices to verify the sufficiency. To see this, we assume that $R_0[S]$ is semiprimitive. With the notations in the proof of Theorem 3.2, $K_0[S]$ is isomorphic to the generalized upper triangular matrix algebra

$$\begin{pmatrix} M_{n_1}(R_0[\overline{M_{11}}]) & M_{n_1,n_2}(R_0[\overline{M_{12}}]) & \cdots & M_{n_1,n_r}(R_0[\overline{M_{1r}}]) \\ 0 & M_{n_2}(R_0[\overline{M_{22}}]) & \cdots & M_{n_2,n_r}(R_0[\overline{M_{2r}}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{n_r}(R_0[\overline{M_{rr}}]) \end{pmatrix}.$$

It follows that the Jacobson radical $J(R_0[S])$ of $R_0[S]$ is equal to

$$\begin{pmatrix} J\left(M_{n_1}(R_0[\overline{M_{11}}])\right) & R_0[\overline{M_{12}}] & \cdots & R_0[\overline{M_{1r}}] \\ 0 & J\left(M_{n_2}(R_0[\overline{M_{22}}])\right) & \cdots & R_0[\overline{M_{2r}}] \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J\left(M_{n_r}(R_0[\overline{M_{rr}}])\right) \end{pmatrix}.$$

So, $J\left(M_{n_i}(R_0[\overline{M_{ii}}])\right) = 0$ and for $i, j = 1, 2, \cdots, r$, $M_{n_i,n_j}(R_0[\overline{M_{ij}}]) = 0$ if $i \neq j$. Thus $R_0[\overline{M_{ii}}]$ is semiprimitive, so that $R_0[M_{ii}]$ is semiprimitive; and $M_{ij} = \varnothing$ if $i \neq j$. Since S is finite, M_{ii} is finite, so that M_{ii} is a subgroup of S. But for all $a \in M_{ii}$, $a\mathcal{L}^* f_i$ giving $a\mathcal{L} f_i$, so $a\mathcal{H} f_i$ since M_{ii} is a subgroup of S, thus M_{ii} is indeed a maximum subgroup of S. By the choice of f_i's, we know that any nonzero idempotent of S is $\mathcal{D}$-related to some f_i, thereby any nonzero maximum subgroup of S is isomorphic to some M_{ii}. Therefore the condition (B) is satisfied.

For the condition (A), we prove only that any nonzero element of S is regular. Let $a \in S\backslash\{0\}$ and $a^{\#} = e, a^{*} = f$. Let $e\mathcal{D} f_i$ and $f\mathcal{D} f_j$. Then there exist $u \in eSf_k, v \in f_kSe, x \in f_jSf, y \in fSf_j$ such that $e = uv, f_i = vu, f = yx, f_j = xy$. It follows that $f_i\mathcal{R}v\mathcal{L}e$ and $f\mathcal{R}y\mathcal{L}f_j$. Thus $vay \in M_{ij}$ since $(v)_{f_i,e}(a)_{e,f}(y)_{f,f_j} = (vay)_{f_i,f_j}$. We consider the following two cases:

- If $f_i = f_j$, then $vay \in M_{ii}$ and so as M_{ii} is a subgroup of S, there exists $b \in S$ such that $vay = vaybvay$, so that $a = u \cdot vay \cdot x = u \cdot vaybvay \cdot x = uv \cdot aybva \cdot yx = a \cdot ybv \cdot a$. It follows that a is regular.
- Assume $f_i \neq f_j$. By the foregoing proof, $M_{ij} = \varnothing$, contrary to the fact: $vay \in M_{ij}$.

Consequently, a is regular, as required. □

Based on Corollary 2.7, any right ample finite monoid is strict RA. Again by Theorem 3.10, the following corollary is obvious.

Corollary 3.11. *Let S be a right ample monoid. If S is finite, then $K_0[S]$ is semiprimitive if and only if the following conditions are satisfied:*
(A) S is an inverse semigroup;
(B) for every maximum subgroup G of S, $R[G]$ is semiprimitive.

Note that inverse semigroups are strict RA. By Theorem 3.10, we can re-obtain the well-known result on inverse semigroup algebras as follows:

Corollary 3.12. *Let S be an inverse semigroup. If S is finite, then $K_0[S]$ is semiprimitive if and only if for every maximum subgroup G of S, $R[G]$ is semiprimitive.*

4 Self-injective algebras

Recall that an algebra $\mathfrak{A}$ (possibly without unity) is *right* (respectively, *left*) *self-injective* if $\mathfrak{A}$ is an injective right (respectively, left) $\mathfrak{A}$-module. Okiniński pointed out that for a semigroup S and a field K, the algebra $K[S]$ is right (left) self-injective if and only if so is $K_0[S]$ (see [15, the arguments before Lemma 3, p.188]). So, in this section we always assume that the semigroup has zero element θ. The aim of this section is to answer when the semigroup algebra of a strict RA semigroup is left self-injective.

To begin with, we recall a known result on left (right) perfect rings, which follows from [27, Theorem (23.20), p.354 and Corollary (24.19), p.365].

Lemma 4.1. *Let R be a ring with unity. If R is left (right) perfect, then*
(A) R does not contain an infinite orthogonal set of nonzero idempotents.
(B) Any quotient of R is left (right) perfect.

We need some known facts on left self-injective semigroup algebras. By the dual of [28, Theorem 1], any left self-injective algebra is a left perfect algebra. Note that, by the argument in [29, Remark], whenever $K_0[S]$ is left perfect, there exist ideals S_i, $i = 0, 1, \cdots, n$, such that

$$\theta = S_0 \sqsubset S_1 \sqsubset \cdots \sqsubset S_n = S$$

and the Rees quotients S_i/S_{i+1} are completely 0-simple or T-nilpotent. So, the following lemma is straight.

Lemma 4.2. *Let S be an arbitrary semigroup and K a field. If $K_0[S]$ is a left self-injective K-algebra, then*
(A) There exist ideals S_i, $i = 0, 1, \cdots, n$, such that $\theta = S_0 \sqsubset S_1 \sqsubset \cdots \sqsubset S_n = S$ and the Rees quotients S_i/S_{i+1} are completely 0-simple or T-nilpotent.
(B) ([15, Lemmas 9 and 10, p.192]) *S satisfies the descending chain condition on principal right ideals and has no infinite subgroups.*
(C) [28, Theorem 1] *$K_0[S]$ is left perfect.*
(D) [28, Lemma 1] $\dfrac{K_0[S]}{Rad(K_0[S])}$ *is regular, where $Rad(K_0[S])$ is the Jacobson radical of $K_0[S]$.*

Moreover, we have

Lemma 4.3. *Let S be a right ample semigroup. If $K_0[S]$ is left self-injective, then*
(A) $|E(S)| < \infty$.
(B) For any $e \in E(S)\backslash\{0\}$, the subset $M_{e,e} = \{x \in S : ex = x, x^* = e\}$ is a finite subgroup of S.

Proof. (A) By Lemma 4.2, we assume that S_i, $i = 0, 1, \cdots, n$, are ideals of S satisfying the conditions:
- $\theta = S_0 \sqsubset S_1 \sqsubset \cdots \sqsubset S_n = S$; and
- the Rees quotients S_i/S_{i+1} are completely 0-simple or T-nilpotent.

So, $S = \{\theta\} \sqcup (\sqcup_{i=1}^n S_i\backslash S_{i-1})$. This shows that for any $e \in E(S)\backslash\{\theta\}$, there exists $i \geq 1$ such that S_i/S_{i-1} is completely 0-simple and $e \in S_i/S_{i-1}$. Now, to verify that $E(S)$ is finite, it suffices to prove that any completely 0-simple semigroup S_i/S_{i-1} is finite.

Now let S_i/S_{i-1} be a completely 0-simple semigroup. By the definition of Rees quotient, any nonzero idempotents of S_i/S_{i-1} is an idempotent of S. But $E(S)$ is a semilattice, so S_i/S_{i-1} is an inverse semigroup. Thus S_i/S_{i-1} is a Brandt semigroup. By the structure theorem of Brandt semigroups in [30], S_i/S_{i-1} is isomorphic to the semigroup $T = I \times G \times I \sqcup \{\theta\}$ whose multiplication is defined by

$$(a, g, x)(b, h, y) = \begin{cases} (a, gh, y) & \text{if } x = b; \\ \theta & \text{otherwise,} \end{cases}$$

where I is a nonempty set and G is a subgroup of S_i/S_{i-1} Clearly, G is a subgroup of S. By Lemma 4.2(B), G is a finite group. By computation, any nonzero idempotent of T is of the form: $(a, 1_G, a)$ where 1_G is the identity of G.

For convenience, we identify S_i/S_{i-1} with T. Now, we need only to show that $|I| < \infty$. By Lemma 4.2, $K_0[S]$ is left perfect, and so $\dfrac{K_0[S]}{Rad(K_0[S])}$ is semisimple. It follows that $\dfrac{K_0[S]}{Rad(K_0[S])}$ has an identity. It is easy to see that $K_0[S_{i-1}]$ is an ideal of $K_0[S]$. Consider the algebra $W := \dfrac{K_0[S]}{Rad(K_0[S]) + K_0[S_{i-1}]}$. Note that

$$\frac{\frac{K_0[S]}{Rad(K_0[S])}}{\frac{Rad(K_0[S])+K_0[S_{i-1}]}{Rad(K_0[S])}} \cong \frac{K_0[S]}{Rad(K_0[S]) + K_0[S_{i-1}]} \cong \frac{\frac{K_0[S]}{K_0[S_{i-1}]}}{\frac{Rad(K_0[S])+K_0[S_{i-1}]}{K_0[S_{i-1}]}},$$

we can observe that
- $(a, 1_G, a)(b, 1_G, b) = \theta$ whenever $a \neq b$;
- W has an unity;
- $(a, 1_G, a) + Rad(K_0[S]) + K_0[S_{i-1}] \neq (b, 1_G, b) + Rad(K_0[S]) + K_0[S_{i-1}]$ whenever $a \neq b$.

Again by the property that $(a, 1_G, a)(b, 1_G, b) = \theta$ whenever $a \neq b$, we get that the set

$$X := \{(a, 1_G, a) + Rad(K_0[S]) + K_0[S_{i-1}] : a \in I\}$$

is an orthogonal set of nonzero idempotents of W. On the other hand, since $\dfrac{K_0[S]}{Rad(K_0[S])}$ is semisimple, we know that $\dfrac{K_0[S]}{Rad(K_0[S])}$ is left perfect, and so by Lemma 4.1, W is left perfect. Again by Lemma 4.1, this shows that W does not contain an infinite orthogonal set of nonzero idempotents. It follows that $|X| < \infty$. Therefore $|I| < \infty$ since $|X| = |I|$. Consequently, S_i/S_{i-1} is finite, and so $|E(S)| < \infty$.

(B) Let $a \in M_{e,e}$. Consider the chain of principal right ideals of S:

$$\cdots a^n S^1 \subseteq a^{n-1} S^1 \subseteq \cdots \subseteq a^2 S^1 \subseteq aS^1.$$

By Lemma 4.2, there exists a positive integer n such that $a^n S^1 = a^{n+1} S^1$. That is, there is $x \in S^1$ such that $a^n = a^{n+1}x$. Hence $a^{n-1} = a^* a^{n-1} = a^* a^n x = a^n x$. Continuing this process, we can obtain $a^* = ax$. But $a = aa^*$, now $a\mathcal{R}a^*$. Thus a is regular. Therefore $a\mathcal{H}a^*$ since $a\mathcal{L}^* a^*$. So, $M_{e,e}$ is a subgroup of S and further by Lemma 4.2, $M_{e,e}$ is finite. □

For convenience, in the rest of this section, we always let S be a strict RA semigroup and K a field. Assume that $K_0[S]$ is a left self-injective K-algebra. Let us turn back to the proof of Theorem 3.2. We know that $\sum_{i=1}^n \overline{e_i}$ is the identity of $K_0[PRA(S)]$. By Theorem 3.4, we have that $M_{ij} = \varnothing$ if $j < i$, and

$$K_0[S] := \mathfrak{B} \cong \begin{pmatrix} M_{n_1}(K_0[\overline{M_{11}}]) & M_{n_1,n_2}(K_0[\overline{M_{12}}]) & \cdots & M_{n_1,n_r}(K_0[\overline{M_{1r}}]) \\ 0 & M_{n_2}(K_0[\overline{M_{22}}]) & \cdots & M_{n_2,n_r}(K_0[\overline{M_{2r}}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{n_r}(K_0[\overline{M_{rr}}]) \end{pmatrix}.$$

Since our aim is to show that S is finite, for convenience, we may assume $n_i = 1$ for $i = 1, 2, \cdots, r$. So, we let

$$\mathfrak{B} = \begin{pmatrix} K_0[\overline{M_{11}}] & K_0[\overline{M_{12}}] & \cdots & K_0[\overline{M_{1r}}] \\ 0 & K_0[\overline{M_{22}}] & \cdots & K_0[\overline{M_{2r}}] \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & K_0[\overline{M_{rr}}] \end{pmatrix}.$$

For $1 \leq j \leq r$, we denote by m_j the smallest positive integer in the set $\{i : M_{ij} \neq \varnothing\}$. By definition, $m_j \leq j$ for any i.

Lemma 4.4. $\overline{M_{kl}} \cdot \overline{M_{m_j,j}} = 0$ *whenever* $M_{kl} \neq M_{m_j,m_j}$.

Proof. By definition, $\overline{M_{k,m_j}} \cdot \overline{M_{m_j,j}} \subseteq \overline{M_{k,j}}$ and

$$\overline{M_{kl}} \cdot \overline{M_{m_j,j}} \begin{cases} =0 & \text{if } l \neq m_i \\ \subseteq \overline{M_{k,j}} & \text{if } l = m_j. \end{cases}$$

In the second case, $k \leq m_j$. This shows that $k = m_j$ by the minimality of m_j. □

Lemma 4.5. $|M_{m_j,j}| < \infty$.

Proof. Lemma 4.3 results the case for $m_j = j$. Assume now that $m_j < j$. By Lemma 4.4, the algebra

$$\mathfrak{C} := \begin{pmatrix} 0 \cdots 0 & 0 & 0 \cdots 0 \\ \vdots \; \vdots \; \vdots & \vdots & \vdots \; \vdots \; \vdots \\ 0 \cdots 0 & K_0[\overline{M_{m_j,j}}] & 0 \cdots 0 \\ \vdots \; \vdots \; \vdots & \vdots & \vdots \; \vdots \; \vdots \\ 0 \cdots 0 & 0 & 0 \cdots 0 \end{pmatrix}$$

is a left ideal of $\mathfrak{B}$. Pick $w \in M_{m_j,j}$ and define

$$\theta : M_{m_j,j} \to M_{m_j,j};\ x \to \begin{cases} x \text{ if } x \in M_{m_j,m_j} w \\ \\ 0 \text{ if otherwise,} \end{cases}$$

and span linearly to $K_0[M_{i,m_i}]$. Further define a map ζ of $\mathfrak{C}$ into $\mathfrak{B}$ by

$$X = \begin{pmatrix} 0 \cdots 0 & 0 & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & (x)_{m_j,j} & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & 0 & 0 \cdots 0 \end{pmatrix} \to \zeta(X) = \begin{pmatrix} 0 \cdots 0 & 0 & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & (\theta(x))_{m_j,j} & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & 0 & 0 \cdots 0 \end{pmatrix}.$$

By Lemma 4.4, a routine computation shows that ζ is a $\mathfrak{B}$-module homomorphism. But $\mathfrak{B}$ is left self-injective, now by Baer condition, there exists $U \in \mathfrak{B}$ such that $\zeta(A) = AU$ for any $A \in \mathfrak{C}$. Especially, $\zeta(X) = XU$. Now let $U = (u_{kl})$. Then $u_{jj} = \sum_{k=1}^{n} r_k \overline{a_k}$ where $r_k \in K$, $a_k \in M_{jj}$. Since $\zeta(X) = XU$, we have $(\theta(x))_{m_j,j} = (\theta(x)_{m_j,j} u_{jj} = (\sum_{k=1}^{n} r_k x a_k)_{m_j,j}$ and

$$x = \sum_{k=1}^{n} r_k x a_k. \tag{3}$$

So, we may let $n_1, n_2, \cdots, n_s$ be positive integers such that
- $n_1 + n_2 + \cdots + n_s = n$;
- $(*)$ $\quad x a_l = x$, for $l = 1, 2, \cdots, n_1$;
- $(**)$ $\quad x a_{n_q+1} = x a_{n_q+2} = \cdots = x a_{n_q+n_{q+1}-1} = b_q$ for $q = 1, 2, \cdots, s$, where $b_1, b_2, \cdots, b_s$ are different elements in M_{jj}.

Now by Eq. (3), we get $r_1 + r_2 + \cdots r_{n_1} = 1$ and $r_{n_l+1} + r_{n_l+2} + \cdots r_{n_l+n_{l+1}-1} = 0$ for $l = 1, 2, \cdots, s$. Note that $f_j = x^* \mathcal{L}^* x$. Therefore by Eq. $(*)$, $a_l = f_j a_l = x^* a_l = x^* = f_j$, for $l = 1, 2, \cdots, n_1$; and by Eq. $(**)$, $x^* a_{n_q+1} = x^* a_{n_q+2} = \cdots = x^* a_{n_q+n_{q+1}-1}$ for $q = 1, 2, \cdots, s$, so that as $x^* = f_j$, $a_{n_q+1} = a_{n_q+2} = \cdots = a_{n_q+n_{q+1}-1}$ for $q = 1, 2, \cdots, s$. Consequently, $u_{jj} = \overline{f_j}$.

Now, for any $x \in M_{m_j,j}$, $(\theta(x))_{m_j,j} = (x)_{m_j,j} u_{jj} = (x f_j)_{m_j,j} = (x)_{m_j,j}$. This shows that $\theta(x) = x$ and $M_{m_j,j} = M_{m_j,m_j} w$ since $\theta(x) = 0$ for any $x \in M_{m_j,j} \backslash M_{m_j,m_j} w$. It follows that $|M_{m_j,j}| < \infty$ since $M_{m_j,m_j} = M_{f_j} \subseteq M_{f_j,f_j}$ and by Lemma 4.3 (B), is a finite subgroup. □

Lemma 4.6. $|M_{ij}| < \infty$.

Proof. If $i = m_j$, nothing is to prove.

If $m_j \neq i$ and $M_{ij} \neq \varnothing$, then $m_j < i$. Let i_0 be the smallest positive integer of the set

$$Y = \{k : \text{ there exist } k = k_0, k_1, k_2, \cdots, k_m = i \text{ such that } M_{K_{l-1},k_l} \neq \varnothing \text{ for } l = 1, 2, \cdots, m-1\}.$$

We shall prove $i_0 = m_j$. Assume on the contrary that $i_0 \neq m_j$. By definition, $M_{k_0 j} \neq \varnothing$ and specially $M_{i_0 j} \neq \varnothing$. Thus $m_j < i_0$. By the minimality of i_0, $M_{ki} = \varnothing$ for $1 \le k < i_0$, and further $\overline{M_{ki_0}} \cdot \overline{M_{i_0 j}} = 0$. This and Lemma 4.4 show that

$$\mathfrak{D} = \begin{pmatrix} 0 \cdots 0 & 0 & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & K_0[\overline{M_{m_j,j}}] & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & K_0[\overline{M_{i_0,j}}] & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & 0 & 0 \cdots 0 \end{pmatrix}$$

is a left ideal of $\mathfrak{B}$ and that the map η defined by

$$A = \begin{pmatrix} 0 \cdots 0 & 0 & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & (a)_{m_j,j} & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & (b)_{i_0,j} & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & 0 & 0 \cdots 0 \end{pmatrix} \to \eta(A) = \begin{pmatrix} 0 \cdots 0 & 0 & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & (a)_{m_j,j} & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & 0 & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & 0 & 0 \cdots 0 \end{pmatrix}$$

is a $\mathfrak{B}$-module homomorphism. Since $\mathfrak{B}$ is left self-injective, it follows from Baer condition that there exists $V \in \mathfrak{B}$ such that $\eta(A) = AV$ for any $A \in \mathfrak{D}$. Hence $(a)_{m_j,j} = (a)_{m_j,j}v_{jj}$ and $0 = (b)_{m_j,j}v_{jj}$ where $V = (v_{ij})$. By the first equality, $av_{jj} = a$ and further by a similar arguments as proving $u_{jj} = \overline{f_j}$ in the proof of Lemma 4.5, $\overline{f_j} = v_{jj} \neq 0$; by the second equality, $0 = (b)_{m_j,j}v_{jj} = (bf_j)_{m_j,j} = (b)_{m_j,j}$ and $b = 0$, so that as $b^* = f_j$, we get $0 = f_j$, thus $v_{jj} = 0$, contrary to the foregoing proof: $v_{jj} \neq 0$. Therefore $m_j = i_0$. We have now proved that $M_{i_0 i}M_{ij} \subseteq M_{m_j,j}$. This shows that $cM_{ij} \subseteq M_{m_j,j}$ for some $c \in M_{i_0 i}$. Note that for $w, z \in M_{ij}$, if $cw = cz$, then $f_i w = f_i z$, so that $w = z$. We can observe that $|cM_{ij}| = |M_{ij}|$, and $|M_{ij}| \leq |M_{m_j,j}| < \infty$. □

Lemma 4.7. *$M_{ij} = \varnothing$ if $i \neq j$.*

Proof. By Lemmas 4.2 and 4.6, $\mathfrak{B}$ is a finite dimensional algebra and further is quasi-Frobenius. Assume on the contrary that there exists $M_{ij} \neq \varnothing$. Let n be the biggest number such that $M_{in} \neq \varnothing$ for some $i \neq n$, and further let m be the biggest number such that $M_{mn} \neq \varnothing$. Obviously, $m < n$. By definition, $M_{km} = \varnothing$ if $m < k$ (if not, then as $M_{mn} \neq \varnothing$ and by definition, $M_{kn} \neq \varnothing$, contrary to the maximality of m); and $M_{nl} = \varnothing$ if $n < l$ (if not, then as $M_{in} \neq \varnothing$ and by definition, $M_{il} \neq \varnothing$, contrary to the maximality of n). By these, a routine computation shows that

$$\mathfrak{C} = \begin{pmatrix} 0 \cdots 0 & K_0[\overline{M_{1n}}] & 0 \cdots 0 \\ 0 \cdots 0 & K_0[\overline{M_{2n}}] & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & K_0[\overline{M_{mn}}] & 0 \cdots 0 \\ 0 \cdots 0 & 0 & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & 0 & 0 \cdots 0 \end{pmatrix}$$

is a left ideal of $\mathfrak{B}$.

Let $x \in K_0[\overline{M_{jl}}]$ and $M_{ij} \neq \varnothing$. If $K_0[\overline{M_{ij}}]x = 0$, then $\overline{a}x = 0$ for some $a \in M_{ij}$, so that by a similar argument as proving $u_{jj} = \overline{f_j}$ in the proof of Lemma 4.5, $\overline{a^*}x = 0$. By $a^* = f_j$, $x = \overline{f_j}x = \overline{a^*}x = 0$. So, $K_0[\overline{M_{ij}}]x = 0$ if and only if $x = 0$. Because $M_{mn} \neq \varnothing$, this can show that the right annihilator $ann_r(\mathfrak{C})$ of $\mathfrak{C}$ in $\mathfrak{B}$ is equal to

$$\begin{pmatrix} K_0[\overline{M_{11}}] & \cdots & K_0[\overline{M_{1,n-1}}] & K_0[\overline{M_{1n}}] & K_0[\overline{M_{1,n+1}}] & \cdots & K_0[\overline{M_{1r}}] \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & K_0[\overline{M_{n-1,n-1}}] & K_0[\overline{M_{n-1,n}}] & K_0[\overline{M_{n-1,n+1}}] & \cdots & K_0[\overline{M_{n-1,r}}] \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & K_0[\overline{M_{n+1,n+1}}] & \cdots & K_0[\overline{M_{n+1,r}}] \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & K_0[\overline{M_{rr}}] \end{pmatrix}.$$

But the left annihilator $ann_\ell(ann_r(\mathfrak{C}))$ of $ann_r(\mathfrak{C})$ in $\mathfrak{B}$ includes

$$\begin{pmatrix} 0 \cdots 0 & K_0[\overline{M_{1n}}] & 0 \cdots 0 \\ 0 \cdots 0 & K_0[\overline{M_{2n}}] & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & K_0[\overline{M_{nn}}] & 0 \cdots 0 \\ 0 \cdots 0 & 0 & 0 \cdots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \cdots 0 & 0 & 0 \cdots 0 \end{pmatrix}.$$

It is clear that $ann_\ell(ann_r(\mathfrak{E})) \neq \mathfrak{E}$. But $\mathfrak{B}$ is a quasi-Frobenius algebra, so that by [31, Theorem 30.7, p.333] we should have $ann_\ell(ann_r(\mathfrak{E})) = \mathfrak{E}$, a contradiction. □

Theorem 4.8. *Let S be a strict RA semigroup and K a field. Then $K_0[S]$ is left self-injective if and only if S is a finite inverse semigroup.*

Proof. To verify Theorem 4.8, by [18, Theorem 4.1], it suffices to prove that if $K_0[S]$ is left self-injective, then S is regular. Assume now that $K_0[S]$ is left self-injective. By Theorem 3.4, $K_0[S]$ is isomorphic to

$$\begin{pmatrix} M_{n_1}(K_0[\overline{M_{11}}]) & M_{n_1,n_2}(K_0[\overline{M_{12}}]) & \cdots & M_{n_1,n_r}(K_0[\overline{M_{1r}}]) \\ M_{n_2,n_1}(K_0[\overline{M_{21}}]) & M_{n_2}(K_0[\overline{M_{22}}]) & \cdots & M_{n_2,n_r}(K_0[\overline{M_{2r}}]) \\ \cdots & \cdots & \cdots & \cdots \\ M_{n_r,n_1}(K_0[\overline{M_{r1}}]) & M_{n_r,n_2}(K_0[\overline{M_{r2}}]) & \cdots & M_{n_r}(K_0[\overline{M_{rr}}]) \end{pmatrix}.$$

By Theorem 3.4 and Lemma 4.7, whenever $i \neq j$, $M_{n_i,n_j}(K_0[\overline{M_{ij}}]) = 0$, so that $M_{ij} = \varnothing$, thus $\{x \in S : x^\# = f_i, x^* = f_j\} = \varnothing$. Note that f_i's are representatives of the partition $\pi = \cup_{i=1}^r E_r$ of $E(S)\backslash\{0\}$ induced by $\mathcal{D}$. Therefore $\{x \in S : x^\# = e, x^* = h\} = \varnothing$ for all $e \in E_i, h \in E_j$ with $i \neq j$. This means that for all $x \in S$, $x^\# \mathcal{D} x^*$.

On the other hand, by Lemma 4.3(B), for any $e \in E(S)\backslash\{0\}$, $M_{e,e} = \{x \in S : x^\# = e, x^* = e\}$ is a subgroup of S and so M_{ii} is a subgroup of S. Let $f \in E(S)\backslash\{0\}$ and $e\mathcal{D}f$. Let $x, y \in S$ with $x^\# = e$, $y^\# = f$, $x^* = f$ and $y^* = e$, and of course $x, y \in M_{e,f}$. Then as $\mathcal{L}^*$ is a right congruence, we have $xy\mathcal{L}^* fy = y$, so that $y \neq 0$. Similarly, $yx \neq 0$. Note that $(x)_{e,f}, (y)_{f,e} \in PRA(S)$. We observe that $(xy)_{e,e} = (x)_{e,f}(y)_{f,e} \in PRA(S)\backslash\{0\}$, and hence $xy \in M_{e,e}$. Thus there is $a \in M_{e,e}$ such that $e = axy$ since $M_{e,e}$ is a subgroup of S with identity e. This and $ye = y$ imply that $e\mathcal{L}y$. Therefore y is regular. We have now proved that y is regular whenever $y^\# \mathcal{D} y^*$.

However, any element of S is regular and S is regular, as required. □

It is a natural problem whether Theorem 4.8 is valid for the case for right self-injectivity. We answer this question next. Firstly, we prove the following lemma.

Lemma 4.9. *Let S be a right ample semigroup and $K_0[S]$ have an unity 1. If $K_0[S]$ is right self-injective, then S is a finite inverse semigroup.*

Proof. For any $a \in S$, S^1a is a left ideal of S, and so $K_0[S^1a]$ is a left ideal of $K_0[S]$. Since $K_0[S]$ is right self-injective, and by [31, Lemma 30.9, p.334], we have $ann_\ell(ann_r(K_0[S^1a])) = K_0[S^1a]$. By the definition of $\mathcal{L}^*$, $ann_r(K_0[S^1a]) = ann_r(a) = ann_r(a^*) = (1 - a^*)K_0[S]$ and so

$$K_0[S^1a] = ann_\ell(ann_r(K_0[S^1a]) = ann_\ell((1 - a^*)K_0[S]) = K_0[S]a^* = K_0[S^1a^*].$$

It follows that $S^1a = S^1a^*$. Thus $a\mathcal{L}a^*$ so that a is a regular element of S. Therefore S is an inverse semigroup, and by the Wenger Theorem, S is a finite inverse semigroup. □

Based on Lemma 4.9, we can verify the following theorem, which illuminates that Theorem 4.8 is valid for right self-injectivity.

Theorem 4.10. *Let S be a strict RA semigroup. Then $K_0[S]$ is right self-injective if and only if S is a finite inverse semigroup.*

Proof. By Theorem 4.8, it suffices to verify the necessity. By Lemma 4.9, we need only to show that $K_0[S]$ has an identity. Now let us turn back to the proof of Lemma 4.3(A). We notice that the proof is valid for right self-injectivity and so $|E(S)| < \infty$ when $K_0[S]$ is right self-injective. By Theorem 3.4, we have that $K_0[S]$ is isomorphic to the generalized upper triangular matrix algebra

$$\begin{pmatrix} M_{n_1}(R_0[\overline{M_{11}}]) & M_{n_1,n_2}(R_0[\overline{M_{12}}]) & \cdots & M_{n_1,n_r}(R_0[\overline{M_{1r}}]) \\ 0 & M_{n_2}(R_0[\overline{M_{22}}]) & \cdots & M_{n_2,n_r}(R_0[\overline{M_{2r}}]) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{n_r}(R_0[\overline{M_{rr}}]) \end{pmatrix}.$$

It is easy to see that $K_0[S]$ has an identity. We complete the proof. □

We now arrive at the main result of this section, which follows immediately from Theorems 4.8 and 4.10.

Theorem 4.11. *Let S be a strict RA semigroup and K a field. Then the following statements are equivalent:*
(1) $K_0[S]$ is left self-injective;
(2) $K_0[S]$ is right self-injective;
(3) S is a finite inverse semigroup;
(4) $K_0[S]$ is quasi-Frobenius;
(5) $K_0[S]$ is Frobenius.

Note that ample semigroups are both strict RA and strict LA. By Theorem 4.11 and its dual, the following corollary is immediate, which is the main result of [18] (see, [18, Theorem 4.1]).

Corollary 4.12. *Let S be an ample semigroup and K a field. Then the following statements are equivalent:*
(1) $K_0[S]$ is left self-injective;
(2) S is a finite inverse semigroup;
(3) $K_0[S]$ is quasi-Frobenius;
(4) $K_0[S]$ is Frobenius;
(5) $K_0[S]$ is right self-injective.

Let S be a right ample monoid. Note that $E(S)\cdot S = S$. By Corollary 2.7, S is a strict RA semigroup. If $K_0[S]$ is left self-injective, then S has only finite idempotents. Now, by Theorem 4.11, the following corollary is immediate.

Corollary 4.13. *Let S be a right ample monoid. Then the following statements are equivalent:*
(1) $K_0[S]$ is left self-injective;
(2) S is a finite inverse semigroup;
(3) $K_0[S]$ is quasi-Frobenius;
(4) $K_0[S]$ is Frobenius;
(5) $K_0[S]$ is right self-injective.

Recall that if $\mathcal{A}$ is a ring and $\mathcal{A}^1$ is the standard extension of $\mathcal{A}$ to a ring with unity, then $\mathcal{A}$ is left self-injective if and only if the left $\mathcal{A}^1$-module $\mathcal{A}$ satisfies the Baer condition (c.f. [32, Chapter 1]). The following example, due to Okniński [28], shows that in Theorem 4.8, the assumption that $E(S)$ is a semilattice, i.e., all idempotents commute, is essential.

Example 4.14. Let $S = \{g, h\}$ be the semigroup of left zeros, and $\mathbb{Q}$ the field of rational numbers. Consider the algebra $\mathbb{Q}[S] = \mathbb{Q}_0[S]$ and the standard extension $\mathbb{Q}[S]^1$ of $\mathbb{Q}[S]$ to a $\mathbb{Q}$-algebra with unity. It may be shown that for any left ideal I of $\mathbb{Q}[S]^1$, any homomorphism of left $\mathbb{Q}[S]^1$-modules $I \to \mathbb{Q}[S]$, extends to a homomorphism of $\mathbb{Q}[S]^1$-modules $I\mathbb{Q}[S]^1 \to \mathbb{Q}[S]$. Moreover, by computing the right ideals of $\mathbb{Q}[S]^1$, one can easily check that $\mathbb{Q}[S]^1$ satisfies Baer's condition. Hence, $\mathbb{Q}[S]$ satisfies Baer's condition as $\mathbb{Q}[S]^1$-module which means that $\mathbb{Q}[S]$ is left self-injective. On the other hand, since $\mathbb{Q}[S]$ has no right identities, it is obvious that $\mathbb{Q}[S]$ is not right self-injective.

Obviously, the polynomial algebra $K[x]$ is indeed the semigroup algebra $K[M]$, where $M = \{1, x, x^2, \cdots\}$. It is easy to see that M is a cancellative monoid, and of course, a strict RA semigroup. By Theorem 4.11 and its dual, $K[x]$ is neither left self-injective nor right self-injective. Recall that a category $\mathcal{C}$ is said to be *finite* if $|Obj(\mathcal{C})| < \infty$ and $|Hom(A, B)| < \infty$, for all $A, B \in Obj(\mathcal{C})$. For the self-injectivity of path algebras and category algebras, we have the following proposition.

Proposition 4.15. (A) *Let Q be a quiver and K a field. Then the path algebra KQ is left self-injective if and only if Q has neither edges nor loops; if and only if KQ is right self-injective.*

(*B*) *Let $\mathcal{C}$ be a left cancellative category. Then the category algebra $K\mathcal{C}$ is left self-injective if and only if $\mathcal{C}$ is a finite groupoid; if and only if $K\mathcal{C}$ is right self-injective.*

Proof. (A) We only need to prove the necessity. If KQ is left (resp. right) self-injective, then $S(Q)$ is a finite inverse semigroup. It follows that Q has no circles, and of course, no loops. On the other hand, since the idempotents of $S(Q)$ are empty paths, it is easy to show that any edge is not regular in the semigroup $S(Q)$, thus Q has no edges.

(B) By Example 2.5, $S(\mathcal{C})$ is an inverse semigroup whenever $\mathcal{C}$ is a groupoid. So, it suffices to verify the necessity. Assume that $K\mathcal{C}$ is left self-injective, then by Theorem 4.8, $S(\mathcal{C})$ is a finite inverse semigroup. It follows that $\mathcal{C}$ is a finite groupoid. □

Recall from [27] that an algebra $\mathfrak{A}$ with unity is semisimple if and only if every left $\mathfrak{A}$-module is injective. So, any semisimple algebra is left self-injective. By Theorem 4.11 and Corollary 3.12, we immediately have

Theorem 4.16. *Let S be a strict RA semigroup and K be a field. If $K_0[S]$ has a unity, then $K_0[S]$ is semisimple if and only if S is a finite inverse semigroup and the order of any maximum subgroup of S is not divided by the characteristic of K.*

By Theorem 4.11, the following proposition is immediate, which answers positively [15, Problem 6, p.328] for strict RA semigroups:

Does the fact that $K[S]$ is a right (respectively, left) self-injective imply that S is finite?

Proposition 4.17. *Let S be a strict RA semigroup. If $K_0[S]$ is left (respectively, right) self-injective, then S is finite.*

Moreover, we have the following corollary.

Corollary 4.18. *Let S be a right ample monoid and K a field. If $K_0[S]$ is left (respectively, right) self-injective, then S is finite.*

Acknowledgement: This research is jointly supported by the National Natural Science Foundation of China (grant: 11361027; 11761034; 11661042); the Natural Science Foundation of Jiangxi Province (grant: 20161BAB201018) and the Science Foundation of the Education Department of Jiangxi Province, China (grant: GJJ14251).

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