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Research Article

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Functional analysis method for the M/G/1 queueing model with single working vacation

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Abstract: In this paper, we study the asymptotic property of underlying operator corresponding to the M/G/1 queueing model with single working vacation, where both service times in a regular busy period and in a working vacation period are function. We obtain that all points on the imaginary axis except zero belong to the resolvent set of the operator and zero is an eigenvalue of both the operator and its adjoint operator with geometric multiplicity one. Therefore, we deduce that the time-dependent solution of the queueing model strongly converges to its steady-state solution. We also study the asymptotic behavior of the time-dependent queueing system's indices for the model.

Keywords: M/G/1 queueing model with single working vacation, C_0 -semigroup, Eigenvalue, Resolvent set, Dirichlet operator

MSC: 47D03, 47A10, 60K25

1 Introduction

Queueing system with server working vacations have arisen many researchers' attention because the working vacation policy is more appropriate to model the real system in which the server has additional task during a vacation[1–3]. Unlike a classical vacation policy, the working vacation policy requires the server working at a lower rate rather than completely stopping service during a vacation. Therefore, compared with the classical vacation model, there are also customers who leave the system due to the completion of the services during working vacation. In this way, the number of customers in the system may be reduced. For example, an agent in a call center is required to do additional work after speaking with a customer. The agent may provide service to the next customer at a lower rate while performing additional tasks. In 2002, Servi and Finn [1] first introduced the M/M/1 queueing system with multiple working vacation. Since then, many researchers have extended their work to various type of queueing system (see Chandrasekaran et al.[4]). Kim et al.[5] and Wu and Takagi [6] extended Servi and Finn's [1] M/M/1 queueing system to an M/G/1 queueing system. Xu et al. [7] and Baba [8] studied a batch arrival $M^X/M/1$ queueing with working vacation. Gao and Yao [9] generalized it to an $M^X/G/1$ queueing system. Baba [10] introduced the general input GI/M/1 queueing model with working vacation. Du [11] and Arivudainambi et al. [12] developed retrial queueing model with the concept of working vacation, etc. In 2012, Zhang and Hou [13] established the mathematical model of the M/G/1 queueing system with single working vacation by using the supplementary variable technique and studied the queueing length

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distribution and service status at the arbitrary epoch in the steady-state case under the following hypothesis:

$$\begin{aligned} \lim_{t \rightarrow \infty} p_{0,v}(t) &= p_{0,v}, & \lim_{t \rightarrow \infty} p_{n,v}(x, t) &= p_{n,v}(x), & n \geq 1, \\ \lim_{t \rightarrow \infty} p_{0,b}(t) &= p_{0,b}, & \lim_{t \rightarrow \infty} p_{n,b}(x, t) &= p_{n,b}(x), & n \geq 1. \end{aligned} \quad (H)$$

According to Zhang and Hou [13], the M/G/1 queueing model with single working vacation can be described by the following partial differential equations:

$$\begin{aligned} \frac{dp_{0,v}(t)}{dt} &= -(\theta + \lambda)p_{0,v}(t) + \int_0^\infty \mu_v(x)p_{1,v}(x, t)dx + \int_0^\infty \mu_b(x)p_{1,b}(x, t)dx, \\ \frac{\partial p_{1,v}(x, t)}{\partial t} + \frac{\partial p_{1,v}(x, t)}{\partial x} &= -[\theta + \lambda + \mu_v(x)]p_{1,v}(x, t), \\ \frac{\partial p_{n,v}(x, t)}{\partial t} + \frac{\partial p_{n,v}(x, t)}{\partial x} &= -[\theta + \lambda + \mu_v(x)]p_{n,v}(x, t) + \lambda p_{n-1,v}(x, t), \quad \forall n \geq 2, \\ \frac{dp_{0,b}(t)}{dt} &= -\lambda p_{0,b}(t) + \theta p_{0,v}(t), \\ \frac{\partial p_{1,b}(x, t)}{\partial t} + \frac{\partial p_{1,b}(x, t)}{\partial x} &= -[\lambda + \mu_b(x)]p_{1,b}(x, t), \\ \frac{\partial p_{n,b}(x, t)}{\partial t} + \frac{\partial p_{n,b}(x, t)}{\partial x} &= -[\lambda + \mu_b(x)]p_{n,b}(x, t) + \lambda p_{n-1,b}(x, t), \quad \forall n \geq 2, \end{aligned} \quad (1.1)$$

with the integral boundary conditions

$$\begin{aligned} p_{1,v}(0, t) &= \lambda p_{0,v}(t) + \int_0^\infty \mu_v(x)p_{2,v}(x)dx, \\ p_{n,v}(0, t) &= \int_0^\infty \mu_v(x)p_{n+1,v}(x)dx, \quad \forall n \geq 2, \\ p_{1,b}(0, t) &= \lambda p_{0,b}(t) + \int_0^\infty \mu_b(x)p_{2,b}(x)dx + \theta \int_0^\infty p_{1,v}(x)dx, \\ p_{n,b}(0, t) &= \int_0^\infty \mu_b(x)p_{n+1,b}(x)dx + \theta \int_0^\infty p_{n,v}(x)dx, \quad \forall n \geq 2. \end{aligned} \quad (1.2)$$

If we assume the system states when there are no customers in the system and the server is in vacation, i.e.,

$$p_{0,v}(0) = 1, p_{0,b}(0) = 0, p_{m,v}(x, 0) = p_{m,b}(x, 0) = 0, \quad \forall m \geq 1, \quad (1.3)$$

where $(x, t) \in [0, \infty) \times [0, \infty)$; $p_{0,v}(t)$ represents the probability that there is no customer in the system and the server is in a working vacation period at time t ; $p_{n,v}(x, t)dx$ ($n \geq 1$) is the probability that at time t the server is in a working vacation period and there are n customers in the system with elapsed service time of the customer undergoing service lying in $(x, x + dx]$; $p_{0,b}(t)$ represents the probability that there is no customer in the system and the server is in a regular busy period at time t ; $p_{n,b}(x, t)dx$ ($n \geq 1$) is the probability that at time t the server is in a regular busy period and there are n customers in the system with elapsed service time of the customer undergoing service lying in $(x, x + dx]$; λ is the mean arrival rate of customers; θ is the vacation duration rate of the server; $\mu_v(x)$ is the service rate of the server while the server is in a working vacation period and satisfies

$$\mu_v(x) \geq 0, \quad \int_0^\infty \mu_v(x)dx = \infty.$$

$\mu_b(x)$ is the service rate of the server while the server is in a regular busy period and satisfying

$$\mu_b(x) \geq 0, \quad \int_0^\infty \mu_b(x)dx = \infty.$$

In fact, the above hypothesis (H) implies the following two hypotheses in view of partial differential equations:

Hypothesis 1. The model has a unique time-dependent solution.

Hypothesis 2. The time-dependent solution converges to its steady-state solution.

In 2016, Kasim and Gupur [14] did the dynamic analysis for the above model and gave the detailed proof of the hypothesis 1. Moreover, when the service rates in a working vacation period and in a regular busy period are constant, by using the C_0 -semigroup theory they obtained that the hypothesis 2 also hold. In the general case, the service rates are function, the hypothesis 2 does not always hold, see Gupur [15] and Kasim and Gupur [16], and it is necessary to study the asymptotic behavior of the time-dependent solution of the model. This paper is an effort on this subject.

The rest of this paper is organized as follows. In Section 2 we convert the model into an abstract Cauchy problem. In Section 3, by investigating the spectral properties of the underlying operator we give the main results of this paper. Firstly, we prove that 0 is an eigenvalue of the underlying operator with geometric multiplicity one by using the probability generating function. Next, to obtain the resolvent set of the underlying operator we apply the boundary perturbation method. We obtain that all points on the imaginary axis except zero belong to the resolvent set of the operator. Last, we determine the adjoint operator and verify that 0 is an eigenvalue of the adjoint operator with geometric multiplicity one. Finally, based on these results we present the desired result in this paper: the time-dependent solution of the model strongly converges to its steady-state solution. In addition, the asymptotic behavior of the queueing system's indices are discussed. A conclusion is given in Section 4. Section 5 provides a detail proof of some lemmas.

2 Abstract Setting for the system

In this section, we reformulate the equation (1.1)-(1.3) as an abstract Cauchy problem. We start by introducing the state space as follows.

$$\begin{aligned} X \times Y &= \left\{ (p_v, p_b) \mid p_v \in X, p_b \in Y, \|(p_v, p_b)\| = \|p_v\| + \|p_b\| \right\}, \\ X &= \left\{ p_v \in \mathbb{R} \times L^1[0, \infty) \times \cdots \mid \|p_v\| = |p_{0,v}| + \sum_{n=1}^{\infty} \|p_{n,v}\|_{L^1[0, \infty)} < \infty \right\}, \\ Y &= \left\{ p_b \in \mathbb{R} \times L^1[0, \infty) \times \cdots \mid \|p_b\| = |p_{0,b}| + \sum_{n=1}^{\infty} \|p_{n,b}\|_{L^1[0, \infty)} < \infty \right\}. \end{aligned}$$

It is obvious that $X \times Y$ is a Banach space. Define an operator and its domain.

$$\begin{aligned} A_m \left(\begin{pmatrix} p_{0,v} \\ p_{1,v}(x) \\ p_{2,v}(x) \\ p_{3,v}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} p_{0,b} \\ p_{1,b}(x) \\ p_{2,b}(x) \\ p_{3,b}(x) \\ \vdots \end{pmatrix} \right) &= \begin{pmatrix} -(\theta + \lambda) & \phi_v & 0 & 0 & \cdots \\ 0 & \psi_v & 0 & 0 & \cdots \\ 0 & \lambda & \psi_v & 0 & \cdots \\ 0 & 0 & \lambda & \psi_v & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{0,v} \\ p_{1,v}(x) \\ p_{2,v}(x) \\ p_{3,v}(x) \\ \vdots \end{pmatrix} + \begin{pmatrix} 0 & \phi_b & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{0,b} \\ p_{1,b}(x) \\ p_{2,b}(x) \\ p_{3,b}(x) \\ \vdots \end{pmatrix}, \\ &\begin{pmatrix} -\lambda & 0 & 0 & 0 & \cdots \\ 0 & \psi_b & 0 & 0 & \cdots \\ 0 & \lambda & \psi_b & 0 & \cdots \\ 0 & 0 & \lambda & \psi_b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{0,b} \\ p_{1,b}(x) \\ p_{2,b}(x) \\ p_{3,b}(x) \\ \vdots \end{pmatrix} + \begin{pmatrix} \theta & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{0,v} \\ p_{1,v}(x) \\ p_{2,v}(x) \\ p_{3,v}(x) \\ \vdots \end{pmatrix}, \end{aligned}$$

where

$$\phi_v f = \int_0^{\infty} \mu_v(x) f(x) dx, \quad \phi_b f = \int_0^{\infty} \mu_b(x) f(x) dx, \quad f \in L^1[0, \infty),$$

$$\begin{aligned}\psi_v g &= -\frac{dg(x)}{dx} - (\theta + \lambda + \mu_v(x))g(x), \quad g \in W^{1,1}[0, \infty), \\ \psi_b g &= -\frac{dg(x)}{dx} - (\lambda + \mu_b(x))g(x), \quad g \in W^{1,1}[0, \infty). \\ D(A_m) &= \left\{ (p_v, p_b) \in X \times Y \left| \begin{array}{l} \frac{dp_{n,v}}{dx} \in L^1[0, \infty), \quad \frac{dp_{n,b}}{dx} \in L^1[0, \infty), \\ p_{n,v}(x) \text{ and } p_{n,b}(x) (n \geq 1) \text{ are absolutely} \\ \text{continuous and } \sum_{n=1}^{\infty} \left\| \frac{dp_{n,v}}{dx} \right\|_{L^1[0, \infty)} < \infty, \\ \sum_{n=1}^{\infty} \left\| \frac{dp_{n,b}}{dx} \right\|_{L^1[0, \infty)} < \infty \end{array} \right. \right\}.\end{aligned}$$

We choose the boundary space of $X \times Y$

$$\partial(X \times Y) = l^1 \times l^1$$

and define two boundary operators as

$$\begin{aligned}L : D(A_m) &\rightarrow \partial(X \times Y), \quad \Phi : D(A_m) \rightarrow \partial(X \times Y), \\ L \left(\begin{pmatrix} p_{0,v} \\ p_{1,v}(x) \\ p_{2,v}(x) \\ p_{3,v}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} p_{0,b} \\ p_{1,b}(x) \\ p_{2,b}(x) \\ p_{3,b}(x) \\ \vdots \end{pmatrix} \right) &= \left(\begin{pmatrix} p_{1,v}(0) \\ p_{2,v}(0) \\ p_{3,v}(0) \\ p_{4,v}(0) \\ \vdots \end{pmatrix}, \begin{pmatrix} p_{1,b}(0) \\ p_{2,b}(0) \\ p_{3,b}(0) \\ p_{4,b}(0) \\ \vdots \end{pmatrix} \right) \\ \Phi \left(\begin{pmatrix} p_{0,v} \\ p_{1,v}(x) \\ p_{2,v}(x) \\ p_{3,v}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} p_{0,b} \\ p_{1,b}(x) \\ p_{2,b}(x) \\ p_{3,b}(x) \\ \vdots \end{pmatrix} \right) &= \left(\begin{pmatrix} \lambda & 0 & \phi_v & 0 & 0 & \cdots \\ 0 & 0 & 0 & \phi_v & 0 & \cdots \\ 0 & 0 & 0 & 0 & \phi_v & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{0,v} \\ p_{1,v}(x) \\ p_{2,v}(x) \\ p_{3,v}(x) \\ \vdots \end{pmatrix}, \right. \\ &\quad \left. \begin{pmatrix} \lambda & 0 & \phi_b & 0 & 0 & \cdots \\ 0 & 0 & 0 & \phi_b & 0 & \cdots \\ 0 & 0 & 0 & 0 & \phi_b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{0,b} \\ p_{1,b}(x) \\ p_{2,b}(x) \\ p_{3,b}(x) \\ \vdots \end{pmatrix} + \begin{pmatrix} 0 & \theta \mathcal{F} & 0 & 0 & \cdots \\ 0 & 0 & \theta \mathcal{F} & 0 & \cdots \\ 0 & 0 & 0 & \theta \mathcal{F} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{0,v} \\ p_{1,v}(x) \\ p_{2,v}(x) \\ p_{3,v}(x) \\ \vdots \end{pmatrix} \right),\end{aligned}$$

where $\mathcal{F}g = \int_0^\infty g(x)dx$, $g \in L^1[0, \infty)$.

Now we define operator A and its domain as

$$\begin{aligned}A(p_v, p_b) &= A_m(p_v, p_b), \\ D(A) &= \{(p_v, p_b) \in D(A_m) \mid L(p_v, p_b) = \Phi(p_v, p_b)\}.\end{aligned}$$

Then the above equations (1.1)-(1.3) can be written as an abstract Cauchy problem:

$$\begin{cases} \frac{d(p_v, p_b)(t)}{dt} = A(p_v, p_b)(t), & t \in (0, \infty) \\ (p_v, p_b)(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases} \quad (2.1)$$

Kasim and Gupur [14] have obtained the following results.

Theorem 2.1. *If $\mu_v(x)$ and $\mu_b(x)$ are measurable functions and satisfy $\overline{\mu_v} = \sup_{x \in [0, \infty)} \mu_v(x) < \infty$ and $\overline{\mu_b} = \sup_{x \in [0, \infty)} \mu_b(x) < \infty$, then A generates a positive contraction C_0 -semigroup $T(t)$. The system (2.1) has a unique positive time-dependent solution $(p_v, p_b)(x, t) = T(t)(p_v, p_b)(0)$ satisfying*

$$p_{0,v}(t) + \sum_{n=1}^{\infty} \int_0^\infty p_{n,v}(x, t)dx + p_{0,b}(t) + \sum_{n=1}^{\infty} \int_0^\infty p_{n,b}(x, t)dx = 1, \quad t \in [0, \infty).$$

3 Main results

In this section, firstly we prove that 0 is an eigenvalue of A with geometric multiplicity one, next we study the resolvent set of operator A by using the Greiner's idea [17] and obtain that all points on the imaginary axis except zero belong to the resolvent set of A . Thirdly, we determine the expression of A^* , the adjoint operator of A , and verify that 0 is an eigenvalue of the adjoint operator A^* with geometric multiplicity one. Thus, we conclude that the time-dependent solution of the system (2.1) strongly converges to its steady-state solution.

Lemma 3.1. *If $\int_0^\infty \lambda x \mu_b(x) e^{-\int_0^x \mu_b(\tau) d\tau} dx < 1$, then 0 is an eigenvalue of A with geometric multiplicity one.*

Proof. We consider the equation $A(p_v, p_b) = 0$, which is equivalent to

$$\begin{aligned} (\theta + \lambda)p_{0,v} &= \int_0^\infty \mu_v(x)p_{1,v}(x)dx + \int_0^\infty \mu_b(x)p_{1,b}(x)dx, \\ \frac{dp_{1,v}(x)}{dx} &= -(\theta + \lambda + \mu_v(x))p_{1,v}(x), \\ \frac{dp_{n,v}(x)}{dx} &= -(\theta + \lambda + \mu_v(x))p_{n,v}(x) + \lambda p_{n-1,v}(x), \quad n \geq 2, \\ \lambda p_{0,b} &= \theta p_{0,v}, \\ \frac{dp_{1,b}(x)}{dx} &= -(\lambda + \mu_b(x))p_{1,b}(x), \\ \frac{dp_{n,b}(x)}{dx} &= -(\lambda + \mu_b(x))p_{n,b}(x) + \lambda p_{n-1,b}(x), \quad n \geq 2, \end{aligned} \quad (3.1)$$

with the boundary conditions

$$\begin{aligned} p_{1,v}(0) &= \lambda p_{0,v} + \int_0^\infty \mu_v(x)p_{2,v}(x)dx, \\ p_{n,v}(0) &= \int_0^\infty \mu_v(x)p_{n+1,v}(x)dx, \quad n \geq 2, \\ p_{1,b}(0) &= \lambda p_{0,b} + \int_0^\infty \mu_b(x)p_{2,b}(x)dx + \theta \int_0^\infty p_{1,v}(x)dx, \\ p_{n,b}(0) &= \int_0^\infty \mu_b(x)p_{n+1,b}(x)dx + \theta \int_0^\infty p_{n,v}(x)dx, \quad n \geq 2. \end{aligned} \quad (3.2)$$

By solving (3.1) we obtain

$$p_{n,v}(x) = e^{-(\theta+\lambda)x - \int_0^x \mu_v(s)ds} \sum_{k=1}^n p_{k,v}(0) \frac{(\lambda x)^{n-k}}{(n-k)!}, \quad n \geq 1, \quad (3.3)$$

$$p_{n,b}(x) = e^{-\lambda x - \int_0^x \mu_b(s)ds} \sum_{k=1}^n p_{k,b}(0) \frac{(\lambda x)^{n-k}}{(n-k)!}, \quad n \geq 1. \quad (3.4)$$

Define

$$\begin{aligned} b_k &= \int_0^\infty \frac{(\lambda x)^k}{k!} \mu_b(x) e^{-\lambda x - \int_0^x \mu_b(s)ds} dx, \quad c_k = \int_0^\infty \frac{(\lambda x)^k}{k!} \mu_v(x) e^{-(\theta+\lambda)x - \int_0^x \mu_v(s)ds} dx, \\ d_k &= \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-(\theta+\lambda)x - \int_0^x \mu_v(s)ds} dx, \quad k \geq 0. \end{aligned}$$

The probability generating functions of these sequences are given by, for $|z| < 1$

$$B(z) = \sum_{i=0}^\infty b_i z^i = \int_0^\infty \mu_b(x) e^{-(\lambda - \lambda z)x - \int_0^x \mu_b(s)ds} dx,$$

$$C(z) = \sum_{i=0}^{\infty} c_i z^i = \int_0^{\infty} \mu_v(x) e^{-(\theta+\lambda-\lambda z)x - \int_0^x \mu_v(s) ds} dx,$$

$$D(z) = \sum_{i=0}^{\infty} d_i z^i = \int_0^{\infty} e^{-(\theta+\lambda-\lambda z)x - \int_0^x \mu_v(s) ds} dx.$$

From (3.1)-(3.4) we have

$$\begin{aligned} p_{0,v} &= [1 - (\theta + \lambda)]p_{0,v} + c_0 p_{1,v}(0) + b_0 p_{1,b}(0), \\ p_{0,b} &= (1 - \lambda)p_{0,b} + \theta p_{0,v}, \\ p_{1,v}(0) &= \lambda p_{0,v} + c_0 p_{2,v}(0) + c_1 p_{1,v}(0), \\ p_{1,b}(0) &= \lambda p_{0,b} + b_0 p_{2,b}(0) + b_1 p_{1,b}(0) + \theta d_0 p_{1,v}(0), \\ p_{2,v}(0) &= c_0 p_{3,v}(0) + c_1 p_{2,v}(0) + c_2 p_{1,v}(0), \\ p_{2,b}(0) &= b_0 p_{3,b}(0) + b_1 p_{2,b}(0) + b_2 p_{1,b}(0) + \theta d_0 p_{2,v}(0) + \theta d_1 p_{1,v}(0), \\ &\dots\dots \\ p_{n,v}(0) &= \sum_{k=0}^n c_{n-k} p_{k+1,v}(0), \quad n \geq 2, \\ p_{n,b}(0) &= \sum_{k=0}^n b_{n-k} p_{k+1,b}(0) + \sum_{k=1}^n \theta d_{n-k} p_{k,v}(0), \quad n \geq 2. \end{aligned} \quad (3.5)$$

Let $q_0 = (p_{0,v}, p_{0,b})$, $q_i = (p_{i,v}(0), p_{i,b}(0))$, $i \geq 1$, and define

$$U_0 = \begin{pmatrix} 1 - (\theta + \lambda) & \theta \\ 0 & 1 - \lambda \end{pmatrix}, \quad U_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad E_0 = \begin{pmatrix} 0 & c_0 \\ 0 & b_0 \end{pmatrix},$$

$$L_0 = \begin{pmatrix} c_0 & 0 \\ 0 & b_0 \end{pmatrix}, \quad L_i = \begin{pmatrix} c_i & \theta d_{i-1} \\ 0 & b_i \end{pmatrix}, \quad i \geq 1.$$

Then (3.5) can be written as

$$q_0 = q_0 U_0 + q_1 E_0, \quad (3.6)$$

$$q_1 = q_0 U_1 + q_1 L_1 + q_2 L_0, \quad (3.7)$$

$$q_2 = q_1 L_2 + q_2 L_1 + q_3 L_0, \quad (3.8)$$

...

$$q_k = \sum_{j=1}^{k+1} q_j L_{k+1-j}, \quad k \geq 2. \quad (3.9)$$

Now, we introduce the row-vector generating functions

$$\Pi(z) = \sum_{i=1}^{\infty} q_i z^i, \quad L(z) = \sum_{i=0}^{\infty} L_i z^i, \quad |z| < 1.$$

Hence, from (3.7)-(3.9) we deduce

$$\begin{aligned} \Pi(z) &= \sum_{i=1}^{\infty} q_i z^i = q_0 U_1 z + \sum_{i=1}^{\infty} z^i \sum_{j=1}^{i+1} q_j L_{i+1-j} \\ &= q_0 U_1 z + \frac{1}{z} \sum_{i=1}^{\infty} q_i z^i \sum_{j=0}^{\infty} L_j z^j - q_1 L_0 \\ &= q_0 U_1 z + \frac{1}{z} \Pi(z) L(z) - q_1 L_0, \\ &\implies \\ \Pi(z) &= z(q_0 U_1 z - q_1 L_0)[zI - L(z)]^{-1}. \end{aligned} \quad (3.10)$$

An easy computation shows that

$$[zI - L(z)] = \begin{pmatrix} z - C(z) & -\theta z D(z) \\ 0 & z - B(z) \end{pmatrix},$$

and

$$[zI - L(z)]^{-1} = \begin{pmatrix} \frac{1}{z - C(z)} & \frac{\theta z D(z)}{[z - C(z)][z - B(z)]} \\ 0 & \frac{1}{z - B(z)} \end{pmatrix}.$$

This together with (3.10) yields

$$\begin{aligned} \Pi(z) = z & \left(\frac{\lambda z p_{0,v} - c_0 p_{1,v}(0)}{z - C(z)}, \right. \\ & \left. \frac{(\lambda z p_{0,v} - c_0 p_{1,v}(0))\theta z D(z) + (\lambda p_{0,b} z - b_0 p_{1,b}(0))[z - C(z)]}{[z - C(z)][z - B(z)]} \right). \end{aligned} \quad (3.11)$$

Thus, we have

$$\Pi(z)e = z \frac{(\lambda z p_{0,v} - c_0 p_{1,v}(0))[\theta z D(z) + z - B(z)] + (\lambda p_{0,b} z - b_0 p_{1,b}(0))[z - C(z)]}{[z - C(z)][z - B(z)]}, \quad (3.12)$$

where $e = (1, 1)^T$.

In the following, by using the Rouché's theorem we conclude that $z - C(z)$ has a unique zero point inside unit circle $|z| = 1$. Let this root be denoted by γ , this must be root of the numerator of the equation (3.12) too. So, substituting $z = \gamma$ into (3.12) we get

$$\begin{aligned} & (\lambda \gamma p_{0,v} - c_0 p_{1,v}(0))[\theta \gamma D(\gamma) + \gamma - B(\gamma)] = 0 \\ & \implies \\ & p_{1,v}(0) = \frac{\lambda \gamma}{c_0} p_{0,v}. \end{aligned} \quad (3.13)$$

(3.6) and (3.13) give

$$p_{1,b}(0) = \frac{\lambda(1 - \gamma) + \theta}{b_0} p_{0,v}. \quad (3.14)$$

By using the L'Hospital rule and (3.12)-(3.14), we determine

$$\begin{aligned} \sum_{i=1}^{\infty} p_{i,v}(0) &= \lim_{z \rightarrow 1} \Pi(z)e_1 = \lim_{z \rightarrow 1} \frac{(\lambda z p_{0,v} - c_0 p_{1,v}(0))z}{z - C(z)} \\ &= \frac{\lambda(1 - \gamma)}{\theta \int_0^{\infty} e^{-\theta x - \int_0^x \mu_v(s) ds} dx} p_{0,v}, \\ \sum_{i=1}^{\infty} p_{i,b}(0) &= \lim_{z \rightarrow 1} \Pi(z)e_2 \\ &= \lim_{z \rightarrow 1} \frac{z \{ (\lambda z p_{0,v} - c_0 p_{1,v}(0))\theta z D(z) + (\lambda z p_{0,b} - b_0 p_{1,b}(0))[z - C(z)] \}}{[z - C(z)][z - B(z)]} \\ &= \lim_{z \rightarrow 1} \left\{ \lambda \theta z D(z) p_{0,v} + (\lambda z p_{0,v} - c_0 p_{1,v}(0))[\theta D(z) + \theta z D'(z)] \right. \\ &\quad \left. + \lambda z p_{0,b}[z - c(z)] + (\lambda z p_{0,b} - b_0 p_{1,b}(0))[1 - C'(z)] \right\} \\ &\quad \times \left\{ [1 - C'(z)][z - B(z)] + [z - C(z)][1 - B'(z)] \right\}^{-1} \end{aligned} \quad (3.15)$$

$$\begin{aligned}
&= \left\{ \lambda \theta D(1) p_{0,v} + \lambda(1-\gamma) [\theta D(1) + \theta D'(1)] p_{0,v} \right. \\
&\quad \left. + \theta [1 - c(1)] p_{0,v} - \lambda(1-\gamma) [1 - C'(z)] p_{0,v} \right\} \\
&\quad \times \left\{ [1 - C'(1)] [1 - B(1)] + [1 - C(1)] [1 - B'(1)] \right\}^{-1} \\
&= \frac{(\theta + \lambda) [\lambda(1-\gamma) + \theta] D(1) - \lambda(1-\gamma)}{[1 - C(1)] [1 - B'(1)]} p_{0,v} \\
&= \frac{(\theta + \lambda) [\lambda(1-\gamma) + \theta] \int_0^\infty e^{-\theta x - \int_0^x \mu_v(s) ds} dx - \lambda(1-\gamma)}{\theta \int_0^\infty e^{-\theta x - \int_0^x \mu_v(s) ds} dx [1 - \int_0^\infty \mu_b(x) \lambda x e^{-\int_0^x \mu_b(s) ds} dx]} p_{0,v}, \tag{3.16}
\end{aligned}$$

where $e_1 = (1, 0)^T$, $e_2 = (0, 1)^T$.

By combining (3.15) and (3.16) with (3.3) and (3.4), we estimate

$$\begin{aligned}
\sum_{n=1}^\infty \int_0^\infty p_{n,v}(x) dx &= \int_0^\infty e^{-(\theta+\lambda)x - \int_0^x \mu_v(s) ds} \sum_{n=1}^\infty \sum_{k=1}^n p_{k,v}(0) \frac{(\lambda x)^{n-k}}{(n-k)!} dx \\
&= \sum_{k=1}^\infty p_{k,v}(0) \int_0^\infty e^{-(\theta+\lambda)x - \int_0^x \mu_v(s) ds} \sum_{n=1}^\infty \frac{(\lambda x)^n}{n!} dx \\
&= \sum_{k=1}^\infty p_{k,v}(0) \int_0^\infty e^{-\theta x - \int_0^x \mu_v(s) ds} dx < \infty, \tag{3.17}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^\infty \int_0^\infty p_{n,b}(x) dx &= \int_0^\infty e^{-\lambda x - \int_0^x \mu_b(s) ds} \sum_{n=1}^\infty \sum_{k=1}^n p_{k,b}(0) \frac{(\lambda x)^{n-k}}{(n-k)!} dx \\
&= \sum_{k=1}^\infty p_{k,b}(0) \int_0^\infty e^{-\int_0^x \mu_b(s) ds} dx < \infty. \tag{3.18}
\end{aligned}$$

(3.17) and (3.18) imply

$$\|(p_v, p_b)\| = \|p_v\| + \|p_b\| < \infty.$$

Thus, 0 is an eigenvalue of A . Moreover, from (3.5) it easy to see that the eigenvectors corresponding to zero span one dimensional linear space, i.e., the geometric multiplicity of 0 is one. $\square$

In order to obtain the asymptotic behavior of the time-dependent solution of the system (2.1) we need to know the spectrum of A on the imaginary axis (see Theorem 14 in Gupur et al.[18]). For that purpose we use boundary perturbation method, which is developed by the Greiner [17], through which the spectrum of the operator can be deduced by discussing the boundary operator. It is related to the resolvent set of operator A_0 and spectrum of ΦD_γ , where D_γ is inverse of L in $\ker(\gamma I - A_m)$. Hence, we first consider the operator

$$\begin{aligned}
A_0(p_v, p_b) &= A_m(p_v, p_b), \\
D(A_0) &= \{(p_v, p_b) \in D(A_m) \mid L(p_v, p_b) = 0\},
\end{aligned}$$

and discuss its inverse. For any given $(y, z) \in X \times Y$, we consider the equation $(\gamma I - A_0)(p_v, p_b) = (y, z)$, i.e.,

$$\begin{aligned}
 (\gamma + \theta + \lambda)p_{0,v} &= y_0 + \int_0^\infty \mu_v(x)p_{1,v}(x)dx + \int_0^\infty \mu_b(x)p_{1,b}(x)dx, \\
 \frac{dp_{1,v}(x)}{dx} &= -(\gamma + \theta + \lambda + \mu_v(x))p_{1,v}(x) + y_1(x), \\
 \frac{dp_{n,v}(x)}{dx} &= -(\gamma + \theta + \lambda + \mu_v(x))p_{n,v}(x) + \lambda p_{n-1,v}(x) + y_n(x), \quad n \geq 2, \\
 (\gamma + \lambda)p_{0,b} &= z_0 + \theta p_{0,v}, \\
 \frac{dp_{1,b}(x)}{dx} &= -(\gamma + \lambda + \mu_b(x))p_{1,b}(x) + z_1(x), \\
 \frac{dp_{n,b}(x)}{dx} &= -(\gamma + \lambda + \mu_b(x))p_{n,b}(x) + \lambda p_{n-1,b}(x) + z_n(x), \quad n \geq 2, \\
 p_{n,v}(0) &= 0, \quad p_{n,b}(0) = 0, \quad n \geq 1.
 \end{aligned} \tag{3.19}$$

By solving (3.19) we have

$$\begin{aligned}
 p_{0,v} &= \frac{1}{\gamma + \theta + \lambda} y_0 + \frac{1}{\gamma + \theta + \lambda} \int_0^\infty \mu_v(x)p_{1,v}(x)dx \\
 &\quad + \frac{1}{\gamma + \theta + \lambda} \int_0^\infty \mu_b(x)p_{1,b}(x)dx,
 \end{aligned} \tag{3.20}$$

$$p_{1,v}(x) = e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} \int_0^x y_1(\tau) e^{\int_0^\tau (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} d\tau, \tag{3.21}$$

$$\begin{aligned}
 p_{n,v}(x) &= \lambda e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} \int_0^x p_{n-1,v}(\tau) e^{\int_0^\tau (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} d\tau \\
 &\quad + e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} \int_0^x y_n(\tau) e^{\int_0^\tau (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} d\tau, \quad n \geq 1,
 \end{aligned} \tag{3.22}$$

$$p_{0,b} = \frac{1}{\gamma + \lambda} z_0 + \frac{\theta}{\gamma + \lambda} p_{0,v}, \tag{3.23}$$

$$p_{1,b}(x) = e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi))d\xi} \int_0^x z_1(\tau) e^{\int_0^\tau (\gamma + \lambda + \mu_b(\xi))d\xi} d\tau, \tag{3.24}$$

$$\begin{aligned}
 p_{n,b}(x) &= \lambda e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi))d\xi} \int_0^x p_{n-1,b}(\tau) e^{\int_0^\tau (\gamma + \lambda + \mu_b(\xi))d\xi} d\tau \\
 &\quad + e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi))d\xi} \int_0^x z_n(\tau) e^{\int_0^\tau (\gamma + \lambda + \mu_b(\xi))d\xi} d\tau, \quad n \geq 1.
 \end{aligned} \tag{3.25}$$

If we introduce the following two operators as

$$E_v f(x) = e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} \int_0^x f(\tau) e^{\int_0^\tau (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} d\tau, \quad f \in L^1[0, \infty),$$

$$E_b f(x) = e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi))d\xi} \int_0^x f(\tau) e^{\int_0^\tau (\gamma + \lambda + \mu_b(\xi))d\xi} d\tau, \quad f \in L^1[0, \infty),$$

then (3.21), (3.22), (3.24) and (3.25) imply

$$p_{1,v}(x) = E_v y_1(x), \tag{3.26}$$

$$\begin{aligned}
p_{n,v}(x) &= \lambda E_v p_{n-1,v}(x) + E_v y_n(x) \\
&= \lambda E_v [\lambda E_v p_{n-2,0}(x) + E_v y_{n-1}(x)] + E_v y_n(x) \\
&= \lambda^2 E_v^2 p_{n-2,0}(x) + \lambda E_v^2 y_{n-1}(x) + E_v y_n(x) \\
&= \lambda^3 E_v^3 p_{n-3,0}(x) + \lambda^2 E_v^3 y_{n-2}(x) + \lambda E_v^2 y_{n-1}(x) + E_v y_n(x) \\
&= \dots \\
&= \lambda^{n-1} E_v^n y_1(x) + \lambda^{n-2} E_v^{n-1} y_2(x) + \lambda^{n-3} E_v^{n-2} y_3(x) \\
&\quad + \dots + \lambda^2 E_v^3 y_{n-2}(x) + \lambda E_v^2 y_{n-1}(x) + E_v y_n(x) \\
&= \sum_{k=0}^{n-1} \lambda^k E_v^{k+1} y_{n-k}(x), \quad n \geq 2.
\end{aligned} \tag{3.27}$$

Similarly, we have

$$p_{1,b}(x) = E_b z_1(x), \tag{3.28}$$

$$p_{n,b}(x) = \sum_{k=0}^{n-1} \lambda^k E_b^{k+1} z_{n-k}(x), \quad n \geq 2. \tag{3.29}$$

By inserting (3.26) and (3.28) into (3.20), we obtain

$$\begin{aligned}
p_{0,v} &= \frac{1}{\gamma + \theta + \lambda} y_0 + \frac{1}{\gamma + \theta + \lambda} \phi_v E_v y_1(x) + \frac{1}{\gamma + \theta + \lambda} \phi_b E_b z_1(x), \\
p_{0,b} &= \frac{1}{\gamma + \lambda} z_0 + \frac{\theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} y_0 \\
&\quad + \frac{\theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} \phi_v E_v y_1(x) + \frac{\theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} \phi_b E_b z_1(x).
\end{aligned} \tag{3.30}$$

(3.26)-(3.30) give the expression of $(\gamma I - A_0)^{-1}$ as follows if $(\gamma I - A_0)^{-1}$ exists.

$$\begin{aligned}
&(\gamma I - A_0)^{-1} \left(\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \end{pmatrix}, \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ \vdots \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} \frac{1}{\gamma + \theta + \lambda} & \frac{1}{\gamma + \theta + \lambda} \phi_v E_v & 0 & 0 & 0 & \dots \\ 0 & E_v & 0 & 0 & 0 & \dots \\ 0 & \lambda E_v^2 & E_v & 0 & 0 & \dots \\ 0 & \lambda^2 E_v^3 & \lambda E_v^2 & E_v & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} y_0 \\ y_1(x) \\ y_2(x) \\ y_3(x) \\ \vdots \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} 0 & \frac{1}{\gamma + \theta + \lambda} \phi_b E_b & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} z_0 \\ z_1(x) \\ z_2(x) \\ z_3(x) \\ \vdots \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} \frac{1}{\gamma + \lambda} & \frac{\theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} \phi_b E_b & 0 & 0 & 0 & \dots \\ 0 & E_b & 0 & 0 & 0 & \dots \\ 0 & \lambda E_b^2 & E_b & 0 & 0 & \dots \\ 0 & \lambda^2 E_b^3 & \lambda E_b^2 & E_b & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} z_0 \\ z_1(x) \\ z_2(x) \\ z_3(x) \\ \vdots \end{pmatrix} \right)
\end{aligned}$$

$$+ \begin{pmatrix} \frac{\theta}{(\gamma+\lambda)(\gamma+\theta+\lambda)} & \frac{\theta}{(\gamma+\lambda)(\gamma+\theta+\lambda)} \phi_v E_v & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} y_0 \\ y_1(x) \\ y_2(x) \\ y_3(x) \\ \vdots \end{pmatrix}.$$

Therefore, we obtain the following two lemmas and their proof given in the appendix.

Lemma 3.2. *If*

$$0 < \underline{\mu_v} = \inf_{x \in [0, \infty)} \mu_v(x) \leq \overline{\mu_v} = \sup_{x \in [0, \infty)} \mu_v(x) < \infty,$$

$$0 < \underline{\mu_b} = \inf_{x \in [0, \infty)} \mu_b(x) \leq \overline{\mu_b} = \sup_{x \in [0, \infty)} \mu_b(x) < \infty,$$

then

$$\left\{ \gamma \in \mathbb{C} \begin{array}{l} \operatorname{Re} \gamma + \lambda > 0 \\ \operatorname{Re} \gamma + \underline{\mu_v} > 0 \\ \operatorname{Re} \gamma + \underline{\mu_b} > 0 \end{array} \right\} \subset \rho(A_0).$$

Lemma 3.3. *Let*

$$0 < \underline{\mu_v} = \inf_{x \in [0, \infty)} \mu_v(x) \leq \overline{\mu_v} = \sup_{x \in [0, \infty)} \mu_v(x) < \infty,$$

$$0 < \underline{\mu_b} = \inf_{x \in [0, \infty)} \mu_b(x) \leq \overline{\mu_b} = \sup_{x \in [0, \infty)} \mu_b(x) < \infty.$$

If $\gamma \in \rho(A_0)$, then

$$\begin{aligned} (p_v, p_b) &\in \ker(\gamma I - A_m) \iff \\ p_{0,v} &= \frac{1}{\gamma + \theta + \lambda} a_{1,v} \int_0^\infty \mu_v(x) e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi)) d\xi} dx \\ &+ \frac{1}{\gamma + \theta + \lambda} a_{1,b} \int_0^\infty \mu_b(x) e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi)) d\xi} dx, \\ p_{n,v}(x) &= e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi)) d\xi} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} a_{n-k,v}, \quad n \geq 1, \\ p_{0,b} &= \frac{\theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} a_{1,v} \int_0^\infty \mu_v(x) e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi)) d\xi} dx \\ &+ \frac{\theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} a_{1,b} \int_0^\infty \mu_b(x) e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi)) d\xi} dx, \\ p_{n,b}(x) &= e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi)) d\xi} \sum_{k=0}^n \frac{(\lambda x)^k}{k!} a_{n-k,b}, \quad n \geq 1, \\ (a_{1,v}, a_{2,v}, a_{3,v}, \dots) &\in l^1, \quad (a_{1,b}, a_{2,b}, a_{3,b}, \dots) \in l^1. \end{aligned} \tag{3.31}$$

Using the results in Greiner [17], observe that the operator L is surjective. So,

$$L|_{\ker(\gamma I - A_m)} : \ker(\gamma I - A_m) \rightarrow \partial(X \times Y)$$

is invertible if $\gamma \in \rho(A_0)$. Its inverse will play an important role in the characterization of the spectrum of A on the imaginary axis and we denote its inverse by

$$D_\gamma = \left(L|_{\ker(\gamma I - A_m)} \right)^{-1} : \partial(X \times Y) \rightarrow \ker(\gamma I - A_m),$$

and call it the Dirichlet operator. Furthermore, Lemma 3.3 gives the explicit formula of D_γ for all $\gamma \in \rho(A_0)$,

$$D_\gamma \left(\begin{pmatrix} a_{1,v} \\ a_{2,v} \\ a_{3,v} \\ \vdots \end{pmatrix}, \begin{pmatrix} a_{1,v} \\ a_{2,v} \\ a_{3,v} \\ \vdots \end{pmatrix} \right) = \left(\begin{pmatrix} \frac{1}{\gamma+\theta+\lambda} \phi_v h_{00} & 0 & 0 & \cdots \\ h_{11} & 0 & 0 & \cdots \\ h_{21} & h_{22} & 0 & \cdots \\ h_{31} & h_{32} & h_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{1,v} \\ a_{2,v} \\ a_{3,v} \\ a_{4,v} \\ \vdots \end{pmatrix} \right. \\ \left. + \begin{pmatrix} \frac{1}{\gamma+\theta+\lambda} \phi_b m_{00} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{1,b} \\ a_{2,b} \\ a_{3,b} \\ a_{4,1} \\ \vdots \end{pmatrix} \right), \\ \left(\begin{pmatrix} \frac{\theta}{(\gamma+\lambda)(\gamma+\theta+\lambda)} \phi_b m_{00} & 0 & 0 & \cdots \\ m_{11} & 0 & 0 & \cdots \\ m_{21} & m_{22} & 0 & \cdots \\ m_{31} & m_{32} & m_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{1,b} \\ a_{2,b} \\ a_{3,b} \\ a_{4,1} \\ \vdots \end{pmatrix} \right. \\ \left. + \begin{pmatrix} \frac{\theta}{(\gamma+\lambda)(\gamma+\theta+\lambda)} \phi_v h_{00} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{1,v} \\ a_{2,v} \\ a_{3,v} \\ a_{4,v} \\ \vdots \end{pmatrix} \right),$$

where

$$h_{00} = e^{-\int_0^x (\gamma+\theta+\lambda+\mu_v(\xi)) d\xi}, \\ h_{11} = e^{-\int_0^x (\gamma+\theta+\lambda+\mu_v(\xi)) d\xi}, \quad h_{21} = \lambda x e^{-\int_0^x (\gamma+\theta+\lambda+\mu_v(\xi)) d\xi}, \quad h_{22} = e^{-\int_0^x (\gamma+\theta+\lambda+\mu_v(\xi)) d\xi}, \\ h_{31} = \frac{(\lambda x)^2}{2} e^{-\int_0^x (\gamma+\theta+\lambda+\mu_v(\xi)) d\xi}, \quad h_{32} = \lambda x e^{-\int_0^x (\gamma+\theta+\lambda+\mu_v(\xi)) d\xi}, \quad h_{33} = e^{-\int_0^x (\gamma+\theta+\lambda+\mu_v(\xi)) d\xi}, \\ \dots \\ h_{ij} = \frac{(\lambda x)^{i-j}}{(i-j)!} e^{-\int_0^x (\gamma+\theta+\lambda+\mu_v(\xi)) d\xi}, \quad i = 1, 2, \dots, j = 1, 2, \dots, i. \\ m_{00} = e^{-\int_0^x (\gamma+\lambda+\mu_b(\xi)) d\xi}, \\ m_{11} = e^{-\int_0^x (\gamma+\lambda+\mu_b(\xi)) d\xi}, \quad m_{21} = \lambda x e^{-\int_0^x (\gamma+\lambda+\mu_b(\xi)) d\xi}, \quad m_{22} = e^{-\int_0^x (\gamma+\lambda+\mu_b(\xi)) d\xi}, \\ m_{31} = \frac{(\lambda x)^2}{2} e^{-\int_0^x (\gamma+\lambda+\mu_b(\xi)) d\xi}, \quad m_{32} = \lambda x e^{-\int_0^x (\gamma+\lambda+\mu_b(\xi)) d\xi}, \quad m_{33} = e^{-\int_0^x (\gamma+\lambda+\mu_b(\xi)) d\xi}, \\ \dots \\ m_{ij} = \frac{(\lambda x)^{i-j}}{(i-j)!} e^{-\int_0^x (\gamma+\lambda+\mu_b(\xi)) d\xi}, \quad i = 1, 2, \dots, j = 1, 2, \dots, i.$$

From the expression of D_γ and the definition of Φ , it is easy to determine the explicit form of ΦD_γ as follows.

$$\Phi D_\gamma \left(\begin{pmatrix} a_{1,v} \\ a_{2,v} \\ a_{3,v} \\ \vdots \end{pmatrix}, \begin{pmatrix} a_{1,b} \\ a_{2,b} \\ a_{3,b} \\ \vdots \end{pmatrix} \right) \\ = \left(\begin{pmatrix} \frac{\lambda}{\gamma+\theta+\lambda} \phi_v h_{00} + \phi_v h_{21} & \phi_v h_{22} & 0 & 0 & \cdots \\ \phi_v h_{31} & \phi_v h_{32} & \phi_v h_{33} & 0 & \cdots \\ \phi_v h_{41} & \phi_v h_{42} & \phi_v h_{43} & \phi_v h_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{1,v} \\ a_{2,v} \\ a_{3,v} \\ \vdots \end{pmatrix} \right)$$

$$\begin{aligned}
& + \begin{pmatrix} \frac{\lambda}{\gamma+\theta+\lambda} \phi_b m_{00} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{1,b} \\ a_{2,b} \\ a_{3,b} \\ \vdots \end{pmatrix}, \\
& \begin{pmatrix} \frac{\lambda\theta}{(\gamma+\lambda)(\gamma+\theta+\lambda)} \phi_b m_{00} + \phi_b m_{21} & \phi_b m_{22} & 0 & 0 & \cdots \\ \phi_b m_{31} & \phi_b m_{32} & \phi_b m_{33} & 0 & \cdots \\ \phi_b m_{41} & \phi_b m_{42} & \phi_b m_{43} & \phi_b m_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{1,b} \\ a_{2,b} \\ a_{3,b} \\ \vdots \end{pmatrix} \\
& + \begin{pmatrix} \frac{\lambda\theta}{(\gamma+\lambda)(\gamma+\theta+\lambda)} \phi_v h_{00} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{1,v} \\ a_{2,v} \\ a_{3,v} \\ \vdots \end{pmatrix} \\
& + \begin{pmatrix} \theta \mathcal{F} h_{11} & 0 & 0 & 0 & \cdots \\ \theta \mathcal{F} h_{21} & \theta \mathcal{F} h_{22} & 0 & 0 & \cdots \\ \theta \mathcal{F} h_{31} & \theta \mathcal{F} h_{32} & \theta \mathcal{F} h_{33} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{1,v} \\ a_{2,v} \\ a_{3,v} \\ \vdots \end{pmatrix}.
\end{aligned}$$

Haji and Radl [19] gave the following result, which indicates the relations between the spectrum of A and spectrum of ΦD_γ .

Lemma 3.4. *If $\gamma \in \rho(A_0)$ and there exists $\gamma_0 \in \mathbb{C}$ such that $1 \notin \sigma(\Phi D_{\gamma_0})$, then*

$$\gamma \in \sigma(A) \iff 1 \in \sigma(\Phi D_\gamma).$$

From Lemma 3.4 and Nagel [20], we obtain the resolvent set of A on the imaginary axis.

Lemma 3.5. *If*

$$\begin{aligned}
0 < \underline{\mu}_v &= \inf_{x \in [0, \infty)} \mu_v(x) \leq \overline{\mu}_v = \sup_{x \in [0, \infty)} \mu_v(x) < \infty, \\
0 < \underline{\mu}_b &= \inf_{x \in [0, \infty)} \mu_b(x) \leq \overline{\mu}_b = \sup_{x \in [0, \infty)} \mu_b(x) < \infty,
\end{aligned}$$

then all points on the imaginary axis except zero belong to the resolvent set of A .

Proof. Let $\gamma = i\beta$, $\beta \in \mathbb{R} \setminus \{0\}$. The Riemann-Lebesgue lemma

$$\lim_{b \rightarrow \infty} \int_0^\infty f(x) \cos \beta x dx = 0, \quad \lim_{b \rightarrow \infty} \int_0^\infty f(x) \sin \beta x dx = 0, \quad f \in L^1[0, \infty)$$

implies that there exists a positive constant $\mathcal{K} > 0$ such that $\forall |\beta| > \mathcal{K}$,

$$\begin{aligned}
\left| \int_0^\infty f(x) e^{-i\beta x} dx \right|^2 &= \left| \int_0^\infty f(x) (\cos \beta x - i \sin \beta x) dx \right|^2 \\
&= \left(\int_0^\infty f(x) \cos \beta x dx \right)^2 + \left(\int_0^\infty f(x) \sin \beta x dx \right)^2 \\
&< \left(\int_0^\infty f(x) dx \right)^2, \quad 0 < f \in L^1[0, \infty).
\end{aligned}$$

In this formula, by replacing $f(x)$ with $\mu_v(x) e^{-\int_0^x (\theta + \lambda + \mu_v(\xi)) d\xi}$, $\mu_b(x) e^{-\int_0^x (\lambda + \mu_b(\xi)) d\xi}$, and

$$\sum_{j=l+1}^\infty \left| \int_0^\infty \mu_v(x) \frac{(\lambda x)^{j-l}}{(j-l)!} e^{-\int_0^x (i\beta + \theta + \lambda + \mu_v(\xi)) d\xi} dx \right|$$

$$\begin{aligned}
&\leq \sum_{j=l+1}^{\infty} \int_0^{\infty} \mu_v(x) \frac{(\lambda x)^{j-1}}{(j-1)!} \left| e^{-i\beta x} \right| \left| e^{-\int_0^x (\theta + \lambda + \mu_v(\xi)) d\xi} \right| dx \\
&= \sum_{j=l+1}^{\infty} \int_0^{\infty} \mu_v(x) \frac{(\lambda x)^{j-1}}{(j-1)!} e^{-\int_0^x (\theta + \lambda + \mu_v(\xi)) d\xi} dx, \quad l \geq 1,
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=l+1}^{\infty} \left| \int_0^{\infty} \mu_b(x) \frac{(\lambda x)^{j-l}}{(j-l)!} e^{-\int_0^x (i\beta + \lambda + \mu_b(\xi)) d\xi} dx \right| \\
&\leq \sum_{j=l+1}^{\infty} \int_0^{\infty} \mu_b(x) \frac{(\lambda x)^{j-1}}{(j-1)!} \left| e^{-i\beta x} \right| \left| e^{-\int_0^x (\lambda + \mu_b(\xi)) d\xi} \right| dx \\
&= \sum_{j=l+1}^{\infty} \int_0^{\infty} \mu_b(x) \frac{(\lambda x)^{j-1}}{(j-1)!} e^{-\int_0^x (\lambda + \mu_b(\xi)) d\xi} dx, \quad l \geq 1,
\end{aligned}$$

and using the fact $\int_0^{\infty} \mu_b(x) e^{-\int_0^x \mu_b(\xi) d\xi} dx = 1$, $\int_0^{\infty} (\theta + \mu_v(x)) e^{-\int_0^x (\theta + \mu_v(\xi)) d\xi} dx = 1$, we estimate for $\vec{a}_v = (a_{1,v}, a_{2,v}, a_{3,v}, \dots) \in l^1$ and $\vec{a}_b = (a_{1,b}, a_{2,b}, a_{3,b}, \dots) \in l^1$,

$$\begin{aligned}
\|\Phi D_{\gamma}(\vec{a}_v, \vec{a}_b)\| &\leq \left| \frac{\lambda}{\gamma + \theta + \lambda} \phi_v h_{00} + \frac{\lambda \theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} \phi_v h_{00} + \phi_v h_{21} \right| |a_{1,v}| \\
&\quad + \sum_{j=3}^{\infty} |\phi_v h_{j1}| |a_{1,v}| + \sum_{j=2}^{\infty} |\phi_v h_{j2}| |a_{2,v}| + \sum_{j=3}^{\infty} |\phi_v h_{j3}| |a_{3,v}| + \dots \\
&\quad + \left| \frac{\lambda}{\gamma + \theta + \lambda} \phi_b m_{00} + \frac{\lambda \theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} \phi_b m_{00} + \phi_b m_{21} \right| |a_{1,b}| \\
&\quad + \sum_{j=3}^{\infty} |\phi_b m_{j1}| |a_{1,b}| + \sum_{j=2}^{\infty} |\phi_b m_{j2}| |a_{2,b}| + \sum_{j=3}^{\infty} |\phi_b m_{j3}| |a_{3,b}| + \dots \\
&\quad + \sum_{j=1}^{\infty} |\theta \mathcal{F} h_{j1}| |a_{1,v}| + \sum_{j=2}^{\infty} |\theta \mathcal{F} h_{j2}| |a_{2,v}| + \sum_{j=3}^{\infty} |\theta \mathcal{F} h_{j3}| |a_{3,v}| + \dots \\
&\leq \frac{\lambda}{|\gamma + \lambda|} |\phi_v h_{00}| |a_{1,v}| + \sum_{j=2}^{\infty} |\phi_v h_{j1}| |a_{1,v}| \\
&\quad + \sum_{j=2}^{\infty} |\phi_v h_{j2}| |a_{2,v}| + \sum_{j=3}^{\infty} |\phi_v h_{j3}| |a_{3,v}| + \dots \\
&\quad + \frac{\lambda}{|\gamma + \lambda|} |\phi_b m_{00}| |a_{1,b}| + \sum_{j=2}^{\infty} |\phi_b m_{j1}| |a_{1,b}| \\
&\quad + \sum_{j=2}^{\infty} |\phi_b m_{j2}| |a_{2,b}| + \sum_{j=3}^{\infty} |\phi_b m_{j3}| |a_{3,b}| + \dots \\
&\quad + \theta \left\{ \sum_{j=1}^{\infty} |\mathcal{F} h_{j1}| |a_{1,v}| + \sum_{j=2}^{\infty} |\mathcal{F} h_{j2}| |a_{2,v}| + \sum_{j=3}^{\infty} |\mathcal{F} h_{j3}| |a_{3,v}| + \dots \right\} \\
&\leq \sum_{j=1}^{\infty} \left| \int_0^{\infty} \mu_v(x) \frac{(\lambda x)^{j-1}}{(j-1)!} e^{-\int_0^x (i\beta + \theta + \lambda + \mu_v(\xi)) d\xi} dx \right| |a_{1,v}| \\
&\quad + \sum_{j=2}^{\infty} \left| \int_0^{\infty} \mu_v(x) \frac{(\lambda x)^{j-2}}{(j-2)!} e^{-\int_0^x (i\beta + \theta + \lambda + \mu_v(\xi)) d\xi} dx \right| |a_{2,v}| \\
&\quad + \sum_{j=3}^{\infty} \left| \int_0^{\infty} \mu_v(x) \frac{(\lambda x)^{j-3}}{(j-3)!} e^{-\int_0^x (i\beta + \theta + \lambda + \mu_v(\xi)) d\xi} dx \right| |a_{3,v}| + \dots \\
&\quad + \sum_{j=1}^{\infty} \left| \int_0^{\infty} \mu_b(x) \frac{(\lambda x)^{j-1}}{(j-1)!} e^{-\int_0^x (i\beta + \lambda + \mu_b(\xi)) d\xi} dx \right| |a_{1,b}|
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=2}^{\infty} \left| \int_0^{\infty} \mu_b(x) \frac{(\lambda x)^{j-2}}{(j-2)!} e^{-\int_0^x (\beta + \lambda + \mu_b(\xi)) d\xi} dx \right| |a_{2,b}| \\
& + \sum_{j=3}^{\infty} \left| \int_0^{\infty} \mu_b(x) \frac{(\lambda x)^{j-3}}{(j-3)!} e^{-\int_0^x (\beta + \lambda + \mu_b(\xi)) d\xi} dx \right| |a_{3,b}| + \dots \\
& + \theta \left\{ \sum_{j=1}^{\infty} \left| \int_0^{\infty} \frac{(\lambda x)^{j-1}}{(j-1)!} e^{-\int_0^x (\beta + \theta + \lambda + \mu_v(\xi)) d\xi} dx \right| |a_{1,v}| \right. \\
& + \sum_{j=2}^{\infty} \left| \int_0^{\infty} \frac{(\lambda x)^{j-2}}{(j-2)!} e^{-\int_0^x (\beta + \theta + \lambda + \mu_v(\xi)) d\xi} dx \right| |a_{2,v}| \\
& + \sum_{j=3}^{\infty} \left| \int_0^{\infty} \frac{(\lambda x)^{j-3}}{(j-3)!} e^{-\int_0^x (\beta + \theta + \lambda + \mu_v(\xi)) d\xi} dx \right| |a_{3,v}| + \dots \Big\} \\
& < \int_0^{\infty} \mu_v(x) \sum_{j=1}^{\infty} \frac{(\lambda x)^{j-1}}{(j-1)!} e^{-\int_0^x (\theta + \lambda + \mu_v(\xi)) d\xi} dx |a_{1,v}| \\
& + \int_0^{\infty} \mu_v(x) \sum_{j=2}^{\infty} \frac{(\lambda x)^{j-2}}{(j-2)!} e^{-\int_0^x (\theta + \lambda + \mu_v(\xi)) d\xi} dx |a_{2,v}| \\
& + \int_0^{\infty} \mu_v(x) \sum_{j=3}^{\infty} \frac{(\lambda x)^{j-3}}{(j-3)!} e^{-\int_0^x (\theta + \lambda + \mu_v(\xi)) d\xi} dx |a_{3,v}| + \dots \\
& + \int_0^{\infty} \mu_b(x) \sum_{j=1}^{\infty} \frac{(\lambda x)^{j-1}}{(j-1)!} e^{-\int_0^x (\lambda + \mu_b(\xi)) d\xi} dx |a_{1,b}| \\
& + \int_0^{\infty} \mu_b(x) \sum_{j=2}^{\infty} \frac{(\lambda x)^{j-2}}{(j-2)!} e^{-\int_0^x (\lambda + \mu_b(\xi)) d\xi} dx |a_{2,b}| \\
& + \int_0^{\infty} \mu_b(x) \sum_{j=3}^{\infty} \frac{(\lambda x)^{j-3}}{(j-3)!} e^{-\int_0^x (\lambda + \mu_b(\xi)) d\xi} dx |a_{3,b}| + \dots \\
& + \theta \left\{ \int_0^{\infty} \sum_{j=1}^{\infty} \frac{(\lambda x)^{j-1}}{(j-1)!} e^{-\int_0^x (\theta + \lambda + \mu_v(\xi)) d\xi} dx |a_{1,v}| \right. \\
& + \int_0^{\infty} \sum_{j=2}^{\infty} \frac{(\lambda x)^{j-2}}{(j-2)!} e^{-\int_0^x (\theta + \lambda + \mu_v(\xi)) d\xi} dx |a_{2,v}| \\
& + \int_0^{\infty} \sum_{j=3}^{\infty} \frac{(\lambda x)^{j-3}}{(j-3)!} e^{-\int_0^x (\theta + \lambda + \mu_v(\xi)) d\xi} dx |a_{3,v}| + \dots \Big\} \\
& = \int_0^{\infty} \mu_v(x) e^{-\int_0^x (\theta + \mu_v(\xi)) d\xi} dx \sum_{n=1}^{\infty} |a_{n,v}| \\
& + \int_0^{\infty} \mu_b(x) e^{-\int_0^x \mu_b(\xi) d\xi} dx \sum_{n=1}^{\infty} |a_{n,b}| \\
& + \theta \int_0^{\infty} e^{-\int_0^x (\theta + \mu_v(\xi)) d\xi} dx \sum_{n=1}^{\infty} |a_{n,v}| \\
& = \int_0^{\infty} (\theta + \mu_v(x)) e^{-\int_0^x (\theta + \mu_v(\xi)) d\xi} dx \sum_{n=1}^{\infty} |a_{n,v}| + \sum_{n=1}^{\infty} |a_{n,b}| \\
& = \sum_{n=1}^{\infty} |a_{n,v}| + \sum_{n=1}^{\infty} |a_{n,b}| = \|(\vec{a}_v, \vec{a}_b)\|
\end{aligned}$$

$$\implies \|\Phi D_\gamma\| < 1. \quad (3.32)$$

(3.32) shows that $1 \notin \sigma(\Phi D_\gamma)$ when $|\beta| > \mathcal{K}$. This together with Lemma 3.4 give

$$\{i\beta \mid |\beta| > \mathcal{K}\} \subset \rho(A), \quad \{i\beta \mid |\beta| \leq \mathcal{K}\} \subset \sigma(A). \quad (3.33)$$

Theorem 2.1 and Lemma 3.1 ensures that $T(t)$ is a positive contraction C_0 -semigroup and its spectral bound is zero. By Nagel [20] we know that $\sigma(A)$ is imaginary additively cyclic (see also Theorem 1.88 in [21]) which states that

$$i\beta \in \sigma(A) \Rightarrow i\beta h \in \sigma(A), \quad \text{all positive integer } h.$$

From which together with (3.33) and Lemma 3.1 it follows that $i\mathbb{R} \cap \sigma(A) = \{0\}$. $\square$

A trivial verification shows that $X^* \times Y^*$, the dual space of $X \times Y$, is as follows.

$$X^* \times Y^* = \left\{ (q_v^*, q_b^*) \mid \begin{array}{l} q_v^* \in X^*, \quad q_b^* \in Y^*, \\ |||(q_v^*, q_b^*)||| = \sup\{|||q_v^*|||, |||q_b^*|||\} \end{array} \right\},$$

here

$$X^* = \left\{ q_v^* \mid \begin{array}{l} q_v^*(x) = (q_{0,v}^*, q_{1,v}^*(x), q_{2,v}^*(x), q_{3,v}^*(x), \dots), \\ |||y^*||| = \sup \left\{ |q_{0,v}^*|, \sup_{n \geq 1} \|q_{n,v}^*\|_{L^\infty[0,\infty)} \right\} < \infty \end{array} \right\},$$

$$Y^* = \left\{ q_b^* \mid \begin{array}{l} q_b^*(x) = (q_{0,b}^*, q_{1,b}^*(x), q_{2,b}^*(x), q_{3,b}^*(x), \dots) \\ |||q_b^*||| = \sup \left\{ |q_{0,b}^*|, \sup_{n \geq 1} \|q_{n,b}^*\|_{L^\infty[0,\infty)} \right\} < \infty \end{array} \right\}.$$

It is evident that $X^* \times Y^*$ is a Banach space.

Lemma 3.6. A^* , the adjoint operator of A , is as follows.

$$\begin{aligned} A^*(q_v^*, q_b^*) &= \begin{pmatrix} -(\theta + \lambda) & 0 & 0 & 0 \cdots \\ 0 & \frac{d}{dx} - (\theta + \lambda + \mu_v(x)) & \lambda & 0 \cdots \\ 0 & 0 & \frac{d}{dx} - (\theta + \lambda + \mu_v(x)) & \lambda \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_{0,v}^* \\ q_{1,v}^*(x) \\ q_{2,v}^*(x) \\ \vdots \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \lambda & 0 & \cdots \\ \mu_v(x) & 0 & 0 & \cdots \\ 0 & \mu_v(x) & 0 & \cdots \\ 0 & 0 & \mu_v(x) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_{0,v}^* \\ q_{1,v}^*(0) \\ q_{2,v}^*(0) \\ q_{3,v}^*(0) \\ \vdots \end{pmatrix} + \begin{pmatrix} \theta & 0 & 0 & 0 \cdots \\ 0 & \theta & 0 & 0 \cdots \\ 0 & 0 & \theta & 0 \cdots \\ 0 & 0 & 0 & \theta \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_{0,b}^* \\ q_{1,b}^*(0) \\ q_{2,b}^*(0) \\ q_{3,b}^*(0) \\ \vdots \end{pmatrix}, \\ &\begin{pmatrix} -\lambda & 0 & 0 & 0 \cdots \\ 0 & \frac{d}{dx} - (\lambda + \mu_b(x)) & \lambda & 0 \cdots \\ 0 & 0 & \frac{d}{dx} - (\lambda + \mu_b(x)) & \lambda \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_{0,b}^* \\ q_{1,b}^*(x) \\ q_{2,b}^*(x) \\ q_{3,b}^*(x) \\ \vdots \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \lambda & 0 & 0 \cdots \\ 0 & 0 & 0 & 0 \cdots \\ 0 & \mu_b(x) & 0 & 0 \cdots \\ 0 & 0 & \mu_b(x) & 0 \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_{0,b}^* \\ q_{1,b}^*(0) \\ q_{2,b}^*(0) \\ q_{3,b}^*(0) \\ \vdots \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \cdots \\ \mu_v(x) & 0 & 0 & 0 \cdots \\ 0 & 0 & 0 & 0 \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_{0,v}^* \\ q_{1,v}^*(0) \\ q_{2,v}^*(0) \\ \vdots \end{pmatrix}, \\ D(A^*) &= \left\{ (q_v^*, q_b^*) \in X^* \times Y^* \mid \begin{array}{l} \frac{dq_{n,v}^*(x)}{dx} \text{ and } \frac{dq_{n,b}^*(x)}{dx} \text{ exist and} \\ q_{n,v}^*(\infty) = q_{n,b}^*(\infty) = \alpha, \quad n \geq 1 \end{array} \right\}, \end{aligned}$$

here α in $D(A^*)$ is a constant which is independent of n .

Proof. By using integration by parts and the boundary conditions on $(p_v, p_b) \in D(A)$, we have, for $(q_v^*, q_b^*) \in D(A^*)$

$$\begin{aligned}
& \langle A(p_v, p_b), (q_v^*, q_b^*) \rangle \\
&= \left\{ -(\theta + \lambda)p_{0,v} + \int_0^\infty \mu_v(x)p_{1,v}(x)dx + \int_0^\infty \mu_b(x)p_{1,b}(x)dx \right\} q_{0,v}^* \\
&\quad + \int_0^\infty \left\{ -\frac{dp_{1,v}(x)}{dx} - (\lambda + \mu_v(x))p_{1,v}(x) \right\} q_{1,v}^*(x)dx \\
&\quad + \sum_{n=2}^\infty \int_0^\infty \left\{ -\frac{dp_{n,v}(x)}{dx} - (\lambda + \mu_v(x))p_{n,v}(x) + \lambda p_{n-1,v}(x) \right\} q_{n,v}^*(x)dx \\
&\quad + \left\{ -\lambda p_{0,b} + \theta p_{0,v} \right\} q_{0,b}^* \\
&\quad + \int_0^\infty \left\{ -\frac{dp_{1,b}(x)}{dx} - (\lambda + \mu_b(x))p_{1,b}(x) \right\} q_{1,b}^*(x)dx \\
&\quad + \sum_{n=2}^\infty \int_0^\infty \left\{ -\frac{dp_{n,b}(x)}{dx} - (\lambda + \mu_b(x))p_{n,b}(x) + \lambda p_{n-1,b}(x) \right\} q_{n,b}^*(x)dx \\
&= -(\theta + \lambda)p_{0,v}q_{0,v}^* + \int_0^\infty p_{1,v}(x)\mu_v(x)q_{0,v}^*dx + \int_0^\infty p_{1,b}(x)\mu_b(x)q_{0,v}^*dx \\
&\quad + \sum_{n=1}^\infty \int_0^\infty -\frac{dp_{n,v}(x)}{dx} q_{n,v}^*(x)dx - \sum_{n=1}^\infty \int_0^\infty (\lambda + \mu_v(x))p_{n,v}(x)q_{n,v}^*(x)dx \\
&\quad + \sum_{n=2}^\infty \int_0^\infty p_{n-1,v}(x)\lambda q_{n,v}^*(x)dx \\
&\quad + p_{0,b}(-\lambda q_{0,b}^*) + p_{0,v}\theta q_{0,b}^* + \sum_{n=1}^\infty \int_0^\infty -\frac{dp_{n,b}(x)}{dx} q_{n,b}^*(x)dx \\
&\quad - \sum_{n=1}^\infty \int_0^\infty p_{n,b}(x)(\lambda + \mu_b(x))q_{n,b}^*(x)dx + \sum_{n=2}^\infty \int_0^\infty p_{n-1,b}(x)\lambda q_{n,b}^*(x)dx \\
&= p_{0,v}[-(\theta + \lambda)q_{0,v}^* + \theta q_{0,b}^*] + \int_0^\infty p_{1,v}(x)\mu_v(x)q_{0,v}^*dx + \int_0^\infty p_{1,b}(x)\mu_b(x)q_{0,v}^*dx \\
&\quad + \sum_{n=1}^\infty \left[-p_{n,v}(x)q_{n,v}^*(x) \Big|_{x=0}^{x=\infty} + \int_0^\infty p_{n,v}(x) \frac{dq_{n,v}^*(x)}{dx} dx \right] \\
&\quad + \sum_{n=1}^\infty \int_0^\infty p_{n,v}(x)[-(\lambda + \mu_v(x))q_{n,v}^*(x)]dx + \sum_{n=1}^\infty \int_0^\infty p_{n,b}(x)[\lambda q_{n+1,v}^*(x)]dx \\
&\quad + p_{0,b}(-\lambda q_{0,b}^*) + \sum_{n=1}^\infty \left[-p_{n,b}(x)q_{n,b}^*(x) \Big|_{x=0}^{x=\infty} + \int_0^\infty p_{n,b}(x) \frac{dq_{n,b}^*(x)}{dx} dx \right] \\
&\quad + \sum_{n=1}^\infty \int_0^\infty p_{n,b}(x)[-(\lambda + \mu_b(x))q_{n,b}^*(x)]dx \\
&\quad + \sum_{n=1}^\infty \int_0^\infty p_{n,b}(x)[\lambda q_{n+1,b}^*(x)]dx \\
&= p_{0,v}[-(\theta + \lambda)q_{0,v}^* + \theta q_{0,b}^*] + \int_0^\infty p_{1,v}(x)\mu_v(x)q_{0,v}^*dx + \int_0^\infty p_{1,b}(x)\mu_b(x)q_{0,v}^*dx
\end{aligned}$$

$$\begin{aligned}
& + \sum_{n=1}^{\infty} p_{n,v}(0) q_{n,v}^*(0) + \sum_{n=1}^{\infty} \int_0^{\infty} p_{n,v}(x) \frac{dq_{n,v}^*(x)}{dx} dx \\
& + \sum_{n=1}^{\infty} \int_0^{\infty} p_{n,v}(x) [-(\lambda + \mu_v(x)) q_{n,v}^*(x)] dx + \sum_{n=1}^{\infty} \int_0^{\infty} p_{n,v}(x) [\lambda q_{n+1,v}^*(x)] dx \\
& + p_{0,b}(-\lambda q_{0,b}^*) + \sum_{n=1}^{\infty} p_{n,b}(0) q_{n,b}^*(0) + \sum_{n=1}^{\infty} \int_0^{\infty} p_{n,b}(x) \frac{dq_{n,b}^*(x)}{dx} dx \\
& + \sum_{n=1}^{\infty} \int_0^{\infty} p_{n,b}(x) [-(\lambda + \mu_b(x)) q_{n,b}^*(x)] dx + \sum_{n=1}^{\infty} \int_0^{\infty} p_{n,b}(x) [\lambda q_{n+1,b}^*(x)] dx \\
& = p_{0,v}[-(\theta + \lambda) q_{0,v}^* + \theta q_{0,b}^*] + \int_0^{\infty} p_{1,v}(x) \mu_v(x) q_{0,v}^* dx + \int_0^{\infty} p_{1,b}(x) \mu_b(x) q_{0,v}^* dx \\
& + p_{0,v} \lambda q_{1,v}^*(0) + \sum_{n=1}^{\infty} \int_0^{\infty} \mu_v(x) p_{n+1,v}(x) q_{n,v}^*(0) dx \\
& + \sum_{n=1}^{\infty} \int_0^{\infty} p_{n,v}(x) \left[\frac{dq_{n,v}^*(x)}{dx} - (\lambda + \mu_v(x)) q_{n,v}^*(x) + \lambda q_{n+1,v}^*(x) \right] dx \\
& + p_{0,b}[-\lambda q_{0,b}^* + \lambda q_{1,b}^*(0)] \\
& + \sum_{n=1}^{\infty} \left[\int_0^{\infty} \mu_b(x) p_{n+1,b}(x) dx + \theta \int_0^{\infty} p_{n,v}(x) dx \right] q_{n,b}^*(0) \\
& + \sum_{n=1}^{\infty} \int_0^{\infty} p_{n,b}(x) \left[\frac{dq_{n,b}^*(x)}{dx} - (\lambda + \mu_b(x)) q_{n,b}^*(x) + \lambda q_{n+1,b}^*(x) \right] dx \\
& = p_{0,v}[-(\theta + \lambda) q_{0,v}^* + \lambda q_{1,v}^*(0) + \theta q_{0,b}^*] \\
& + \int_0^{\infty} p_{1,v}(x) \left[\frac{dq_{1,v}^*(x)}{dx} - (\lambda + \mu_v(x)) q_{1,v}^*(x) + \lambda q_{2,v}^*(x) + \mu_v(x) q_{0,v}^* + \theta q_{1,b}^*(0) \right] dx \\
& + \sum_{n=2}^{\infty} \int_0^{\infty} p_{n,v}(x) \left[\frac{dq_{n,v}^*(x)}{dx} - (\lambda + \mu_v(x)) q_{n,v}^*(x) + \lambda q_{n+1,v}^*(x) \right. \\
& \quad \left. + \mu_v(x) q_{n-1,v}^*(0) + \theta q_{n,b}^*(0) \right] dx \\
& + p_{0,b}[-\lambda q_{0,b}^* + \lambda q_{1,b}^*(0)] \\
& + \int_0^{\infty} p_{1,b}(x) \left[\frac{dq_{1,b}^*(x)}{dx} - (\lambda + \mu_b(x)) q_{1,b}^*(x) + \lambda q_{2,b}^*(x) + \mu_b(x) q_{0,v}^* \right] dx \\
& + \sum_{n=2}^{\infty} \int_0^{\infty} p_{n,b}(x) \left[\frac{dq_{n,b}^*(x)}{dx} - (\lambda + \mu_b(x)) q_{n,b}^*(x) + \lambda q_{n+1,b}^*(x) + \mu_b(x) q_{n-1,b}^*(0) \right] dx \\
& = \langle (p_v, p_b), A^*(q_v^*, q_b^*) \rangle.
\end{aligned}$$

From this together with the definition of adjoint operator the assertion follows. $\square$

From Theorem 2.1, Lemma 3.1 and Arendt et al. [22], we know that 0 is an eigenvalue of A^* . Furthermore, we deduce the following result.

Lemma 3.7. *If $\int_0^{\infty} \lambda x \mu_b(x) e^{-\int_0^x \mu_b(\tau) d\tau} dx < 1$, then 0 is an eigenvalue of A^* with geometric multiplicity one.*

Now, combining the Theorem 2.1, Lemma 3.1, Lemma 3.5 and Lemma 3.7 with Theorem 14 in Gupur et al. [18] we obtain the following main result.

Theorem 3.8. *If*

$$0 < \underline{\mu_v} = \inf_{x \in [0, \infty)} \mu_v(x) \leq \overline{\mu_v} = \sup_{x \in [0, \infty)} \mu_v(x) < \infty,$$

$$0 < \underline{\mu_b} = \inf_{x \in [0, \infty)} \mu_b(x) \leq \overline{\mu_b} = \sup_{x \in [0, \infty)} \mu_b(x) < \infty.$$

then the time-dependent solution of the system (2.1) strongly converges to its steady-state solution, i.e.,

$$\lim_{t \rightarrow \infty} \|(p_v, p_b)(\cdot, t) - \omega(p_v, p_b)(\cdot)\| = 0,$$

here $(p_v, p_b)(x)$ is the eigenvector in Lemma 3.1 and ω is decided by the eigenvector in Lemma 3.7 and the initial value $(p_v, p_b)(0)$.

In the following, by applying the Theorem 3.8 we briefly discuss the queueing system's indices. It is easily seen that the time-dependent queueing size at the departure point converges to a positive number, i.e.,

$$\begin{aligned} \lim_{t \rightarrow \infty} \pi_j(t) &= \lim_{t \rightarrow \infty} \left\{ K_0 \int_0^\infty \mu_v(x) p_{j+1,v}(x, t) dx + K_0 \int_0^\infty \mu_b(x) p_{j+1,b}(x, t) dx \right\} \\ &= \left\{ K_0 \int_0^\infty \mu_v(x) p_{j+1,v}(x) dx + K_0 \int_0^\infty \mu_b(x) p_{j+1,b}(x) dx \right\} = \pi_j, \quad j \geq 0. \end{aligned}$$

and the time-dependent queueing length $L(t)$ converges to the steady-state queueing length L , that is,

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathcal{L}(t) &= \lim_{t \rightarrow \infty} \left\{ \sum_{n=1}^\infty n \int_0^\infty p_{n,v}(x, t) dx + \sum_{n=1}^\infty n \int_0^\infty p_{n,b}(x, t) dx \right\} \\ &= \sum_{n=1}^\infty n \int_0^\infty p_{n,v}(x) dx + \sum_{n=1}^\infty n \int_0^\infty p_{n,b}(x) dx = L. \end{aligned}$$

From this we can obtain that other queueing indices $L_q(t)$, $W(t)$ and $W_q(t)$ also converge to a positive number L_q , W and W_q respectively.

4 Conclusion

In this paper, we study an M/G/1 queueing model with single working vacation, in which the service time is generally distributed. The system is described by infinite number of partial differential equations with integral boundary conditions which we have converted into an abstract Cauchy problem in the Banach space. Then, by investigating the spectrum of the operator on the imaginary axis, which corresponds to the M/G/1 queueing model with single working vacation, we proved that the time-dependent solution of the model strongly converges to its steady-state solution. In other words, we verified that the hypothesis 2 holds in the sense of strong convergence.

In this paper and our previous paper, we only studied spectra of the operator on the right half complex plane and imaginary axis, which corresponds to the M/G/1 queueing model with single working vacation, so it is worth studying spectra of the operator on the left half complex plane.

5 Appendix

Proof of Lemma 3.2. For any $f \in L^1[0, \infty)$, by using integration by parts, we have

$$\|E_v f\|_{L^1[0, \infty)} = \int_0^\infty |E_v f(x)| dx$$

$$\begin{aligned}
&= \int_0^\infty \left| e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi)) d\xi} \int_0^x f(\tau) e^{\int_0^\tau (\gamma + \theta + \lambda + \mu_v(\xi)) d\xi} d\tau \right| dx \\
&\leq \int_0^\infty e^{-\int_0^x (\operatorname{Re} \gamma + \theta + \lambda + \mu_v(\xi)) d\xi} \int_0^x |f(\tau)| e^{\int_0^\tau (\operatorname{Re} \gamma + \theta + \lambda + \mu_v(\xi)) d\xi} d\tau dx \\
&= \int_0^\infty \frac{-1}{\operatorname{Re} \gamma + \theta + \lambda + \mu_v(x)} \int_0^x |f(\tau)| e^{\int_0^\tau (\operatorname{Re} \gamma + \theta + \lambda + \mu_v(\xi)) d\xi} d\tau d e^{-\int_0^x (\operatorname{Re} \gamma + \theta + \lambda + \mu_v(\xi)) d\xi} \\
&\leq \int_0^\infty \frac{-1}{\operatorname{Re} \gamma + \theta + \lambda + \underline{\mu}_v} \int_0^x |f(\tau)| e^{\int_0^\tau (\operatorname{Re} \gamma + \theta + \lambda + \mu_v(\xi)) d\xi} d\tau d e^{-\int_0^x (\operatorname{Re} \gamma + \theta + \lambda + \mu_v(\xi)) d\xi} \\
&= -\frac{1}{\operatorname{Re} \gamma + \theta + \lambda + \underline{\mu}_v} \left\{ e^{-\int_0^x (\operatorname{Re} \gamma + \theta + \lambda + \mu_v(\xi)) d\xi} \int_0^x |f(\tau)| e^{\int_0^\tau (\operatorname{Re} \gamma + \theta + \lambda + \mu_v(\xi)) d\xi} d\tau \right\}_{x=0}^{x=\infty} \\
&\quad - \int_0^\infty |f(x)| e^{\int_0^x (\operatorname{Re} \gamma + \theta + \lambda + \mu_v(\xi)) d\xi} e^{-\int_0^x (\operatorname{Re} \gamma + \theta + \lambda + \mu_v(\xi)) d\xi} dx \Big\} \\
&= \frac{1}{\operatorname{Re} \gamma + \theta + \lambda + \underline{\mu}_v} \|f\|_{L^1[0, \infty)} \\
&\implies \\
\|E_v\| &\leq \frac{1}{\operatorname{Re} \gamma + \theta + \lambda + \underline{\mu}_v}. \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
\|E_b f\| &= \int_0^\infty |E_b f(x)| dx \\
&= \int_0^\infty \left| e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi)) d\xi} \int_0^x f(\tau) e^{\int_0^\tau (\gamma + \lambda + \mu_b(\xi)) d\xi} d\tau \right| dx \\
&\leq \frac{1}{\operatorname{Re} \gamma + \lambda + \underline{\mu}_b} \|f\|_{L^1[0, \infty)} \\
&\implies \\
\|E_b\| &\leq \frac{1}{\operatorname{Re} \gamma + \lambda + \underline{\mu}_b}. \tag{A.2}
\end{aligned}$$

From (A.1) and (A.2) together with condition of this lemma and using $\|\phi_v\| \leq \overline{\mu}_v$, $\|\phi_b\| \leq \overline{\mu}_b$ we deduce, for any $(y, z) \in X \times Y$

$$\begin{aligned}
&\|(\gamma I - A_0)^{-1}(y, z)\| \\
&= \left\| \frac{1}{\gamma + \theta + \lambda} y_0 + \frac{1}{\gamma + \theta + \lambda} \phi_v E_v y_1 + \frac{1}{\gamma + \theta + \lambda} \phi_b E_b z_1 \right\| \\
&\quad + \|E_v y_1\|_{L^1[0, \infty)} + \|\lambda E_v^2 y_1 + E_v y_2\|_{L^1[0, \infty)} \\
&\quad + \|\lambda^2 E_v^3 y_1 + \lambda E_v^2 y_2 + E_v y_3\|_{L^1[0, \infty)} + \left\| \sum_{k=0}^3 \lambda^k E_v^{k+1} y_{4-k} \right\|_{L^1[0, \infty)} \\
&\quad + \dots + \left\| \sum_{k=0}^{n-1} \lambda^k E_v^{k+1} y_{n-k} \right\|_{L^1[0, \infty)} + \dots \\
&\quad + \left\| \frac{1}{\gamma + \lambda} z_0 + \frac{\theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} \phi_b E_b z_1 + \frac{\theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} y_0 \right. \\
&\quad \left. + \frac{\theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} \phi_v E_v y_1 \right\| + \|E_b z_1\|_{L^1[0, \infty)} + \|\lambda E_b^2 z_1 + E_b z_2\|_{L^1[0, \infty)}
\end{aligned}$$

$$\begin{aligned}
& + \left\| \lambda^2 E_b^3 z_1 + \lambda E_b^2 z_2 + E_b z_3 \right\|_{L^1[0, \infty)} + \left\| \sum_{k=0}^3 \lambda^k E_b^{k+1} z_{4-k} \right\|_{L^1[0, \infty)} \\
& + \cdots + \left\| \sum_{k=0}^{n-1} \lambda^k E_b^{k+1} z_{n-k} \right\|_{L^1[0, \infty)} + \cdots \\
& \leq \frac{1}{|\gamma + \theta + \lambda|} |y_0| + \frac{1}{|\gamma + \theta + \lambda|} \|\phi_v\| \|E_v\| \|y_1\|_{L^1[0, \infty)} \\
& + \frac{1}{|\gamma + \theta + \lambda|} \|\phi_b\| \|E_b\| \|z_1\|_{L^1[0, \infty)} + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \lambda^k \|E_v\|^{k+1} \|y_{n-k}\|_{L^1[0, \infty)} \\
& + \frac{1}{|\gamma + \lambda|} z_0 + \frac{\theta}{|\gamma + \lambda| |\gamma + \theta + \lambda|} \|\phi_b\| \|E_b\| \|z_1\|_{L^1[0, \infty)} \\
& + \frac{\theta}{|\gamma + \lambda| |\gamma + \theta + \lambda|} |y_0| + \frac{\theta}{|\gamma + \lambda| |\gamma + \theta + \lambda|} \|\phi_v\| \|E_v\| \|y_1\|_{L^1[0, \infty)} \\
& + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \lambda^k \|E_b\|^{k+1} \|z_{n-k}\|_{L^1[0, \infty)} \\
& \leq \frac{1}{|\gamma + \theta + \lambda|} |y_0| + \frac{\bar{\mu}_v}{|\gamma + \theta + \lambda| (\operatorname{Re} \gamma + \theta + \lambda + \bar{\mu}_v)} \|y_1\|_{L^1[0, \infty)} \\
& + \frac{\bar{\mu}_b}{|\gamma + \theta + \lambda| (\operatorname{Re} \gamma + \lambda + \bar{\mu}_b)} \|z_1\|_{L^1[0, \infty)} \\
& + \frac{1}{\operatorname{Re} \gamma + \theta + \lambda + \bar{\mu}_v} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\operatorname{Re} \gamma + \theta + \lambda + \bar{\mu}_v} \right)^k \sum_{n=1}^{\infty} \|y_n\|_{L^1[0, \infty)} \\
& + \frac{1}{|\gamma + \lambda|} z_0 + \frac{\bar{\mu}_b}{|\gamma + \lambda| (\operatorname{Re} \gamma + \lambda + \bar{\mu}_b)} \|z_1\|_{L^1[0, \infty)} \\
& + \frac{1}{|\gamma + \lambda|} |y_0| + \frac{\bar{\mu}_v}{|\gamma + \lambda| (\operatorname{Re} \gamma + \theta + \lambda + \bar{\mu}_v)} \|y_1\|_{L^1[0, \infty)} \\
& + \frac{1}{\operatorname{Re} \gamma + \lambda + \bar{\mu}_b} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\operatorname{Re} \gamma + \lambda + \bar{\mu}_b} \right)^k \sum_{n=1}^{\infty} \|z_n\|_{L^1[0, \infty)} \\
& \leq \sup \left\{ \frac{1}{|\gamma + \lambda|}, \frac{\bar{\mu}_v}{|\gamma + \lambda| (\operatorname{Re} \gamma + \theta + \lambda + \bar{\mu}_v)} \right. \\
& \quad \left. + \frac{1}{\operatorname{Re} \gamma + \theta + \bar{\mu}_v} \right\} \left(|y_0| + \sum_{n=1}^{\infty} \|y_n\|_{L^1[0, \infty)} \right) \\
& + \sup \left\{ \frac{1}{|\gamma + \lambda|}, \frac{\bar{\mu}_b}{|\gamma + \lambda| (\operatorname{Re} \gamma + \lambda + \bar{\mu}_b)} \right. \\
& \quad \left. + \frac{1}{\operatorname{Re} \gamma + \bar{\mu}_b} \right\} \left(|z_0| + \sum_{n=1}^{\infty} \|z_n\|_{L^1[0, \infty)} \right) \\
& \leq \sup \left\{ \frac{1}{|\gamma + \lambda|}, \frac{\bar{\mu}_v}{|\gamma + \lambda| (\operatorname{Re} \gamma + \theta + \lambda + \bar{\mu}_v)} + \frac{1}{\operatorname{Re} \gamma + \theta + \bar{\mu}_v}, \right. \\
& \quad \left. + \frac{\bar{\mu}_b}{|\gamma + \lambda| (\operatorname{Re} \gamma + \lambda + \bar{\mu}_b)} + \frac{1}{\operatorname{Re} \gamma + \bar{\mu}_b} \right\} \|(\gamma, z)\| \\
& < \infty.
\end{aligned}$$

This shows that the result of this lemma is right. $\square$

Proof of Lemma 3.3. If $(p_v, p_b) \in \ker(\gamma I - A_m)$, then $(\gamma I - A_m)(p_v, p_b) = 0$, which is equivalent to

$$(\gamma + \theta + \lambda) p_{0,v} = \int_0^{\infty} \mu_v(x) p_{1,v}(x) dx + \int_0^{\infty} \mu_v(x) p_{1,b}(x) dx, \quad (\text{A.3})$$

$$\frac{dp_{1,v}(x)}{dx} = -(\gamma + \theta + \lambda + \mu_v(x)) p_{1,v}(x), \quad (\text{A.4})$$

$$\frac{dp_{n,v}(x)}{dx} = -(\gamma + \theta + \lambda + \mu_v(x))p_{n,v}(x) + \lambda p_{n-1,v}(x), \quad n \geq 2, \quad (\text{A.5})$$

$$(\gamma + \lambda)p_{0,b} = \theta p_{0,v}, \quad (\text{A.6})$$

$$\frac{dp_{1,b}(x)}{dx} = -(\gamma + \lambda + \mu_v(x))p_{1,b}(x), \quad (\text{A.7})$$

$$\frac{dp_{n,b}(x)}{dx} = -(\gamma + \lambda + \mu_v(x))p_{n,b}(x) + \lambda p_{n-1,b}(x), \quad n \geq 2. \quad (\text{A.8})$$

By solving (A.4), (A.5) and (A.7), (A.8), we have

$$p_{1,v}(x) = a_{1,v}e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi}, \quad (\text{A.9})$$

$$p_{n,v}(x) = a_{n,v}e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} + \lambda e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} \int_0^x p_{n-1,v}(\tau) e^{\int_0^\tau (\gamma + \theta + \lambda + \mu_v(\tau))d\tau} d\tau, \quad n \geq 2, \quad (\text{A.10})$$

$$p_{1,b}(x) = a_{1,b}e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi))d\xi}, \quad (\text{A.11})$$

$$p_{n,b}(x) = a_{n,b}e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi))d\xi} + \lambda e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi))d\xi} \int_0^x p_{n-1,b}(\tau) e^{\int_0^\tau (\gamma + \lambda + \mu_b(\xi))d\xi} d\tau, \quad n \geq 2. \quad (\text{A.12})$$

By using (A.9) and (A.10) repeatedly, we obtain

$$\begin{aligned} p_{2,v}(x) &= a_{2,v}e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} + \lambda e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} \int_0^x a_{1,v}d\tau \\ &= e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} [a_{2,v} + \lambda x a_{1,v}], \\ p_{3,v}(x) &= a_{3,v}e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} + \lambda e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} \int_0^x [a_{2,v} + \lambda \tau a_{1,v}]d\tau \\ &= e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} \left[a_{3,v} + \lambda x a_{2,v} + \frac{(\lambda x)^2}{2} a_{1,v} \right], \\ &\dots\dots\dots \\ p_{n,v}(x) &= e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} \left[a_{n,v} + \lambda x a_{n-1,v} + \frac{(\lambda x)^2}{2} a_{n-2,v} + \dots + \frac{(\lambda x)^n}{n!} a_{1,v} \right] \\ &= e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} a_{n-k,v}, \quad n \geq 1. \end{aligned} \quad (\text{A.13})$$

Similarly, by applying (A.11) and (A.12) repeatedly, we deduce

$$p_{n,b}(x) = e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi))d\xi} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} a_{n-k,b}, \quad n \geq 1. \quad (\text{A.14})$$

Through inserting (A.9) and (A.11) into (A.3), we derive

$$\begin{aligned} p_{0,v} &= \frac{1}{\gamma + \theta + \lambda} a_{1,v} \int_0^\infty \mu_v(x) e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} dx \\ &\quad + \frac{1}{\gamma + \theta + \lambda} a_{1,b} \int_0^\infty \mu_b(x) e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi))d\xi} dx, \\ p_{0,b} &= \frac{\theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} a_{1,v} \int_0^\infty \mu_v(x) e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi))d\xi} dx \end{aligned} \quad (\text{A.15})$$

$$+ \frac{\theta}{(\gamma + \lambda)(\gamma + \theta + \lambda)} a_{1,b} \int_0^\infty \mu_b(x) e^{-\int_0^x (\gamma + \lambda + \mu_b(\xi)) d\xi} dx. \quad (\text{A.16})$$

Since $(p_v, p_b) \in \ker(\gamma I - D(A_m))$, $(p_v, p_b) \in D(A_m)$ implies by the imbedding theorem in Adams [23],

$$\begin{aligned} \sum_{n=1}^\infty |a_{n,v}| &= \sum_{n=1}^\infty |p_{n,v}(0)| \leq \sum_{n=1}^\infty \|p_{n,v}\|_{L^\infty[0,\infty)} \\ &\leq \sum_{n=1}^\infty \|p_{n,v}\|_{L^1[0,\infty)} + \sum_{n=1}^\infty \left\| \frac{dp_{n,v}}{dx} \right\|_{L^1[0,\infty)} < \infty, \\ \sum_{n=1}^\infty |a_{n,b}| &= \sum_{n=1}^\infty |p_{n,b}(0)| \leq \sum_{n=1}^\infty \|p_{n,b}\|_{L^\infty[0,\infty)} \\ &\leq \sum_{n=1}^\infty \|p_{n,b}\|_{L^1[0,\infty)} + \sum_{n=1}^\infty \left\| \frac{dp_{n,b}}{dx} \right\|_{L^1[0,\infty)} < \infty. \end{aligned}$$

from which together with (A.13)-(A.16) we know that (2.55) holds.

Conversely, if (2.55) is right, then by using $\int_0^\infty x^k e^{-Mx} dx = \frac{k!}{M^{k+1}}$, $k \geq 1$, $M > 0$, integration by parts and the Fubini theorem we estimate

$$\begin{aligned} \|p_{n,v}\|_{L^1[0,\infty)} &= \int_0^\infty \left| e^{-\int_0^x (\gamma + \theta + \lambda + \mu_v(\xi)) d\xi} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} a_{n-k,v} \right| dx \\ &\leq \sum_{k=0}^{n-1} |a_{n-k,v}| \frac{\lambda^k}{k!} \int_0^\infty x^k e^{-(\operatorname{Re} \gamma + \theta + \lambda + \underline{\mu}_v)x} dx \\ &= \sum_{k=0}^{n-1} \frac{\lambda^k}{(\operatorname{Re} \gamma + \theta + \lambda + \underline{\mu}_v)^{k+1}} |a_{n-k,v}| \\ &\implies \\ |p_{0,v}| + \sum_{n=1}^\infty \|p_{n,v}\|_{L^1[0,\infty)} &= \frac{\overline{\mu}_v}{|\gamma + \theta + \lambda|(\operatorname{Re} \gamma + \theta + \lambda + \underline{\mu}_v)} |a_{1,v}| \\ &\quad + \frac{\overline{\mu}_b}{|\gamma + \theta + \lambda|(\operatorname{Re} \gamma + \lambda + \underline{\mu}_b)} |a_{1,b}| \\ &\quad + \frac{1}{\operatorname{Re} \gamma + \theta + \underline{\mu}_v} \sum_{n=1}^\infty |a_{n,v}| < \infty. \end{aligned} \quad (\text{A.17})$$

Similarly, we get

$$\begin{aligned} |p_{0,b}| + \sum_{n=1}^\infty \|p_{n,b}\|_{L^1[0,\infty)} &= \frac{\overline{\mu}_b}{|\gamma + \theta|(\operatorname{Re} \gamma + \theta + \lambda + \underline{\mu}_b)} |a_{1,v}| \\ &\quad + \frac{\overline{\mu}_b}{|\gamma + \lambda|(\operatorname{Re} \gamma + \lambda + \underline{\mu}_b)} |a_{1,b}| \\ &\quad + \frac{1}{\operatorname{Re} \gamma + \theta + \underline{\mu}_b} \sum_{n=1}^\infty |a_{n,b}| < \infty. \end{aligned} \quad (\text{A.18})$$

(A.17) and (A.18) gives

$$|p_{0,v}| + |p_{0,b}| + \sum_{n=1}^\infty \|p_{n,v}\|_{L^1[0,\infty)} + \sum_{n=1}^\infty \|p_{n,b}\|_{L^1[0,\infty)} < \infty.$$

Since

$$\begin{aligned} \frac{dp_{1,v}(x)}{dx} &= -(\gamma + \theta + \lambda + \mu_v(x)) p_{1,v} \\ \frac{dp_{n,v}(x)}{dx} &= -(\gamma + \theta + \lambda + \mu_v(x)) p_{n,v}(x) + \lambda p_{n-1,v}(x), \quad n \geq 2, \\ \frac{dp_{1,b}(x)}{dx} &= -(\gamma + \lambda + \mu_b(x)) p_{1,b}(x), \end{aligned}$$

$$\frac{dp_{n,b}(x)}{dx} = -(\gamma + \lambda + \mu_b(x))p_{n,b}(x) + \lambda p_{n-1,b}(x), \quad n \geq 2.$$

It is immediately obtained

$$\begin{aligned} \sum_{n=1}^{\infty} \left\| \frac{dp_{n,v}}{dx} \right\|_{L^1[0,\infty)} &\leq (\operatorname{Re}\gamma + \theta + \lambda + \overline{\mu_v}) \sum_{n=0}^{\infty} \|p_{n,v}\|_{L^1[0,\infty)} \\ &\quad + \lambda \sum_{n=2}^{\infty} \|p_{n-1,v}\|_{L^1[0,\infty)} \\ &\leq \left(\frac{\operatorname{Re}\gamma + \theta + \lambda + \overline{\mu_v}}{\operatorname{Re}\gamma + \theta + \underline{\mu_v}} + \frac{\lambda}{\operatorname{Re}\gamma + \theta + \underline{\mu_v}} \right) \sum_{n=1}^{\infty} |a_{n,v}| \\ &< \infty, \end{aligned} \quad (\text{A.19})$$

$$\begin{aligned} \sum_{n=1}^{\infty} \left\| \frac{dp_{n,b}}{dx} \right\|_{L^1[0,\infty)} &\leq (\operatorname{Re}\gamma + \lambda + \overline{\mu_b}) \sum_{n=1}^{\infty} \|p_{n,b}\|_{L^1[0,\infty)} \\ &\quad + \lambda \sum_{n=2}^{\infty} \|p_{n-1,b}\|_{L^1[0,\infty)} \\ &\leq \left(\frac{\operatorname{Re}\gamma + \lambda + \overline{\mu_b}}{\operatorname{Re}\gamma + \underline{\mu_b}} + \frac{\lambda}{\operatorname{Re}\gamma + \underline{\mu_b}} \right) \sum_{n=0}^{\infty} |a_{n,b}| \\ &< \infty. \end{aligned} \quad (\text{A.20})$$

(A.17)–(A.20) show that $(p_v, p_b) \in \ker(\gamma I - A_m)$. $\square$

Proof of Lemma 3.7. We consider the equation $A^*(q_v^*, q_b^*) = 0$, which is equivalent to

$$\begin{aligned} -(\theta + \lambda)q_{0,v}^* + \lambda q_{1,v}^*(0) + \theta q_{0,b}^* &= 0, \\ \frac{dq_{1,v}^*(x)}{dx} - (\lambda + \mu_v(x))q_{1,v}^*(x) + \lambda q_{2,v}^*(x) + \mu_v(x)q_{0,v}^* + \theta q_{1,b}^*(0) &= 0, \\ \frac{dq_{n,v}^*(x)}{dx} - (\lambda + \mu_v(x))q_{n,v}^*(x) + \lambda q_{n+1,v}^*(x) + \mu_v(x)q_{n-1,v}^*(0) + \theta q_{n,b}^*(0) &= 0, \quad n \geq 2, \\ -\lambda q_{0,b}^* + \lambda q_{1,b}^*(0) &= 0, \\ \frac{dq_{1,b}^*(x)}{dx} - (\lambda + \mu_b(x))q_{1,b}^*(x) + \lambda q_{2,b}^*(x) + \mu_b(x)q_{0,v}^* &= 0, \\ \frac{dq_{n,b}^*(x)}{dx} - (\lambda + \mu_b(x))q_{n,b}^*(x) + \lambda q_{n+1,b}^*(x) + \mu_b(x)q_{n-1,b}^* &= 0, \quad n \geq 2, \\ q_{n,v}^*(\infty) = q_{n,b}^*(\infty) &= \alpha, \quad n \geq 1. \end{aligned} \quad (\text{A.21})$$

It is easy to see that

$$(q_v^*, q_b^*) = \left(\begin{pmatrix} \alpha \\ \alpha \\ \vdots \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ \vdots \end{pmatrix} \right) \in D(A^*)$$

is a solution of (A.21). In addition, (A.21) is equivalent to

$$\begin{aligned} q_{0,v}^* &= \frac{\lambda}{\theta + \lambda} q_{1,v}^*(0) + \frac{\theta}{\theta + \lambda} q_{0,b}^*, \\ q_{2,v}^*(x) &= \frac{1}{\lambda} \left\{ -\frac{dq_{1,v}^*(x)}{dx} + (\lambda + \mu_v(x))q_{1,v}^*(x) - \mu_v(x)q_{0,v}^* - \theta q_{1,b}^*(0) \right\}, \\ q_{n+1,0}^*(x) &= \frac{1}{\lambda} \left\{ -\frac{dq_{n,v}^*(x)}{dx} + (\lambda + \mu_v(x))q_{n,v}^*(x) - \mu_v(x)q_{n-1,v}^*(0) - \theta q_{n,b}^*(0) \right\}, \quad n \geq 2, \\ q_{0,b}^* &= q_{1,b}^*(0), \\ q_{2,b}^*(x) &= \frac{1}{\lambda} \left\{ -\frac{dq_{1,b}^*(x)}{dx} + (\lambda + \mu_b(x))q_{1,b}^*(x) - \mu_b(x)q_{0,v}^* \right\}, \\ q_{n+1,b}^*(x) &= \frac{1}{\lambda} \left\{ -\frac{dq_{n,b}^*(x)}{dx} + (\lambda + \mu_b(x))q_{n,b}^*(x) - \mu_b(x)q_{n-1,b}^* \right\}, \quad n \geq 2. \end{aligned} \quad (\text{A.22})$$

(A.22) show that we can determine each $q_{n,v}^*(x)$ and $q_{n,b}^*(x)$ for all $n \geq 1$ if $q_{1,v}^*(x)$ and $q_{1,b}^*(x)$ are given. That is to say, geometric multiplicity of zero is one. $\square$

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References

- [1] Servi L.D., Finn S.G., $M/M/1$ queues with working vacations ($M/M/1/WV$), *Perform. Eval.*, 2002, 50, 41-52
- [2] Servi L.D., Average delay approximation of $M/G/1$ cyclic service queues with Bernoulli schedules, *IEEE J. Select. Areas Commun.*, 1986, 4(6), 813-822
- [3] Takagi H., Queueing analysis of polling models, *ACM Computing Surveys (CSUR)*, 1988, 20(1), 5-28
- [4] Chandrasekaran V.M., Indhira K., Saravananarajan M.C., A survey on working vacation queueing models, *Int. J. Pure Appl. Math.*, 2016, 106, 33-41
- [5] Kim J.D., Choi D.W., Chae K.C., Analysis of queue-length distribution of the $M/G/1$ queue with working vacations, *Hawaii Int. Conf. Stat. Relat. Fields*, 2003, 6, 5-8
- [6] Wu D., Takagi H., $M/G/1$ queue with multiple working vacations, *Perform. Eval.*, 2006, 63, 654-681
- [7] Xu X., Zhang Z., Tian N., Analysis for the $M^X/M/1$ working vacations queue, *Inf. Manag. Sci.*, 2009, 20, 379-394
- [8] Baba Y., Analysis of a $GI/M/1$ queue with multiple working vacations, *Oper. Res. Lett.*, 2005, 33, 201-209
- [9] Gao S., Yao Y., An $M^X/G/1$ queue with randomized working vacations and at most J vacation, *Int. J. Comput. Math.*, 2014, 91, 368-383
- [10] Baba Y., The $M^X/M/1$ queue with multiple working vacations, *Am. J. Oper. Res.*, 2012, 2, 217-224
- [11] Do T.V., $M/M/1$ retrial queue with working vacations, *Acta Inform.*, 2010, 47, 67-75
- [12] Arivudainambi D., Godhandaraman P., Rajadurai P., Performance analysis of a single server retrial queue with working vacation, *Opsearch.*, 2014, 51, 434-462
- [13] Zhang M., Hou Z., $M/G/1$ queue with single working vacation, *J. Appl. Math. Comput.* 2012, 39, 221-234
- [14] Kasim E., Gupur G., Dynamic analysis of the $M/G/1$ queueing model with single working vacation, *Int. J. Appl. Comput. Math.*, 2017, 3, 2803-2833
- [15] Gupur G., Analysis of the $M/G/1$ retrial queueing model with server breakdown, *J.Pseudo-Differ.Oper.Appl.*, 2010, 1, 313-340
- [16] Kasim E., Gupur G., Further Research on the $M/G/1$ Retrial Queueing Model with Server Breakdowns, *J. Appl. Math.*, 2012, Art.ID 890243, 16 pp.
- [17] Greiner G., Perturbing the boundary conditions of a generator, *Houston J. Math.*, 1987, 13, 213-229
- [18] Gupur G., Li X.Z., Zhu G.T., *Functional Analysis Method in Queueing Theory*, Hertfordshire: Research Information Ltd., 2001
- [19] Haji A., Radl A., Asymptotic stability of the solution of the $M/M^B/1$ queueing model, *Comput. Math. Appl.*, 2007, 53, 1411-1420
- [20] Nagel R., *One-parameter semigroups of positive operators*, Berlin: Springer, 1986
- [21] Gupur G., *Functional analysis methods for reliability models*, Basel: Springer, 2011
- [22] Arendt W., Batty C.K., Tauberian theorems and stability of one-parameter semigroups, *T. Am. Math. Soc.*, 1988, 306, 837-852
- [23] Adams R.A., *Sobolev Spaces*, New York: Academic Press, 1975