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Chao-Hong Tang and Hong-Xu Li*

Stepanov-like pseudo almost periodic functions on time scales and applications to dynamic equations with delay

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Abstract: In this paper, we introduce the concept of S^p-pseudo almost periodicity on time scales and present some basic properties of it, including the translation invariance, uniqueness of decomposition, completeness and composition theorem. Moreover, we prove the seemingly simple but nontrivial result that pseudo almost periodicity implies Stepanov-like pseudo almost periodicity. As an application of the abstract results, we present some existence and uniqueness results on the pseudo almost periodic solutions of dynamic equations with delay.

Keywords: S^p-pseudo almost periodicity, Time scales, Dynamic equations

MSC: 42A75, 34N05

1 Introduction

The theory of time scales was established by S. Hilger in 1988 (see [1]). This theory unifies continuous and discrete problems and provides a powerful tool for applications to economics, populations models, quantum physics among others, and hence has been attracting the attention of many mathematicians (see [2, 3] and the references therein). In 2011, Li and Wang [4, 5] introduced the concept of almost periodic functions on time scales. Since then, many generalized forms of almost periodicity have been introduced on time scales, such as pseudo almost periodicity [6], almost automorphy[7], weighted pseudo almost periodicity [8] etc.

To consider the almost periodicity of integrable functions on the real line, Stepanov [9] and Wiener [10] introduced Stepanov almost periodicity in 1926. Then this concept was extended to Stepanov-like pseudo almost periodicity by Diagana [11] in 2007. On the other hand, Li and Wang [12] extended Stepanov almost periodicity on time scales in 2017. Motivated by the above works, the main purpose of this paper is to consider Stepanov-like pseudo almost periodicity on time scales.

The definition and some basic properties of Stepanov-like pseudo almost periodicity are given in Section 3, including the translation invariance, uniqueness of decomposition, completeness and composition theorem. Moreover, we prove the seemingly simple but nontrivial result that pseudo almost periodicity implies Stepanov-like pseudo almost periodicity. As an application of the abstract results, we present some results on the existence and uniqueness of pseudo almost periodic solutions of dynamic equations with delay in Section 4.

Chao-Hong Tang: Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China and School of Mathematics and Physics, Mianyang Teachers' College, Mianyang, Sichuan 621000, China, E-mail: 836188248@qq.com
***Corresponding Author: Hong-Xu Li:** Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China, E-mail: hoxuli@scu.edu.cn

2 Preliminaries

The concepts and results in this section can be found in [2, 3, 5–7, 12–15], or deduced simply from the results given there. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{R}^+$ the sets of positive integers, integers, real numbers and nonnegative real numbers, respectively. $\mathbb{E}^n$ denotes the Euclidian space $\mathbb{R}^n$ or $\mathbb{C}^n$ with Euclidian norm $|\cdot|$, and $\mathbb{R}^{n\times n}$ the space of all $n \times n$ real-valued matrices with matrix norm $\|\cdot\|$.

Let $\mathbb{T}$ be a time scale, that is, a closed and nonempty subset of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

If $\sigma(t) > t$, we say that t is right-scattered. Otherwise, t is called right-dense. Analogously, if $\rho(t) < t$, then t is called left-scattered. Otherwise, t is left-dense. We always denote $\mathbb{T}$ a time scale from now on.

Let $a, b \in \mathbb{T}$ with $a \le b$, $[a, b]$, $[a, b)$, $(a, b]$, (a, b) being the usual intervals on the real line. The intervals $[a, a)$, $(a, a]$, (a, a) are understood as the empty set, and we use the following symbols:

$$[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}, \quad [a, b)_{\mathbb{T}} = [a, b) \cap \mathbb{T}, \quad (a, b]_{\mathbb{T}} = (a, b] \cap \mathbb{T}, \quad (a, b)_{\mathbb{T}} = (a, b) \cap \mathbb{T}.$$

Note that in this paper we use the above symbols only if $a, b \in \mathbb{T}$.

Denote

$$\begin{aligned}
C(\mathbb{T}; \mathbb{E}^n) &= \{f : \mathbb{T} \to \mathbb{E}^n : f \text{ is continuous}\},\\
C(\mathbb{T} \times D; \mathbb{E}^n) &= \{f : \mathbb{T} \times D \to \mathbb{E}^n : f \text{ is continuous}\},\\
BC(\mathbb{T}; \mathbb{E}^n) &= \{f : \mathbb{T} \to \mathbb{E}^n : f \text{ is bounded and continuous}\},\\
L^p_{loc}(\mathbb{T}; \mathbb{E}^n) &= \{f : \mathbb{T} \to \mathbb{E}^n : f \text{ is locally } L^p\ \Delta\text{-integrable}\}.
\end{aligned}$$

It is easy to see that $BC(\mathbb{T}; \mathbb{E}^n)$ is a Banach space with supremum norm.

If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.1. *For $f : \mathbb{T} \to \mathbb{E}^n$ and $t \in \mathbb{T}^\kappa$, $f^\Delta(t) \in \mathbb{E}^n$ is called the delta derivative of $f(t)$ if for a given $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$. Moreover, f is said to be delta differentiable on $\mathbb{T}$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}$.

We note that the integral $\int_a^b f(t)\Delta t$ always means $\int_{[a,b)_{\mathbb{T}}} f(t)\Delta t$ in this paper, and all the theorems of the general Lebesgue integration theory also hold for the Δ-integrals on $\mathbb{T}$. For more details of continuity, differentiable, Δ-measure and Δ-integral on $\mathbb{T}$, Jordan Δ-measure and multi-Riemann Δ-integrable on $\mathbb{T}^2$, we refer the readers to [2, 3, 13, 15, 16].

2.1 Almost periodicity and pseudo almost periodicity on $\mathbb{T}$

Definition 2.2 ([5, 7])**.** *A time scale $\mathbb{T}$ is called invariant under translations if*

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \ne \{0\}.$$

From now on, we always assume that $\mathbb{T}$ is invariant under translations.

Definition 2.3 ([5, 14])**.** *(i) A function $f \in C(\mathbb{T}; \mathbb{E}^n)$ is called almost periodic on $\mathbb{T}$ if for every $\varepsilon > 0$, the set*

$$T(f, \varepsilon) = \{\tau \in \Pi : |f(t + \tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}\}$$

is relatively dense in Π. $T(f, \varepsilon)$ is called the ε-translation set of f, τ is called the ε-translation number of f. Denote by $AP(\mathbb{T}; \mathbb{E}^n)$ the set of all almost periodic functions.

(ii) *Let $\Omega \subset \mathbb{E}^n$ be open. The set $AP(\mathbb{T}\times\Omega;\mathbb{E}^n)$ consists of all continuous functions $f:\mathbb{T}\times\Omega\to\mathbb{E}^n$ such that $f(\cdot,x)\in AP(\mathbb{T};\mathbb{E}^n)$ uniformly for each $x\in S$, where S is any compact subset of Ω. That is, for $\varepsilon>0$, $\bigcap_{x\in S}T(f(\cdot,x),\varepsilon)$ is relatively dense in Π.*

Let $f\in BC(\mathbb{T};\mathbb{E}^n)$. Then set

$$PAP_0(\mathbb{T};\mathbb{E}^n)=\left\{f\in BC(\mathbb{T};\mathbb{E}^n):\lim_{r\to+\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}|f(s)|\Delta s=0,\ \text{where } t_0\in\mathbb{T}, r\in\Pi\right\}.$$

Definition 2.4 ([6]). *A function $f\in BC(\mathbb{T};\mathbb{E}^n)$ is called pseudo almost periodic if $f=g+\phi$, where $g\in AP(\mathbb{T};\mathbb{E}^n)$ and $\phi\in PAP_0(\mathbb{T};\mathbb{E}^n)$. We denote by $PAP(\mathbb{T};\mathbb{E}^n)$ the set of all pseudo almost periodic functions.*

Proposition 2.5. *$f(\mathbb{T})$ is relatively compact if $f\in AP(\mathbb{T};\mathbb{E}^n)$.*

Proof. It follows from [17, Theorem 3.4] that for $f\in C(\mathbb{T};\mathbb{E}^n)$, $f\in AP(\mathbb{T};\mathbb{E}^n)$ if and only if there exists $g\in AP(\mathbb{R};\mathbb{E}^n)$ such that $f(t)=g(t)$ for $t\in\mathbb{T}$. Meanwhile, it is well known that $g(\mathbb{R})$ is relatively compact if $g\in AP(\mathbb{R};\mathbb{E}^n)$. Therefore $f(\mathbb{T})$ is relatively compact if $f\in AP(\mathbb{T};\mathbb{E}^n)$. □

Proposition 2.6 ([6]). (i) *If $f\in PAP(\mathbb{T};\mathbb{E}^n)$ and $\phi\in PAP_0(\mathbb{T};\mathbb{E}^n)$, then for any $\tau\in\Pi$, $f(\cdot+\tau)\in PAP(\mathbb{T};\mathbb{E}^n)$ and $\phi(\cdot+\tau)\in PAP_0(\mathbb{T};\mathbb{E}^n)$.*
(ii) *$PAP(\mathbb{T},\mathbb{E}^n)$ and $PAP_0(\mathbb{T},\mathbb{E}^n)$ are Banach spaces under the sup norm.*

2.2 S^p-almost periodic functions on $\mathbb{T}$

We always assume that $p\geq 1$ afterwards without any further comments. Let

$$\mathcal{K}:=\begin{cases}\inf\{|\tau|:\tau\in\Pi,\tau\neq 0\}, & \text{if } \mathbb{T}\neq\mathbb{R},\\ 1, & \text{if } \mathbb{T}=\mathbb{R}.\end{cases}$$

Define $\|\cdot\|_{S^p}:L^p_{loc}(\mathbb{T};\mathbb{E}^n)\to\mathbb{R}^+$ as

$$\|f\|_{S^p}:=\sup_{t\in\mathbb{T}}\left(\frac{1}{\mathcal{K}}\int_t^{t+\mathcal{K}}|f(s)|^p\Delta s\right)^{\frac{1}{p}}\quad\text{for } f\in L^p_{loc}(\mathbb{T};\mathbb{E}^n).$$

$f\in L^p_{loc}(\mathbb{T};\mathbb{E}^n)$ is called S^p-bounded if $\|f\|_{S^p}<\infty$. Denote by $BS^p(\mathbb{T};\mathbb{E}^n)$ the space of all these functions.

Definition 2.7 ([12]). *(i) A function $f\in BS^p(\mathbb{T};\mathbb{E}^n)$ is called S^p-almost periodic on $\mathbb{T}$ if for every $\varepsilon>0$, the ε-translation set of f*

$$T(f,\varepsilon)=\{\tau\in\Pi:\|f(\cdot+\tau)-f\|_{S^p}<\varepsilon\}$$

is relatively dense in Π. Denote the set of all these functions by $S^pAP(\mathbb{T},\mathbb{E}^n)$.
(ii) A function $f:\mathbb{T}\times\Omega\to\mathbb{E}^n$ with $\Omega\subset\mathbb{E}^n$ is called S^p-almost periodic in $t\in\mathbb{T}$ if $f(\cdot,x)\in S^pAP(\mathbb{T};\mathbb{E}^n)$ uniformly for each $x\in S$, where S is an arbitrary compact subset of Ω. That is, for $\varepsilon>0$, $\bigcap_{x\in S}T(f(\cdot,x),\varepsilon)$ is relatively dense in Π. Denote the set of all such functions by $S^pAP(\mathbb{T}\times\Omega;\mathbb{E}^n)$.

Proposition 2.8 ([12, 18]). (i) *If $f\in S^pAP(\mathbb{T};\mathbb{E}^n)$, then for any $\tau\in\Pi$, $f(\cdot+\tau)\in S^pAP(\mathbb{T};\mathbb{E}^n)$.*
(ii) *$BS^p(\mathbb{T};\mathbb{E}^n)$, $S^pAP(\mathbb{T},\mathbb{E}^n)$ are Banach spaces under the S^p-norm $\|\cdot\|_{S^p}$.*
(iii) *Let $1\leq q\leq p<\infty$. Then $BS^p(\mathbb{T};\mathbb{E}^n)\subset BS^q(\mathbb{T};\mathbb{E}^n)$, $S^pAP(\mathbb{T};\mathbb{E}^n)\subset S^qAP(\mathbb{T};\mathbb{E}^n)$ and $\|f\|_{S^q}\leq\|f\|_{S^p}$ for $f\in BS^p(\mathbb{T};\mathbb{E}^n)$.*
(iv) *$AP(\mathbb{T};\mathbb{E}^n)\subset S^pAP(\mathbb{T};\mathbb{E}^n)$.*

3 S^p-pseudo almost periodic functions

Now we introduce the concept of S^p-pseudo almost periodicity on time scales and present the main properties of it.

3.1 Definitions

We define the norm operator $\mathcal{N}$ on $BS^p(\mathbb{T};\mathbb{E}^n)$ as follows:

$$\mathcal{N}(f)(t) := \left(\frac{1}{\mathcal{K}}\int_t^{t+\mathcal{K}} |f(s)|^p \Delta s\right)^{\frac{1}{p}} \quad \text{for } f \in BS^p(\mathbb{T};\mathbb{E}^n),\ t \in \mathbb{T}.$$

Lemma 3.1. *The norm operator $\mathcal{N}$ maps $BS^p(\mathbb{T};\mathbb{E}^n)$ into $BC(\mathbb{T};\mathbb{R}^+)$ and maps $S^pAP(\mathbb{T};\mathbb{E}^n)$ into $AP(\mathbb{T};\mathbb{R}^+)$. Moreover, for $f, g \in BS^p(\mathbb{T};\mathbb{E}^n)$, $t \in \mathbb{T}$,*

$$\|\mathcal{N}(f)\|_\infty = \|f\|_{S^p}, \quad |\mathcal{N}(f)(t) - \mathcal{N}(g)(t)| \le \mathcal{N}(f \pm g)(t) \le \mathcal{N}(f)(t) + \mathcal{N}(g)(t). \tag{1}$$

Proof. It is obvious that $\|\mathcal{N}(f)\|_\infty = \|f\|_{S^p}$. Then $\mathcal{N}(f)$ is bounded when $f \in BS^p(\mathbb{T};\mathbb{E}^n)$. The second part of (1) can be got from Minkowski inequality immediately.

Let $f \in BS^p(\mathbb{T};\mathbb{E}^n)$. Then $f \in L^p_{loc}(\mathbb{T};\mathbb{E}^n)$, and the absolute continuity of integral follows that for $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any Δ-measurable set e with $\mu_\Delta(e) < \delta$,

$$\int_e |f(s)|^p \Delta s < \frac{\mathcal{K}\varepsilon}{2}.$$

Thus, for $t_1, t_2 \in \mathbb{T}$, $t_1 < t_2$, $|t_1 - t_2| < \delta$,

$$\begin{aligned}|(\mathcal{N}(f))^p(t_1) - (\mathcal{N}(f))^p(t_2)| &\le \left|\frac{1}{\mathcal{K}}\int_{t_1}^{t_2} |f(s)|^p \Delta s - \frac{1}{\mathcal{K}}\int_{t_1+\mathcal{K}}^{t_2+\mathcal{K}} |f(s)|^p \Delta s\right| \\ &\le \frac{1}{\mathcal{K}}\int_{t_1}^{t_2} |f(s)|^p \Delta s + \frac{1}{\mathcal{K}}\int_{t_1+\mathcal{K}}^{t_2+\mathcal{K}} |f(s)|^p \Delta s \\ &< \varepsilon.\end{aligned}$$

This implies that $(\mathcal{N}(f))^p$ is continuous, and $\mathcal{N}(f)$ is continuous. So $\mathcal{N} : BS^p(\mathbb{T};\mathbb{E}^n) \to BC(\mathbb{T};\mathbb{R}^+)$.

Let $f \in S^pAP(\mathbb{T};\mathbb{E}^n)$. Then $\mathcal{N}(f) \in BC(\mathbb{T};\mathbb{R}^+)$ by the proof above. For $\tau \in T(f,\varepsilon)$, by (1),

$$\begin{aligned}&\|\mathcal{N}(f)(\cdot+\tau) - \mathcal{N}(f)\|_\infty = \sup_{t\in\mathbb{T}} |\mathcal{N}(f)(t+\tau) - \mathcal{N}(f)(t)| = \sup_{t\in\mathbb{T}} |\mathcal{N}(f(\cdot+\tau))(t) - \mathcal{N}(f)(t)| \\ &\le \sup_{t\in\mathbb{T}} \mathcal{N}(f(\cdot+\tau) - f)(t) = \|\mathcal{N}(f(\cdot+\tau) - f)\|_\infty = \|f(\cdot+\tau) - f\|_{S^p} < \varepsilon,\end{aligned}$$

which implies that $T(f,\varepsilon) \subset T(\mathcal{N}(f),\varepsilon)$. Hence $\mathcal{N}(f) \in AP(\mathbb{T};\mathbb{R}^+)$. □

Definition 3.2. *A function $f \in BS^p(\mathbb{T};\mathbb{E}^n)$ is said to be ergodic if $\mathcal{N}(f) \in PAP_0(\mathbb{T};\mathbb{R}^+)$, i.e.*

$$\lim_{r\to+\infty} \frac{1}{2r}\int_{t_0-r}^{t_0+r} \left(\frac{1}{\mathcal{K}}\int_t^{t+\mathcal{K}} |f(s)|^p \Delta s\right)^{\frac{1}{p}} \Delta t = 0, \text{ where } t_0 \in \mathbb{T}, r \in \Pi.$$

We denote by $S^pPAP_0(\mathbb{T};\mathbb{E}^n)$ the set of all ergodic functions from $\mathbb{T}$ to $\mathbb{E}^n$.

Definition 3.3. *(i) A function $f \in BS^p(\mathbb{T};\mathbb{E}^n)$ is called Stepanov-like pseudo almost periodic (S^p-pseudo almost periodic) if $f = g + \phi$, where $g \in S^pAP(\mathbb{T};\mathbb{E}^n)$ and $\phi \in S^pPAP_0(\mathbb{T};\mathbb{E}^n)$. g and ϕ are called the almost periodic component and the ergodic perturbation of f, respectively. We denote by $S^pPAP(\mathbb{T};\mathbb{E}^n)$ the set of all such functions f.*

(ii) *A function $f : \mathbb{T} \times \Omega \to \mathbb{E}^n$ with $\Omega \subset \mathbb{E}^n$ is called Stepanov-like pseudo almost periodic (S^p-pseudo almost periodic) in $t \in \mathbb{T}$ if $f(\cdot, x) = g(\cdot, x) + \phi(\cdot, x) \in S^p PAP(\mathbb{T}; \mathbb{E}^n)$ for each $x \in \Omega$ and $g(\cdot, x) \in S^p AP(\mathbb{T}; \mathbb{E}^n)$ uniformly for each $x \in S$, where S is an arbitrary compact subset of Ω. Denote the set of all these functions by $S^p PAP(\mathbb{T} \times \Omega; \mathbb{E}^n)$.*

Remark 3.4. (i) *We note that Definition 3.3 is a generalization of S^p-pseudo almost periodicity on $\mathbb{R}$ introduced by Diagana [19].*
(ii) *It is easy to check that $S^q PAP(\mathbb{T}; \mathbb{E}^n) \subset S^p PAP(\mathbb{T}; \mathbb{E}^n)$ for $1 \le p \le q$.*

3.2 Some basic properties

From now on, we write $f = g + \phi \in S^p PAP(\mathbb{T}; \mathbb{E}^n)$ implies $g \in S^p AP(\mathbb{T}; \mathbb{E}^n)$ and $\phi \in S^p PAP_0(\mathbb{T}; \mathbb{E}^n)$. In this subsection, we give some basic properties of S^p-pseudo almost periodicity, including the uniqueness of decomposition, the translation invariance and the completeness.

Proposition 3.5. *The decomposition of S^p-pseudo almost periodic functions is unique.*

Proof. If $f = g_1 + \phi_1 = g_2 + \phi_2 \in S^p PAP(\mathbb{T}; \mathbb{E}^n)$, then $g_1 - g_2 = \phi_2 - \phi_1$. This implies $\mathcal{N}(g_1 - g_2) = \mathcal{N}(\phi_1 - \phi_2)$. Note that $\mathcal{N}(\phi_1 - \phi_2) \le \mathcal{N}(\phi_1) + \mathcal{N}(\phi_2)$ by (1), it follows that $\mathcal{N}(\phi_1 - \phi_2) \in PAP_0(\mathbb{T}; \mathbb{R}^+)$, and then $\mathcal{N}(g_1 - g_2) \in PAP_0(\mathbb{T}; \mathbb{R}^+)$. Meanwhile, $g_1 - g_2 \in S^p AP(\mathbb{T}; \mathbb{E}^n)$ since $g_1, g_2 \in S^p AP(\mathbb{T}; \mathbb{E}^n)$. Thus $\mathcal{N}(g_1 - g_2) \in AP(\mathbb{T}; \mathbb{R}^+)$ by Lemma 3.1. Now it follows from [6, Theorem 3.5] that $\mathcal{N}(g_1 - g_2) = 0$. This yields that $g_1 = g_2$ in $S^p AP(\mathbb{T}; \mathbb{E}^n)$, and consequently, $\phi_1 = \phi_2$ in $S^p PAP_0(\mathbb{T}; \mathbb{E}^n)$. □

By Proposition 2.6 (i), Proposition 2.8 (i) and Lemma 3.1, we can easily obtain the following translation invariance of S^p-pseudo almost periodic functions. Here we omit the details.

Proposition 3.6. *Let $f \in S^p PAP(\mathbb{T}; \mathbb{E}^n)$. Then $f(\cdot + \tau) \in S^p PAP(\mathbb{T}; \mathbb{E}^n)$ for $\tau \in \Pi$.*

Proposition 3.7. *$(S^p PAP_0(\mathbb{T}; \mathbb{E}^n), \|\cdot\|_{S^p})$ is a Banach space.*

Proof. By Proposition 2.8 (ii), we only need to prove the closedness of $S^p PAP_0(\mathbb{T}; \mathbb{E}^n)$ in $BS^p(\mathbb{T}; \mathbb{E}^n)$. In fact, let $\{\phi_k\} \subset S^p PAP_0(\mathbb{T}; \mathbb{E}^n)$ and $\phi \in BS^p(\mathbb{T}; \mathbb{E}^n)$ with $\|\phi_k - \phi\|_{S^p} \to 0$ as $k \to \infty$. By Lemma 3.1, $\{\mathcal{N}(\phi_k)\} \subset PAP_0(\mathbb{T}; \mathbb{R}^+)$ and

$$\|\mathcal{N}(\phi_k) - \mathcal{N}(\phi)\|_\infty \le \|\mathcal{N}(\phi_k - \phi)\|_\infty = \|\phi_k - \phi\|_{S^p} \to 0 \text{ as } k \to \infty.$$

This implies $\mathcal{N}(\phi) \in PAP_0(\mathbb{T}; \mathbb{R}^+)$ since $PAP_0(\mathbb{T}; \mathbb{R}^+)$ is a Banach space by Proposition 2.6 (ii). Hence $\phi \in S^p PAP_0(\mathbb{T}; \mathbb{E}^n)$ and $S^p PAP_0(\mathbb{T}; \mathbb{E}^n)$ is closed. □

To prove the completeness of the space $S^p PAP(\mathbb{T}; \mathbb{E}^n)$ we need the following lemma.

Lemma 3.8. *If $f = g + \phi \in S^p PAP(\mathbb{T}; \mathbb{E}^n)$, then $\|g\|_{S^p} \le \|f\|_{S^p}$.*

Proof. Let $Q(t) := \mathcal{N}(g)(t) - \mathcal{N}(\phi)(t)$, $t \in \mathbb{T}$. Then by (1),

$$|Q(t)| \le \mathcal{N}(g + \phi)(t) = \mathcal{N}(f)(t), \quad t \in \mathbb{T}.$$

This implies that $\|Q\|_\infty \le \|\mathcal{N}(f)\|_\infty$. On the other hand, $\mathcal{N}(g) \in AP(\mathbb{T}; \mathbb{R}^+)$ by Lemma 3.1, and clearly, $-\mathcal{N}(\phi) \in PAP_0(\mathbb{T}; \mathbb{R})$. Then $Q = \mathcal{N}(g) - \mathcal{N}(\phi) \in PAP(\mathbb{T}; \mathbb{R})$. Thus, by [6, Theorem 4.2] and (1),

$$\|g\|_{S^p} = \|\mathcal{N}(g)\|_\infty \le \|Q\|_\infty \le \|\mathcal{N}(f)\|_\infty = \|f\|_{S^p}.$$

□

Proposition 3.9. *$(S^p PAP(\mathbb{T}; \mathbb{E}^n), \|\cdot\|_{S^p})$ is a Banach space.*

Proof. It suffices to prove that $S^pPAP(\mathbb{T};\mathbb{E}^n)$ is closed in $BS^p(\mathbb{T};\mathbb{E}^n)$. Let $\{f_k\} = \{g_k + \phi_k\} \subset S^pPAP(\mathbb{T};\mathbb{E}^n)$ and $f \in BS^p(\mathbb{T};\mathbb{E}^n)$ with $\|f_k - f\|_{S^p} \to 0$ as $k \to \infty$. Then $\|f_k - f_j\|_{S^p} \to 0$ as $k, j \to \infty$. It follows from Lemma 3.8 that $\|g_k - g_j\|_{S^p} \le \|f_k - f_j\|_{S^p} \to 0$ as $k, j \to \infty$. This together with Proposition 2.8 (ii) implies that there exists $g \in S^pAP(\mathbb{T};\mathbb{E}^n)$ such that $\|g_k - g\|_{S^p} \to 0$ as $k \to \infty$. Meanwhile, by Lemma 3.8,

$$\|\phi_k - \phi_j\|_{S^p} = \|f_k - f_j + g_j - g_k\|_{S^p} \le \|f_k - f_j\|_{S^p} + \|g_k - g_j\|_{S^p} \to 0 \text{ as } k, j \to \infty,$$

which implies that $\phi_k \to \phi$ as $k \to \infty$ for some $\phi \in S^pPAP_0(\mathbb{T};\mathbb{E}^n)$ by Proposition 3.7. Let $f = g + \phi$. Then $f \in S^pPAP(\mathbb{T};\mathbb{E}^n)$ and $f_k \to f$ as $k \to \infty$. That is $S^pPAP(\mathbb{T};\mathbb{E}^n)$ is closed in $BS^p(\mathbb{T};\mathbb{E}^n)$. □

3.3 $PAP(\mathbb{T};\mathbb{E}^n) \subset S^pPAP(\mathbb{T};\mathbb{E}^n)$

We prove the seemingly simple but nontrivial result that $PAP(\mathbb{T};\mathbb{E}^n) \subset S^pPAP(\mathbb{T};\mathbb{E}^n)$ in this subsection.

For $t_0 \in \mathbb{T}$, let

$$E_P = \{(t,s) \in \mathbb{T} \times \mathbb{T} : t_0 \le t < t_0 + \mathcal{K}, t \le s < t + \mathcal{K}\},$$

Lemma 3.10. *(i) E_P is Jordan Δ-measurable.*
(ii) If $f : E_P \to \mathbb{R}$ is bounded continuous, then f is Riemann Δ-integrable over E_P, and

$$\int_{t_0}^{t_0+\mathcal{K}} \int_{t}^{t+\mathcal{K}} f(t,s)\Delta s\Delta t = \int_{t_0}^{t_0+\mathcal{K}} \int_{t_0}^{s} f(t,s)\Delta t\Delta s + \int_{t_0+\mathcal{K}}^{t_0+2\mathcal{K}} \int_{s-\mathcal{K}}^{t_0+\mathcal{K}} f(t,s)\Delta t\Delta s. \tag{2}$$

Proof. By a fundamental calculation, we can prove that E_P is Jordan Δ-measurable and f is Riemann Δ-integrable over E_P. Here we omit the details, and we only prove that (2) holds. Let $R = [t_0, t_0 + \mathcal{K})_{\mathbb{T}} \times [t_0, t_0 + 2\mathcal{K})_{\mathbb{T}}$ and $F : R \to \mathbb{R}$ be defined as

$$F(t,s) = \begin{cases} f(t,s), & \text{if } (t,s) \in E_P, \\ 0, & \text{if } (t,s) \in R \setminus E_P. \end{cases}$$

Then for $t \in [t_0, t_0 + \mathcal{K})_{\mathbb{T}}$,

$$F(t,s) = \begin{cases} f(t,s), & \text{if } s \in [t, t+\mathcal{K})_{\mathbb{T}}, \\ 0, & \text{if } s \in [t_0, t)_{\mathbb{T}} \cup [t+\mathcal{K}, t_0 + 2\mathcal{K})_{\mathbb{T}}, \end{cases}$$

and $F(t,\cdot)$ can only be discontinuous at t and $t + \mathcal{K}$ since $f : E_P \to \mathbb{R}$ is bounded continuous. It follows from [20, Theorem 5.8] that $F(t,\cdot)$ is Δ-integrable on $[t_0, t_0 + 2\mathcal{K})_{\mathbb{T}}$. Thus, by [21, Theorem 2.15],

$$\iint_R F(t,s)\Delta t\Delta s = \int_{t_0}^{t_0+\mathcal{K}} \left(\int_{t_0}^{t_0+2\mathcal{K}} F(t,s)\Delta s \right) \Delta t = \int_{t_0}^{t_0+\mathcal{K}} \int_{t}^{t+\mathcal{K}} f(t,s)\Delta s\Delta t. \tag{3}$$

On the other hand, for $s \in [t_0, t_0 + \mathcal{K})_{\mathbb{T}}$,

$$F(t,s) = \begin{cases} f(t,s), & \text{if } t \in [t_0, s)_{\mathbb{T}}, \\ 0, & \text{if } t \in [s, t_0 + \mathcal{K})_{\mathbb{T}}, \end{cases}$$

and for $s \in [t_0 + \mathcal{K}, t_0 + 2\mathcal{K})_{\mathbb{T}}$,

$$F(t,s) = \begin{cases} 0, & \text{if } t \in [t_0, s - \mathcal{K})_{\mathbb{T}}, \\ f(t,s), & \text{if } t \in [s - \mathcal{K}, t_0 + \mathcal{K})_{\mathbb{T}}. \end{cases}$$

Similarly, we can get that for every $s \in [t_0, t_0 + 2\mathcal{K})_{\mathbb{T}}$, $F(\cdot, s)$ is Δ-integrable on $[t_0, t_0 + \mathcal{K})_{\mathbb{T}}$. Thus, by [21, Remark 2.16],

$$\iint_R F(t,s)\Delta t\Delta s = \int_{t_0}^{t_0+2\mathcal{K}} \left(\int_{t_0}^{t_0+\mathcal{K}} F(t,s)\Delta t \right) \Delta s$$

$$= \int_{t_0}^{t_0+\mathcal{K}} \int_{t_0}^{s} f(t,s)\Delta t \Delta s + \int_{t_0+\mathcal{K}}^{t_0+2\mathcal{K}} \int_{s-\mathcal{K}}^{t_0+\mathcal{K}} f(t,s)\Delta t \Delta s.$$

This together with (3) leads to the conclusion. □

Proposition 3.11. $PAP(\mathbb{T};\mathbb{E}^n) \subset S^p PAP(\mathbb{T};\mathbb{E}^n)$.

Proof. Let $f = g+\phi \in PAP(\mathbb{T};\mathbb{E}^n)$ with $g \in AP(\mathbb{T};\mathbb{E}^n)$ and $\phi \in PAP_0(\mathbb{T};\mathbb{E}^n)$. Then $f \in BS^p(\mathbb{T};\mathbb{E}^n)$, $AP(\mathbb{T};\mathbb{E}^n) \subset S^p AP(\mathbb{T};\mathbb{E}^n)$ by Proposition 2.8 (iii), and we need only to prove that $\phi \in PAP_0(\mathbb{T};\mathbb{E}^n)$, i.e. $\mathcal{N}(\phi) \in PAP_0(\mathbb{T};\mathbb{R})$. In fact, let $q > 0$ with $1/p + 1/q = 1$, for fixed $t_0 \in \mathbb{T}$ and $m \in \mathbb{N}$,

$$\begin{aligned}
\frac{1}{m\mathcal{K}} \int_{t_0}^{t_0+m\mathcal{K}} \mathcal{N}(\phi)(t)\Delta t &\le (m\mathcal{K})^{\frac{1}{q}-1} \left(\int_{t_0}^{t_0+m\mathcal{K}} (\mathcal{N}(\phi)(t))^p \Delta t \right)^{\frac{1}{p}} \\
&= \left(\frac{1}{m\mathcal{K}^2} \int_{t_0}^{t_0+m\mathcal{K}} \int_{t}^{t+\mathcal{K}} |\phi(s)|^p \Delta s \Delta t \right)^{\frac{1}{p}} \\
&\le \|\phi\|_\infty^{\frac{p-1}{p}} \left(\frac{1}{m\mathcal{K}^2} \int_{t_0}^{t_0+m\mathcal{K}} \int_{t}^{t+\mathcal{K}} |\phi(s)| \Delta s \Delta t \right)^{\frac{1}{p}} \\
&= \|\phi\|_\infty^{\frac{1}{q}} \left(\frac{1}{m\mathcal{K}^2} \sum_{i=0}^{m-1} \int_{t_0}^{t_0+\mathcal{K}} \int_{t}^{t+\mathcal{K}} |\phi(s+i\mathcal{K})| \Delta s \Delta t \right)^{\frac{1}{p}}.
\end{aligned}$$

Meanwhile, by Lemma 3.10 and the fact that $|\phi| + |\phi(\cdot+\mathcal{K})| \in PAP_0(\mathbb{T};\mathbb{R}^n)$,

$$\begin{aligned}
&\frac{1}{m\mathcal{K}^2} \sum_{i=0}^{m-1} \int_{t_0}^{t_0+\mathcal{K}} \int_{t}^{t+\mathcal{K}} |\phi(s+i\mathcal{K})| \Delta s \Delta t \\
&= \frac{1}{m\mathcal{K}^2} \sum_{i=0}^{m-1} \left(\int_{t_0}^{t_0+\mathcal{K}} \int_{t_0}^{s} |\phi(s+i\mathcal{K})| \Delta t \Delta s + \int_{t_0+\mathcal{K}}^{t_0+2\mathcal{K}} \int_{s-\mathcal{K}}^{t_0+\mathcal{K}} |\phi(s+i\mathcal{K})| \Delta t \Delta s \right) \\
&\le \frac{1}{m\mathcal{K}} \sum_{i=0}^{m-1} \left(\int_{t_0}^{t_0+\mathcal{K}} |\phi(s+i\mathcal{K})| \Delta s + \int_{t_0+\mathcal{K}}^{t_0+2\mathcal{K}} |\phi(s+i\mathcal{K})| \Delta s \right) \\
&= \frac{1}{m\mathcal{K}} \int_{t_0}^{t_0+m\mathcal{K}} (|\phi(s)| + |\phi(s+\mathcal{K})|)\Delta s \\
&\to 0 \quad \text{as } m \to +\infty.
\end{aligned}$$

Then

$$\frac{1}{m\mathcal{K}} \int_{t_0}^{t_0+m\mathcal{K}} \mathcal{N}(\phi)(t)\Delta t \to 0 \quad \text{as } m \to +\infty.$$

This implies that $\mathcal{N}(\phi) \in PAP_0(\mathbb{T};\mathbb{R})$. □

3.4 Composition theorems

We will use the following S^p-Lipschitz condition for $f \in S^p AP(\mathbb{T} \times \mathbb{E}^n;\mathbb{E}^n)$:
(H) There exists a constant $L_f > 0$ such that for any $x, y \in BS^p(\mathbb{T};\mathbb{E}^n)$ and $t \in \mathbb{T}$,

$$\mathcal{N}(f(\cdot,x(\cdot)) - f(\cdot,y(\cdot)))(t) \le L_f \mathcal{N}(x-y)(t).$$

Remark 3.12. *Obviously, (H) implies that $\|f(\cdot, x(\cdot)) - f(\cdot, y(\cdot))\|_{S^p} \leq L_f\|x-y\|_{S^p}$. Moreover, f satisfies (H) if $f(t,x)$ is Lipschitz continuous in $x \in \mathbb{E}^n$ uniformly in $t \in \mathbb{T}$, i.e. $|f(t,x) - f(t,y)| \leq L|x-y|$ for every $x, y \in \mathbb{E}^n$, $t \in \mathbb{T}$ and some constant L.*

Theorem 3.13. *Assume that $f \in S^pAP(\mathbb{T}\times\mathbb{E}^n;\mathbb{E}^n)$ satisfies* (H), *and $u \in S^pAP(\mathbb{T};\mathbb{E}^n)$ with $\overline{u(\mathbb{T})}$ compact. Then $f(\cdot, u(\cdot)) \in S^pAP(\mathbb{T};\mathbb{E}^n)$.*

Proof. By (H),

$$\|f(\cdot, u(\cdot)) - f(\cdot, 0)\|_{S^p} \leq L_f\|u\|_{S^p}.$$

Then

$$\|f(\cdot, u(\cdot))\|_{S^p} \leq \|f(\cdot, 0)\|_{S^p} + L_f\|u\|_{S^p} < \infty.$$

That is

$$f(\cdot, u(\cdot)) \in BS^p(\mathbb{T};\mathbb{E}^n). \tag{4}$$

Since $\overline{u(\mathbb{T})}$ is compact, for $\varepsilon > 0$, there exist finite open balls O_k, $k = 1, 2, \ldots, m$, with center $u_k \in \overline{u(\mathbb{T})}$ and radius $\frac{\varepsilon}{8L_f}$ such that $\overline{u(\mathbb{T})} \subset \bigcup_{k=1}^m O_k$. Set $B_k := \{s \in \mathbb{T} : u(s) \in O_k\}$, $k = 1, 2, \ldots, m$. Then $\mathbb{T} = \bigcup_{k=1}^m B_k$. Moreover, let $E_1 := B_1$, $E_k := B_k \setminus (\bigcup_{i=1}^{k-1} B_i)$, $k = 2, \ldots, m$. Then $E_i \cap E_j = \varnothing$ for $i \neq j$ and $\mathbb{T} = \bigcup_{k=1}^m E_k$. Define a step function $\hat{u} : \mathbb{T} \to \mathbb{E}^n$ by $\hat{u}(s) := u_k$, $s \in E_k$, $k = 1, 2, \ldots, m$. It is clear that $|u(s) - \hat{u}(s)| < \frac{\varepsilon}{8L_f}$ for all $s \in \mathbb{T}$. Then by (H), for $\tau \in \bigcap_{k=1}^m T(f(\cdot, u_k), \frac{\varepsilon}{4m})$,

$$\begin{aligned}
&\|f(\cdot+\tau, u(\cdot)) - f(\cdot, u(\cdot))\|_{S^p} \\
&\leq \|f(\cdot+\tau, u(\cdot)) - f(\cdot+\tau, \hat{u}(\cdot))\|_{S^p} + \|f(\cdot+\tau, \hat{u}(\cdot)) - f(\cdot, \hat{u}(\cdot))\|_{S^p} + \|f(\cdot, \hat{u}(\cdot)) - f(\cdot, u(\cdot))\|_{S^p} \\
&\leq 2L_f\|u - \hat{u}\|_{S^p} + \sup_{t\in\mathbb{T}} \left(\frac{1}{\mathcal{K}} \int_t^{t+\mathcal{K}} |f(s+\tau, \hat{u}(s)) - f(s, \hat{u}(s))|^p \Delta s \right)^{\frac{1}{p}} \\
&< \frac{\varepsilon}{4} + \sup_{t\in\mathbb{T}} \left(\frac{1}{\mathcal{K}} \sum_{k=1}^m \int_{[t,t+\mathcal{K})_{\mathbb{T}}\cap E_k} |f(s+\tau, u_k) - f(s, u_k)|^p \Delta s \right)^{\frac{1}{p}} \\
&\leq \frac{\varepsilon}{4} + \sum_{k=1}^m \|f(\cdot+\tau, u_k) - f(\cdot, u_k)\|_{S^p} \leq \frac{\varepsilon}{4} + m \cdot \frac{\varepsilon}{4m} = \frac{\varepsilon}{2}.
\end{aligned}$$

Notice that $\mathcal{G} := \bigcap_{k=1}^m T(f(\cdot, u_k), \frac{\varepsilon}{4m}) \cap T(u, \frac{\varepsilon}{2L_f})$ is relatively dense by [22, Lemma 4.9]. Thus, for $\tau \in \mathcal{G}$, by (H),

$$\begin{aligned}
&\|f(\cdot+\tau, u(\cdot+\tau)) - f(\cdot, u(\cdot))\|_{S^p} \\
&\leq \|f(\cdot+\tau, u(\cdot+\tau)) - f(\cdot+\tau, u(\cdot))\|_{S^p} + \|f(\cdot+\tau, u(\cdot)) - f(\cdot, u(\cdot))\|_{S^p} \\
&< L_f\|u(\cdot+\tau) - u\|_{S^p} + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

This implies that $\mathcal{G} \subset T(f(\cdot, u(\cdot)), \varepsilon)$, and $T(f(\cdot, u(\cdot)), \varepsilon)$ is relatively dense. Therefore, $f(\cdot, u(\cdot)) \in S^pAP(\mathbb{T};\mathbb{E}^n)$. ☐

Theorem 3.14. *Let $f = g + \phi \in S^pPAP(\mathbb{T}\times\mathbb{E}^n;\mathbb{E}^n)$ and $u = x + y \in S^pPAP(\mathbb{T};\mathbb{E}^n)$ with $\overline{x(\mathbb{T})}$ compact. Assume that f and g satisfy* (H) *with Lipschitz constants L_f and L_g, respectively. Then $f(\cdot, u(\cdot)) \in S^pPAP(\mathbb{T};\mathbb{E}^n)$.*

Proof. Let $I_1(t) = g(t, x(t))$, $I_2(t) = f(t, u(t)) - f(t, x(t))$ and $I_3(t) = \phi(t, x(t))$, $t \in \mathbb{T}$. Then

$$f(t, u(t)) = I_1(t) + I_2(t) + I_3(t), \quad t \in \mathbb{T}.$$

By Theorem 3.13, we have $I_1 \in S^pAP(\mathbb{T};\mathbb{E}^n)$. So it suffices to prove that $I_2, I_3 \in S^pPAP_0(\mathbb{T};\mathbb{E}^n)$.

Since f satisfies (H) with L_f,

$$\|I_2\|_{S^p} = \|f(\cdot, u(\cdot)) - f(\cdot, x(\cdot))\|_{S^p} \leq L_f\|u - x\|_{S^p} = L_f\|y\|_{S^p} < \infty.$$

which implies that $BS^p(\mathbb{T};\mathbb{E}^n)$. Moreover, since $y \in S^pPAP_0(\mathbb{T};\mathbb{E}^n)$, for given $t_0 \in \mathbb{T}$,

$$\begin{aligned}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}(I_2)(s)\Delta s &= \frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}(f(\cdot,u(\cdot))-f(\cdot,x(\cdot)))(s)\Delta s\\ &\le \frac{L_f}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}(y)(s)\Delta s \to 0, \quad \text{as } r\to+\infty.\end{aligned}$$

This shows that $I_2 \in S^pPAP_0(\mathbb{T};\mathbb{E}^n)$.

By the same arguments as to get (4), we can get $I_3 \in BS^p(\mathbb{T};\mathbb{E}^n)$. Since $\overline{x(\mathbb{T})}$ is compact, for any $\varepsilon > 0$, as the proof of Theorem 3.13, we can find $x_i \in \overline{x(\mathbb{T})}, E_i \subset \mathbb{T}, i = 1,2,\dots,l$ and $\hat{x} : \mathbb{T} \to \mathbb{E}^n$ such that $\hat{x}(s) := x_i, s \in E_i, i = 1,2,\dots,l$, $E_i \cap E_j = \varnothing$ for $i \ne j$, $\mathbb{T} = \bigcup_{i=1}^{l} E_i$, and $|x(s)-\hat{x}(s)| < \frac{\varepsilon}{2(L_f+L_g)}$ for all $s \in \mathbb{T}$. Since $\phi(\cdot,x) \in S^pPAP_0(\mathbb{T};\mathbb{E}^n)$ for every $x \in \mathbb{E}^n$, there exists $r_0 > 0$ such that for $r > r_0$, $1 \le i \le l$,

$$\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}(\phi(\cdot,x_i))(s)\Delta s < \frac{\varepsilon}{2l}. \tag{5}$$

Note that ϕ satisfies (H) with $L_f + L_g$ since f and g satisfy (H) with L_f and L_g, respectively, then by (H) and (5), for $r > r_0$,

$$\begin{aligned}&\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}(\phi(\cdot,x(\cdot)))(t)\Delta t\\ &\le \frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}(\phi(\cdot,x(\cdot))-\phi(\cdot,\hat{x}(\cdot)))(t)\Delta t + \frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}(\phi(\cdot,\hat{x}(\cdot)))(t)\Delta t\\ &\le \frac{1}{2r}\int_{t_0-r}^{t_0+r}(L_f+L_g)\mathcal{N}(x-\hat{x})(t)\Delta t + \frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{\mathcal{K}}\sum_{i=1}^{l}\int_{[t,t+\mathcal{K})_{\mathbb{T}}\cap E_i}|\phi(s,x_i)|^p\Delta s\right)^{\frac{1}{p}}\Delta t\\ &\le \frac{\varepsilon}{2} + \sum_{i=1}^{l}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}(\phi(\cdot,x_i))(t)\Delta t < \varepsilon.\end{aligned}$$

This yields that $I_3 \in S^pPAP_0(\mathbb{T};\mathbb{E}^n)$. □

By Proposition 2.5, $\overline{x(\mathbb{T})}$ is compact for $x \in AP(\mathbb{T};\mathbb{E}^n)$. Then by Theorem 3.11 and 3.14, we have the following corollary.

Corollary 3.15. *Let $f = g + \phi \in S^pPAP(\mathbb{T}\times\mathbb{E}^n;\mathbb{E}^n)$ and $u \in PAP(\mathbb{T};\mathbb{E}^n)$. Assume that f and g satisfy* (H). *Then $f(\cdot,u(\cdot)) \in S^pPAP(\mathbb{T};\mathbb{E}^n)$.*

4 Dynamic equations

4.1 Exponential functions

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \ne 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T};\mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T};\mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for } t \in \mathbb{T}\}$. The set of all regressive functions on time scales forms an Abelian group under the addition $\oplus$ defined by $p \oplus q \triangleq p + q + \mu(t)pq$. Meanwhile, the additive inverse in this group is denoted by $\ominus p \triangleq -\dfrac{p}{1+\mu(t)p}$.

Definition 4.1 ([2]). *If $p \in \mathcal{R}$ then the exponential function is defined by*

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right),$$

for $s,t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{1}{h}Log(1+hz), & \text{if } h \neq 0, \\ z, & \text{if } h = 0, \end{cases}$$

where Log is the principal logarithm.

Definition 4.2 ([2]). *A matrix-valued function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is called regressive if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^\kappa$, and the class of all such regressive and rd-continuous functions is denoted, similarly to the scalar case, by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T};\mathbb{R}^{n\times n})$.*

Definition 4.3 ([2]). *Let $t_0 \in \mathbb{T}$ and $A \in \mathcal{R}(\mathbb{T};\mathbb{R}^{n\times n})$. The unique matrix-valued solution of the initial value problem (IVP)*

$$X^\Delta(t) = A(t)X(t), \quad X(t_0) = I, \tag{6}$$

where I denotes the $n \times n$ identity matrix, is called the matrix exponential function (at t_0), which is denoted by $e_A(\cdot, t_0)$.

We note that the existence and uniqueness of IVP (6) can be obtained by [2, Theorem 5.8].

Lemma 4.4 ([2]). *Let $t, s \in \mathbb{T}$.*
(i) $e_p(t,t) = 1$, $e_A(t,t) = I$.
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$.
(iii) $e_p(t,s)e_p(s,r) = e_p(t,r)$, $e_A(t,s)e_A(s,r) = e_A(t,r)$.

Lemma 4.5. *Let $a > 0$ be a constant and $t, s \in \mathbb{T}$.*
(i) $e_{\ominus a}(t,s) \leq 1$ *if* $t \geq s$.
(ii) $e_{\ominus a}(t+\tau, s+\tau) = e_{\ominus a}(t,s)$ *for* $\tau \in \Pi$.
(iii) *There exists $N > 0$ such that $(t-s)e_{\ominus a}(t,s) \leq N$ for $t \geq s$.*
(iv) *For $t_0 \in \mathbb{T}$, $e_{\ominus a}(t_0, \cdot)$ is increasing on $(-\infty, t_0]_\mathbb{T}$.*
(v) *The series $\sum_{j=1}^\infty e_{\ominus a}(t, \sigma(t) - (j-1)\mathcal{K})$ converges uniformly for $t \in \mathbb{T}$. Moreover, for all $t \in \mathbb{T}$,*

$$\sum_{j=1}^\infty e_{\ominus a}(t, \sigma(t) - (j-1)\mathcal{K}) \leq \lambda_a := \begin{cases} \frac{1}{1-e^{-a\mathcal{K}}}, & \mathbb{T} = \mathbb{R}, \\ 2 + a\bar{\mu} + \frac{1}{a\bar{\mu}}, & \mathbb{T} \neq \mathbb{R}, \end{cases}$$

where $\bar{\mu} := \sup_{t\in\mathbb{T}} \mu(t)$.

Proof. (i) is obvious. (ii) can be readily obtained by the fact that $\mu(t+\tau) = \mu(t)$ for all $t \in \mathbb{T}$ and $\tau \in \Pi$. If $\mathbb{T} = \mathbb{R}$, $(t-s)e_{\ominus a}(t,s) = (t-s)e^{-a(t-s)} \leq \frac{1}{ae}$. That is (iii) holds with $N = \frac{1}{ae}$. If $\mathbb{T} \neq \mathbb{R}$, let $\{t_i\}_{i\in I}$, $I \subseteq \mathbb{N}$, be all right-scattered points in $\mathbb{T}$. By [13, Theorem 5.2],

$$\begin{aligned} e_{\ominus a}(t,s) &= \exp\left(\int_{[s,t)_\mathbb{T}} (-a)d\tau - \sum_{t_i\in[s,t)_\mathbb{T}} \mathrm{Log}(1 + a\mu(t_i))\right) \\ &= e^{-a\mu_L([s,t)_\mathbb{T})} \prod_{t_i\in[s,t)_\mathbb{T}} \frac{1}{1+a\mu(t_i)} \leq \prod_{t_i\in[s,t)_\mathbb{T}} \frac{1}{1+a\mu(t_i)}, \end{aligned} \tag{7}$$

where μ_L denotes the Lebesgue measure. For $t > s$, there exists a unique $n_{ts} \in \mathbb{N}$ such that $t \in [s+(n_{ts}-1)\mathcal{K}, s+n_{ts}\mathcal{K})_\mathbb{T}$. Let $t_0 \in \mathbb{T}$ be right-scattered, then for every $n \in \mathbb{Z}$, $t_0 + n\mathcal{K}$ is right-scattered and $\mu(t_0 + n\mathcal{K}) = \mu(t_0)$.

Moreover, for $s \in \mathbb{T}$, $[s, s + (n_{ts} - 1)\mathcal{K})_{\mathbb{T}}$ contains $n_{ts} - 1$ right-scattered points with form $t_0 + n\mathcal{K}$. Denote $\Gamma = 1 + a\mu(t_0)$ for the convenient of writing. Then by (7),

$$\begin{aligned}(t-s)e_{\ominus a}(t,s) &\le n_{ts}\mathcal{K} \prod_{t_i \in [s, s+(n_{ts}-1)\mathcal{K})_{\mathbb{T}}} \frac{1}{1 + a\mu(t_i)} \\ &\le n_{ts}\mathcal{K}\left(\frac{1}{1 + a\mu(t_0)}\right)^{n_{ts}-1} = n_{ts}\mathcal{K}\Gamma^{1-n_{ts}} \le \frac{\mathcal{K}\Gamma^{1-1/\ln\Gamma}}{\ln\Gamma}.\end{aligned}$$

So (iii) holds with $N = \mathcal{K}\Gamma^{1-1/\ln\Gamma}/\ln\Gamma$.

(iv) can be verified easily by the definition.

If $\mathbb{T} = \mathbb{R}$, for $t \in \mathbb{T}$,

$$\sum_{j=1}^{\infty} e_{\ominus a}(t, \sigma(t) - (j-1)\mathcal{K}) = \sum_{j=1}^{\infty} e^{-a(j-1)} = \frac{1}{1 - e^{-a}}.$$

That is (v) holds for $\mathbb{T} = \mathbb{R}$. If $\mathbb{T} \neq \mathbb{R}$, then $\mathcal{K} \ge \bar{\mu} = \sup_{t\in\mathbb{T}} \mu(t) > 0$, and it is easy to see that there exists a right-scattered point t_0 such that $\mu(t_0) = \bar{\mu}$. In addition, for $t \in \mathbb{T}$ and $j \ge 3$, $[t, \sigma(t) - (j-1)\mathcal{K})_{\mathbb{T}}$ contains at least $j - 2$ right-scattered points with forms $t_0 + n_t\mathcal{K}$, $n_t \in \mathbb{Z}$, $\mu(t_0 + n_t\mathcal{K}) = \mu(t_0) = \bar{\mu}$, and

$$e_{\ominus a}(t, \sigma(t) - (j-1)\mathcal{K}) \le (e_{\ominus a}(\sigma(t_0), t_0))^{j-2} = (1 + a\bar{\mu})^{2-j}.$$

Then for any $t \in \mathbb{T}$,

$$\begin{aligned}\sum_{j=1}^{\infty} e_{\ominus a}(t, \sigma(t) - (j-1)\mathcal{K}) &\le e_{\ominus a}(t, \sigma(t)) + e_{\ominus a}(t, \sigma(t) - \mathcal{K}) + \sum_{j=3}^{\infty}(1 + a\bar{\mu})^{2-j} \\ &\le (1 + a\bar{\mu}) + 1 + \frac{1}{a\bar{\mu}} = 2 + a\bar{\mu} + \frac{1}{a\bar{\mu}}.\end{aligned}$$

That is (v) holds for $\mathbb{T} \neq \mathbb{R}$. □

Lemma 4.6. *Assume that $A \in \mathcal{R}(\mathbb{T}; \mathbb{R}^{n\times n})$ is almost periodic and*

$$\|e_A(t,s)\| \le Ce_{\ominus\alpha}(t,s), \quad t \ge s, \tag{8}$$

where C and α are positive real numbers. Let $M = (1 + \alpha\mathcal{K})C^2N$ with N the constant in Lemma 4.5 (iii), and for $\varepsilon > 0$,

$$\Upsilon(\varepsilon) = \{r \in \Pi : \|e_A(t + r, \sigma(s) + r) - e_A(t, \sigma(s))\| < \varepsilon, t, s \in \mathbb{T}, t \ge \sigma(s)\}.$$

Then $T(A, \varepsilon/M) \subset \Upsilon(\varepsilon)$, which implies that $\Upsilon(\varepsilon)$ is relatively dense in Π.

Proof. For $\varepsilon > 0$, let $r \in T(A, \varepsilon/M)$ and $U(t, \sigma(s)) := e_A(t + r, \sigma(s) + r) - e_A(t, \sigma(s))$. Differentiate U with respect to t and denote by $\frac{\partial_\Delta U}{\partial_\Delta t}$ the partial derivative, then

$$\begin{aligned}\frac{\partial_\Delta U}{\partial_\Delta t} &= A(t + r)e_A(t + r, s + r) - A(t)e_A(t, \sigma(s)) \\ &= A(t)U(t, \sigma(s)) + (A(t + r) - A(t))e_A(t + r, \sigma(s) + r).\end{aligned}$$

Note that $U(\sigma(s), \sigma(s)) = 0$, then by the variation of constants formula ([2, Theorem 5.24]),

$$U(t, \sigma(s)) = \int_{\sigma(s)}^{t} e_A(t, \sigma(\tau))(A(\tau + r) - A(\tau))e_A(\tau + r, \sigma(s) + r)\Delta\tau.$$

Therefore, by (8), Lemma 4.4, 4.5 and the fact that $\mu(\tau) \le \mathcal{K}$, $\tau \in \mathbb{T}$, for $t, s \in \mathbb{T}$ with $t \ge \sigma(s)$,

$$\|U(t, \sigma(s))\| \le \int_{\sigma(s)}^{t} \|e_A(t, \sigma(\tau))\|\|(A(\tau + r) - A(\tau))\|\|e_A(\tau + r, \sigma(s) + r)\|\Delta\tau$$

$$
\begin{aligned}
&\leq \frac{\varepsilon}{M} C^2 \int_{\sigma(s)}^{t} e_{\ominus\alpha}(t, \sigma(\tau)) e_{\ominus\alpha}(\tau + r, \sigma(s) + r) \Delta\tau \\
&= \frac{\varepsilon}{M} C^2 e_{\ominus\alpha}(t, \sigma(s)) \int_{\sigma(s)}^{t} e_{\ominus\alpha}(\tau, \sigma(\tau)) \Delta\tau \\
&= \frac{\varepsilon}{M} C^2 e_{\ominus\alpha}(t, \sigma(s)) \int_{\sigma(s)}^{t} (1 + \alpha\mu(\tau)) \Delta\tau \\
&\leq \frac{\varepsilon}{M} C^2 (1 + \alpha\mathcal{K})(t - \sigma(s)) e_{\ominus\alpha}(t, \sigma(s)) \\
&\leq \frac{\varepsilon}{M} C^2 (1 + \alpha\mathcal{K}) N = \varepsilon.
\end{aligned}
$$

This implies that $T(A, \varepsilon/M) \subset \Upsilon(\varepsilon)$, and $\Upsilon(\varepsilon)$ is relatively dense in Π. □

4.2 Dynamic equations with delay

As an application of the results obtained in the above sections, we consider the following nonlinear dynamic equation with delay:

$$x^{\Delta}(t) = A(t)x(t) + f(t, x(t - \omega)), \quad t \in \mathbb{T}, \tag{9}$$

where $A(t)$ is an $n \times n$ almost periodic matrix function, $\omega \in \Pi$, $\omega > 0$ and $f \in S^p PAP(\mathbb{T} \times \mathbb{E}^n; \mathbb{E}^n) \cap C(\mathbb{T} \times \mathbb{E}^n; \mathbb{E}^n)$.

To consider (9), we first consider its corresponding linear equation:

$$x^{\Delta}(t) = A(t)x(t) + f(t), \quad t \in \mathbb{T}, \tag{10}$$

where $f = g + \phi \in S^p PAP(\mathbb{T}; \mathbb{E}^n) \cap C(\mathbb{T}; \mathbb{E}^n)$.

Lemma 4.7. *Assume that $A \in \mathcal{R}(\mathbb{T}; \mathbb{R}^{n \times n})$ with (8) satisfied. Then (10) admits a unique bounded continuous solution $u(t)$ given by*

$$u(t) = \int_{-\infty}^{t} e_A(t, \sigma(s)) f(s) \Delta s, \quad t \in \mathbb{T}. \tag{11}$$

Proof. For $t \in \mathbb{T}$, $j \geq 1$, by Hölder inequality,

$$\int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} |f(s)| \Delta s \leq \mathcal{K}^{1/q} \left(\int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} |f(s)|^p \Delta s \right)^{1/p} = \mathcal{K} N(f)(t - j\mathcal{K}) \leq \mathcal{K} \|f\|_{S^p}. \tag{12}$$

Then by (8), (11) and Lemma 4.5 (iv), (v), for $t \in \mathbb{T}$,

$$
\begin{aligned}
|u(t)| &\leq \sum_{j=1}^{\infty} \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} \|e_A(t, \sigma(s))\| |f(s)| \Delta s \\
&\leq C \sum_{j=1}^{\infty} \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_{\ominus\alpha}(t, \sigma(s)) |f(s)| \Delta s \\
&\leq C \sum_{j=1}^{\infty} e_{\ominus\alpha}(t, \sigma(t) - (j-1)\mathcal{K}) \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} |f(s)| \Delta s \\
&\leq C \lambda_{\alpha} \mathcal{K} \|f\|_{S^p}.
\end{aligned}
$$

Thus u is well defined and bounded continuous. Moreover, by Lemma 4.4, fix $t_0 \in \mathbb{T}$,

$$u(t) = \int_{-\infty}^{t} e_A(t, \sigma(s)) f(s) \Delta s = e_A(t, t_0) \int_{-\infty}^{t} e_A(t_0, \sigma(s)) f(s) \Delta s.$$

Then by Lemma 4.4 and [2, Theorem 5.3 (iii)],

$$u^{\Delta}(t) = A(t)e_A(t,t_0)\int_{-\infty}^{t} e_A(t_0,\sigma(s))f(s)\Delta s + e_A(\sigma(t),t_0)e_A(t_0,\sigma(t))f(t) = A(t)u(t) + f(t),$$

which implies that u is a solution of (10). Assume that $v : \mathbb{T} \to \mathbb{E}^n$ is another bounded solution of (10). For $r \in \mathbb{T}$, by the variation of constants formula ([2, Theorem 5.24]),

$$v(t) = e_A(t,r)v(r) + \int_{r}^{t} e_A(t,\sigma(s))f(s)\Delta s, \quad t \in \mathbb{T}.$$

Since v is bounded, (8) implies that $e_A(t,r)v(r) \to 0$ as $r \to -\infty$. Letting $r \to -\infty$,

$$v(t) = \int_{-\infty}^{t} e_A(t,\sigma(s))f(s)\Delta s = u(t).$$

That is the bounded solution of (10) is unique. □

Theorem 4.8. *Assume that $A \in \mathcal{R}(\mathbb{T};\mathbb{R}^{n\times n})$ is almost periodic and* (8) *holds. Then* (10) *admits a unique pseudo almost periodic solution $u(t)$ given by* (11).

Proof. By Lemma 4.7, it suffices to prove that $u \in PAP(\mathbb{T};\mathbb{E}^n)$. In fact, for $t \in \mathbb{T}$, let

$$u(t) = \int_{-\infty}^{t} e_A(t,\sigma(s))f(s)\Delta s = \sum_{j=1}^{\infty} u_j(t),$$

where

$$\begin{aligned} u_j(t) &= \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_A(t,\sigma(s))f(s)\Delta s \\ &= \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_A(t,\sigma(s))g(s)\Delta s + \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_A(t,\sigma(s))\phi(s)\Delta s \\ &:= \varphi_j(t) + \psi_j(t), \quad j \in \mathbb{N}. \end{aligned}$$

For $\varepsilon > 0$, it follows from [22, Lemma 4.9] that $T\left(A, \frac{\varepsilon}{2M\mathcal{K}(1+\|g\|_{S^p})}\right) \cap T\left(g, \frac{\varepsilon}{2C\mathcal{K}}\right)$ is relatively dense in Π. Denote

$$\mathcal{G}_1 = \Upsilon\left(\frac{\varepsilon}{2\mathcal{K}(1+\|g\|_{S^p})}\right) \cap T\left(g, \frac{\varepsilon}{2C\mathcal{K}}\right),$$

where Υ the one given in Lemma 4.6. Then $\mathcal{G}_1$ is relatively dense in Π by Lemma 4.6. Let $\tau \in \mathcal{G}_1$, $t, s \in \mathbb{T}$ with $t \geq \sigma(s)$,

$$\begin{aligned} &|e_A(t+\tau,\sigma(s+\tau))g(s+\tau) - e_A(t,\sigma(s))g(s)| \\ &\leq \|e_A(t+\tau,\sigma(s)+\tau) - e_A(t,\sigma(s))\||g(s+\tau)| + \|e_A(t,\sigma(s))\||g(s+\tau)-g(s)| \\ &\leq \frac{\varepsilon|g(s+\tau)|}{2\mathcal{K}(1+\|g\|_{S^p})} + Ce_{\ominus\alpha}(t,\sigma(s))|g(s+\tau)-g(s)| \\ &\leq \frac{\varepsilon|g(s+\tau)|}{2\mathcal{K}(1+\|g\|_{S^p})} + C|g(s+\tau)-g(s)|. \end{aligned}$$

Now by the same calculation of (12), we can get for $j \in \mathbb{N}$,

$$|\varphi_j(t+\tau) - \varphi_j(t)|$$

$$
\begin{aligned}
&= \left| \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} (e_A(t+\tau,\sigma(s+\tau))g(s+\tau) - e_A(t,\sigma(s))g(s))\,\Delta s \right| \\
&\le \frac{\varepsilon}{2\mathcal{K}(1+\|g\|_{S^p})} \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} |g(s+\tau)|\Delta s + C \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} |g(s+\tau)-g(s)|\Delta s \\
&\le \frac{\varepsilon\mathcal{K}\|g\|_{S^p}}{2\mathcal{K}(1+\|g\|_{S^p})} + C\mathcal{K}\|g(\cdot+\tau)-g\|_{S^p} \\
&< \frac{\varepsilon}{2} + C\mathcal{K}\frac{\varepsilon}{2C\mathcal{K}} = \varepsilon.
\end{aligned}
$$

This implies that $\mathcal{G}_1 \subset T(\varphi_j,\varepsilon)$. Then $T(\varphi_j,\varepsilon)$ is relatively dense in Π and φ_j is almost periodic for $j \in \mathbb{N}$.

Meanwhile, by Lemma 4.5 (iv), for $j \in \mathbb{N}$ and $t \in \mathbb{T}$,

$$
e_{\ominus\alpha}(t,\sigma(t)-(j-1)\mathcal{K}) \le e_{\ominus\alpha}(t,\sigma(t)) = 1+\alpha\mu(t) \le 1+\alpha\bar{\mu}.
$$

Then by (8) and the same calculation of (12),

$$
\begin{aligned}
|\psi_j(t)| &\le \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} \|e_A(t,\sigma(s))\||\phi(s)|\Delta s \\
&\le C \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_{\ominus\alpha}(t,\sigma(s))|\phi(s)|\Delta s \\
&\le C e_{\ominus\alpha}(t,\sigma(t)-(j-1)\mathcal{K}) \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} |\phi(s)|\Delta s \\
&\le C(1+\alpha\bar{\mu})\mathcal{K}\mathcal{N}(\phi)(t-j\mathcal{K}),
\end{aligned}
$$

which implies that $\psi_j \in BC(\mathbb{T};\mathbb{E}^n)$. Notice that $\phi \in S^pPAP_0(\mathbb{T};\mathbb{E}^n)$. Thus for a fixed $t_0 \in \mathbb{T}$,

$$
\lim_{r\to+\infty} \frac{1}{2r}\int_{t_0-r}^{t_0+r} |\psi_j(t)|\Delta t \le C(1+\alpha\bar{\mu})\mathcal{K} \lim_{r\to+\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r} \mathcal{N}(\phi)(t-j\mathcal{K})\Delta t = 0.
$$

Hence $\psi_j \in PAP_0(\mathbb{T};\mathbb{E}^n)$ and $u_j = \varphi_j + \psi_j \in PAP(\mathbb{T};\mathbb{E}^n)$. Consequently, $u \in PAP(\mathbb{T};\mathbb{E}^n)$. □

Remark 4.9. *When $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$ for all $t \in \mathbb{T}$ and hence $A \in \mathcal{R}$ automatically. Then Theorem 4.8 is an extension of [19, Theorem 3.2] to time scales.*

For nonlinear dynamics equation (9), we have the following result.

Theorem 4.10. *Assume that $A \in \mathcal{R}(\mathbb{T};\mathbb{R}^{n\times n})$ with (8) satisfied, and $f = g + \phi \in S^pPAP(\mathbb{T}\times\mathbb{E}^n;\mathbb{E}^n) \cap C(\mathbb{T}\times\mathbb{E}^n;\mathbb{E}^n)$ with f and g satisfying* (H) *with Lipschitz constants L_f and L_g, respectively. Then* (9) *has a unique pseudo almost periodic solution u satisfying*

$$
u(t) = \int_{-\infty}^{t} e_A(t,\sigma(s))f(s,u(s-\omega))\Delta s, \quad t \in \mathbb{T}, \tag{13}
$$

provided that $C\mathcal{K}L_f\lambda_\alpha < 1$, where λ_α is as in Lemma 4.5.

Proof. Let $\varphi \in PAP(\mathbb{T};\mathbb{E}^n)$. It follows from Proposition 2.6 (i) and Corollary 3.15 that $f(\cdot,\varphi(\cdot-\omega)) \in S^pPAP(\mathbb{T};\mathbb{E}^n)$. Let

$$
T(\varphi)(t) := \int_{-\infty}^{t} e_A(t,\sigma(s))f(s,\varphi(s-\omega))\Delta s, \quad t \in \mathbb{T}.
$$

Then $T(\varphi) \in PAP(\mathbb{T};\mathbb{E}^n)$ by Theorem 4.8. That is $T : PAP(\mathbb{T};\mathbb{E}^n) \to PAP(\mathbb{T};\mathbb{E}^n)$. By (H), (8), Lemma 4.5 and the same calculation of (12), for $\varphi, \theta \in PAP(\mathbb{T};\mathbb{E}^n)$, $t \in \mathbb{T}$,

$$
\begin{aligned}
|T(\varphi)(t) - T(\theta)(t)| &\le \int_{-\infty}^{t} \|e_A(t,\sigma(s))\| |f(s,\varphi(s-\omega)) - f(s,\theta(s-w))| \Delta s \\
&\le C \sum_{j=1}^{\infty} \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_{\ominus\alpha}(t,\sigma(s)) |f(s,\varphi(s-\omega)) - f(s,\theta(s-\omega))| \Delta s \\
&\le C \sum_{j=1}^{\infty} e_{\ominus\alpha}(t,\sigma(t)-(j-1)\mathcal{K}) \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} |f(s,\varphi(s-\omega)) - f(s,\theta(s-\omega))| \Delta s \\
&\le C\mathcal{K} \sum_{j=1}^{\infty} e_{\ominus\alpha}(t,\sigma(t)-(j-1)\mathcal{K}) \mathcal{N}(f(\cdot,\varphi(\cdot-\omega)) - f(\cdot,\theta(\cdot-\omega)))(t-j\mathcal{K}) \\
&\le C\mathcal{K}L_f \sum_{j=1}^{\infty} e_{\ominus\alpha}(t,\sigma(t)-(j-1)\mathcal{K}) \mathcal{N}(\varphi(\cdot-\omega) - \theta(\cdot-\omega))(t-j\mathcal{K}) \\
&\le C\mathcal{K}L_f\lambda_\alpha \|\varphi - \theta\|_\infty .
\end{aligned}
$$

This implies that

$$\|T(\varphi) - T(\theta)\|_\infty \le C\mathcal{K}L_f\lambda_\alpha \|\varphi - \theta\|_\infty .$$

Thus T is a contraction operator since $C\mathcal{K}L_f\lambda_\alpha < 1$, and then T has a unique fixed point $u \in PAP(\mathbb{T};\mathbb{E}^n)$. This means that (9) has a unique pseudo almost periodic solution u satisfying (13). □

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