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Domination in 4-regular Knödel graphs

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Abstract: A subset D of vertices of a graph G is a *dominating set* if for each $u \in V(G) \setminus D$, u is adjacent to some vertex $v \in D$. The *domination number*, $\gamma(G)$ of G, is the minimum cardinality of a dominating set of G. For an even integer $n \geq 2$ and $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$, a *Knödel graph* $W_{\Delta,n}$ is a Δ-regular bipartite graph of even order n, with vertices (i, j), for $i = 1, 2$ and $0 \leq j \leq n/2 - 1$, where for every j, $0 \leq j \leq n/2 - 1$, there is an edge between vertex $(1, j)$ and every vertex $(2, (j + 2^k - 1) \bmod (n/2))$, for $k = 0, 1, \cdots, \Delta - 1$. In this paper, we determine the domination number in 4-regular Knödel graphs $W_{4,n}$.

Keywords: Knödel graph, Domination number, Pigoenhole Principal

MSC: 05C69, 05C30

1 Introduction

For graph theory notation and terminology not given here, we refer to [1]. Let $G = (V, E)$ denote a simple graph of order $n = |V(G)|$ and size $m = |E(G)|$. Two vertices $u, v \in V(G)$ are *adjacent* if $uv \in E(G)$. The *open neighborhood* of a vertex $u \in V(G)$ is denoted by $N(u) = \{v \in V(G) | uv \in E(G)\}$ and for a vertex set $S \subseteq V(G)$, $N(S) = \underset{u \in S}{\cup} N(u)$. The cardinality of $N(u)$ is called the *degree* of u and is denoted by $\deg(u)$, (or $\deg_G(u)$ to refer it to G). The *closed neighborhood* of a vertex $u \in V(G)$ is denoted by $N[u] = N(u) \cup \{u\}$ and for a vertex set $S \subseteq V(G)$, $N[S] = \underset{u \in S}{\cup} N[u]$. The *maximum degree* and *minimum degree* among all vertices in G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A graph G is a *bipartite graph* if its vertex set can be partitioned to two disjoint sets X and Y such that each edge in $E(G)$ connects a vertex in X with a vertex in Y. A set $D \subseteq V(G)$ is a *dominating set* if for each $u \in V(G) \setminus D$, u is adjacent to some vertex $v \in D$. The *domination number*, $\gamma(G)$ of G, is the minimum cardinality of a dominating set of G. The concept of domination theory is a widely studied concept in graph theory and for a comprehensive study see, for example [1].

An interesting family of graphs namely *Knödel graphs* was introduced about 1975 [2]. On a one-page note, Walter Knödel introduced a special graph as a minimum gossip graph [2]. For an even integer $n \geq 2$, the graph KG_n is a regular bipartite graph with degree $\lfloor \log_2 n \rfloor$. Each vertex $2j + 1$ is adjacent to vertices $2j + 2^r$, where $j = 0, 1, 2, \cdots, n/2 - 1$ and $r = 1, 2, \cdots, \lfloor \log_2 n \rfloor$. The graphs KG_n are called *modified Knödel graphs* in the literature. In 1995, Bermond et al. presented some methods for constructing new broadcast graphs. Their constructions are based on graph compounding operation. For example, the modified Knödel graph KG_{2n} is the compound of KG_n and K_2 [3]. In 1997, Bermond et al. showed that the edges of the modified Knödel graph can be grouped into dimensions which are similar to the dimensions of hypercubes. In particular, routing, broadcasting and gossiping, can be done easily in modified Knödel graphs using these dimensions

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[4]. The general definition of generalized Knödel Graphs were introduced in 2001 [5]. Since then, they have been widely studied by some authors.

Fraigniaud and Peters formally defined the generalized family of Knödel graphs [5].

Definition 1.1 ([5]). *For an even integer $n \geq 2$ and $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$, a Knödel graph $W_{\Delta,n}$ is a Δ-regular bipartite graph of even order n, with vertices (i, j), for $i = 1, 2$ and $0 \leq j \leq n/2 - 1$, where for every j, $0 \leq j \leq n/2 - 1$, there is an edge between vertex $(1, j)$ and each vertex $(2, (j + 2^k - 1) \bmod (n/2))$, for $k = 0, 1, \cdots, \Delta - 1$.*

Knödel graphs, $W_{\Delta,n}$, are one of the three important families of graphs that have nice properties in terms of broadcasting and gossiping. There exist many important papers presenting graph-theoretic and communication properties of the Knödel graphs, see for example [4, 6–9]. It is worth noting that any Knödel graph is a Cayley graph and so it is a vertex-transitive graph [10].

Xueliang et al. [11] studied the domination number in 3-regular Knödel graphs $W_{3,n}$. They obtained exact domination number for $W_{3,n}$. Mojdeh et al. [14] determined the total domination number in 3-regular Knödel graphs $W_{3,n}$. In this paper, we determine the domination number in 4-regular Knödel graphs $W_{4,n}$. The following is useful.

Theorem 1.2 ([12, 13]). *For any graph G of order n with maximum degree $\Delta(G)$, $\frac{n}{1+\Delta(G)} \leq \gamma(G) \leq n - \Delta(G)$.*

We need also the following simple observation from number theory.

Observation 1.3. *If a, b, c, d and x are positive integers such that $x^a - x^b = x^c - x^d \neq 0$, then $a = c$ and $b = d$.*

2 Properties in the Knödel graphs

In this section we review some properties in the Knödel graphs that are proved in [14]. Mojdeh et al. considered a re-labeling on the vertices of a Knödel graph as follows: we label $(1, i)$ by u_{i+1} for each $i = 0, 1, \ldots, n/2 - 1$, and $(2, j)$ by v_{j+1} for $j = 0, 1, \ldots, n/2 - 1$. Let $U = \{u_1, u_2, \cdots, u_{\frac{n}{2}}\}$ and $V = \{v_1, v_2, \cdots, v_{\frac{n}{2}}\}$. From now on, the vertex set of each Knödel graph $W_{\Delta,n}$ is $U \cup V$ such that U and V are the two partite sets of the graph. If S is a set of vertices of $W_{\Delta,n}$, then clearly, $S \cap U$ and $S \cap V$ partition S, $|S| = |S \cap U| + |S \cap V|$, $N(S \cap U) \subseteq V$ and $N(S \cap V) \subseteq U$. Note that two vertices u_i and v_j are adjacent if and only if $j \in \{i+2^0-1, i+2^1-1, \cdots, i+2^{\Delta-1}-1\}$, where the addition is taken modulo $n/2$. For any subset $\{u_{i_1}, u_{i_2}, \cdots, u_{i_k}\}$ of U with $1 \leq i_1 < i_2 < \cdots < i_k \leq \frac{n}{2}$, by the indices i_js of the elements of A, we want to correspond a sequence to the set A.

Definition 2.1. *For any subset $A = \{u_{i_1}, u_{i_2}, \cdots, u_{i_k}\}$ of U with $1 \leq i_1 < i_2 < \cdots < i_k \leq \frac{n}{2}$ we define a sequence $n_1, n_2, \cdots, n_k$, called a **cyclic sequence**, where $n_j = i_{j+1} - i_j$ for $1 \leq j \leq k-1$ and $n_k = \frac{n}{2} + i_1 - i_k$. For two vertices $u_{i_j}, u_{i_{j'}} \in A$, we define **index-distance** of u_{i_j} and $u_{i_{j'}}$ by $id(u_{i_j}, u_{i_{j'}}) = min\{|i_j - i_{j'}|, \frac{n}{2} - |i_j - i_{j'}|\}$.*

Observation 2.2. *Let $A = \{u_{i_1}, u_{i_2}, \cdots, u_{i_k}\} \subseteq U$ be a set such that $1 \leq i_1 < i_2 < \cdots < i_k \leq \frac{n}{2}$ and let $n_1, n_2, \cdots, n_k$ be the corresponding cyclic-sequence of A. Then,*
(1) $n_1 + n_2 + \cdots + n_k = \frac{n}{2}$.
(2) If $u_{i_j}, u_{i_{j'}} \in A$, then $id(u_{i_j}, u_{i_{j'}})$ equals sum of some consecutive elements of the cyclic-sequence of A and $\frac{n}{2} - id(u_{i_j}, u_{i_{j'}})$ is the sum of the remaining elements of the cyclic-sequence. Furthermore, $\{id(u_{i_j}, u_{i_{j'}}), \frac{n}{2} - id(u_{i_j}, u_{i_{j'}})\} = \{|i_j - i_{j'}|, \frac{n}{2} - |i_j - i_{j'}|\}$.

Proof. (1) By definition of cyclic-sequence, we have $n_1 + n_2 + \cdots + n_k = (i_2 - i_1) + (i_3 - i_2) + \cdots + (i_k - i_{k-1}) + (\frac{n}{2} + i_1 - i_k) = \frac{n}{2}$ as desired.

(2) By vertex transitivity of Knödel graphs, without loss of generality, we assume that $j' = 1$. Since $1 = j' < j \leq k$, $|i_j - i_{j'}| = i_j - i_1 = (i_j - i_{j-1}) + (i_{j-1} - i_{j-2}) + \cdots + (i_2 - i_1)$. So, we have $|i_j - i_1| = n_{j-1} + n_{j-2} + \cdots + n_1$ and by (1), we have $\frac{n}{2} - |i_j - i_1| = n_j + n_{j+1} + \cdots + n_k$. This shows that both of $|i_j - i_1|$ and $\frac{n}{2} - |i_j - i_1|$ and consequently $id(u_{i_j}, u_{i_1})$

are the sum of some consequent elements of cyclic-sequence. The proof of $\{id(u_{i_j}, u_{i_{j'}}), \frac{n}{2} - id(u_{i_j}, u_{i_{j'}})\} = \{|i_j - i_{j'}|, \frac{n}{2} - |i_j - i_{j'}|\}$ is straightforward. □

Let $\mathcal{M}_\Delta = \{2^a - 2^b : 0 \le b < a < \Delta\}$ for $\Delta \ge 2$.

Lemma 2.3. *In the Knödel graph $W_{\Delta,n}$ with vertex set $U \cup V$, for every $i \ne j$ and $1 \le i, j \le n/2$, $N(u_i) \cap N(u_j) \ne \varnothing$ if and only if $id(u_i, u_j) \in \mathcal{M}_\Delta$ or $\frac{n}{2} - id(u_i, u_j) \in \mathcal{M}_\Delta$.*

Proof. Without loss of generality, assume that $i > j$.
First, suppose that $v_t \in N(u_i) \cap N(u_j)$. Therefore, there exist two integers $0 \le a, b \le \Delta - 1$ such that $t \equiv i + 2^a - 1 \equiv j + 2^b - 1 \pmod{n/2}$. Hence, $(i - j) - (2^b - 2^a) \equiv 0 \pmod{n/2}$. Since $0 < i - j < n/2$ and $-n/2 < 2^b - 2^a < n/2$, we have only two different cases: (*) $(i - j) - (2^b - 2^a) = 0$ and so $|i - j| = 2^b - 2^a \in \mathcal{M}_\Delta$ and (**) $(i - j) - (2^b - 2^a) = n/2$ and so $n/2 - |i - j| = 2^a - 2^b \in \mathcal{M}_\Delta$. Now, by Observation 2.2(2), we have $id(u_i, u_j) \in \mathcal{M}_\Delta$ or $\frac{n}{2} - id(u_i, u_j) \in \mathcal{M}_\Delta$ as desired.

To show the if part, suppose that $id(u_i, u_j) \in \mathcal{M}_\Delta$ or $\frac{n}{2} - id(u_i, u_j) \in \mathcal{M}_\Delta$. By Observation 2.2(2), we have $|i - j| = 2^a - 2^b \in \mathcal{M}_\Delta$ or $n/2 - |i - j| = 2^a - 2^b \in \mathcal{M}_\Delta$ for some integers $0 \le b < a \le \Delta - 1$. In the first case, $i + 2^b - 1 = j + 2^a - 1$ and so $v_t \in N(u_i) \cap N(u_j)$, where $t \equiv i + 2^b - 1 = j + 2^a - 1 \pmod{n/2}$. In the second case, $i + 2^a - 1 = n/2 + j + 2^b - 1$ and so $v_t \in N(u_i) \cap N(u_j)$, where $t \equiv i + 2^a - 1 = n/2 + j + 2^b - 1 \pmod{n/2}$. In each case we have $N(u_i) \cap N(u_j) \ne \varnothing$ and proof is completed. □

3 4-regular Knödel graphs

In this section we determine the domination number in 4-regular Knödel graphs $W_{4,n}$. Note that $n \ge 16$ by the definition. For this purpose, we prove the following lemmas namely Lemma 3.1, 3.2, 3.3, 3.4 and 3.5.

Lemma 3.1. *For each even integer $n \ge 16$, we have* $\gamma(W_{4,n}) = 2\lfloor \frac{n}{10} \rfloor + \begin{cases} 0 & n \equiv 0 \pmod{10} \\ 2 & n \equiv 2, 4 \pmod{10} \end{cases}$.

Proof. First assume that $n \equiv 0 \pmod{10}$. Let $n = 10t$, where $t \ge 2$. By Theorem 1.2, $\gamma(W_{4,n}) \ge \frac{n}{5} = 2t$. On the other hand, we can see that the set $D = \{u_1, u_6, \cdots, u_{5t-4}\} \cup \{v_5, v_{10}, \cdots, v_{5t}\}$ is a dominating set with $2t$ elements, and we have $\gamma(W_{4,n}) = 2t = 2\lfloor \frac{n}{10} \rfloor$, as desired.

Next assume that $n \equiv 2 \pmod{10}$. Let $n = 10t + 2$, where $t \ge 2$. By Theorem 1.2, we have $\gamma(W_{4,n}) \ge \frac{n}{5} > 2t$. Suppose that $\gamma(W_{4,n}) = 2t + 1$. Let D be a minimum dominating set of $W_{4,n}$. Then by the Pigeonhole Principal either $|D \cap U| \le t$ or $|D \cap V| \le t$. Without loss of generality, assume that $|D \cap U| \le t$. Let $|D \cap U| = t - a$, where $a \ge 0$. Then $|D \cap V| = t + 1 + a$. Observe that $D \cap U$ dominates at most $4t - 4a$ vertices of V and therefore D dominates at most $(4t - 4a) + (t + 1 + a) = 5t - 3a + 1$ vertices of V. Since D dominates all vertices of V, we have $5t - 3a + 1 \ge 5t + 1$ and so $a = 0$, $|D \cap U| = t$ and $|D \cap V| = t + 1$. Let $D \cap U = \{u_{i_1}, u_{i_2}, \cdots, u_{i_t}\}$ and $n_1, n_2, \cdots, n_t$ be the cyclic-sequence of $D \cap U$. By Observation 2.2, we have $\sum_{k=1}^{t} n_k = 5t + 1$ and, therefore, there exists some k such that $n_k \in \mathcal{M}_4 = \{1, 2, 3, 4, 6, 7\}$. Then by Lemma 2.3, $|N(u_k) \cap N(u_{k+1})| \ge 1$. Hence, $D \cap U$ dominates at most $4t - 1$ vertices of V, that is, D dominates at most $(4t - 1) + (t + 1) = 5t$ vertices of V, a contradiction. Now we deduce that $\gamma(W_{4,n}) \ge 2t + 2$. On the other hand the set $D = \{u_1, u_6, \cdots, u_{5t+1}\} \cup \{v_5, v_{10}, \cdots, v_{5t}\} \cup \{v_{5t+1}\}$ is a dominating set for $W_{4,n}$ with $2t + 2$ elements. Consequently, $\gamma(W_{4,n}) = 2t + 2$.

It remains to assume that $n \equiv 4 \pmod{10}$. Let $n = 10t + 4$, where $t \ge 2$. By Theorem 1.2, we have $\gamma(W_{4,n}) \ge \frac{n}{5} > 2t$. Suppose that $\gamma(W_{4,n}) = 2t + 1$. Let D be a minimum dominating set of $\gamma(W_{4,n})$. Then by the Pigeonhole Principal either $|D \cap U| \le t$ or $|D \cap V| \le t$. Without loss of generality, assume that $|D \cap U| = t - a$ and $a \ge 0$. Then $|D \cap V| = t + 1 + a$. Observe that $D \cap U$ dominates at most $4(t - a)$ elements of V and therefore D dominates at most $4(t - a) + (t + 1 + a) = 5t - 3a + 1$ vertices of V. Since D dominates all vertices of V, we have $5t - 3a + 1 \ge |V| = 5t + 2$ and so $-3a \ge 1$, a contradiction. Thus $\gamma(W_{4,n}) > 2t + 1$. On the other hand, the set $\{u_1, u_6, \cdots, u_{5t+1}\} \cup \{v_5, v_{10}, \cdots, v_{5t}\} \cup \{v_3\}$ is a dominating set with $2t + 2$ elements. Consequently, $\gamma(W_{4,n}) = 2t + 2$. □

Lemma 3.2. *For each even integer $n \geq 46$ with $n \equiv 6 \pmod{10}$, we have $\gamma(W_{4,n}) = 2\lfloor \frac{n}{10} \rfloor + 3$.*

Proof. Let $n = 10t+6$ and $t \geq 4$. By Theorem 1.2, we have $\gamma(W_{4,n}) \geq \frac{n}{5} > 2t+1$. The set $D = \{u_1, u_6, \cdots, u_{5t+1}\} \cup \{v_5, v_{10}, \cdots, v_{5t}\} \cup \{v_2, v_3\}$ is a dominating set with $2t+3$ elements. Thus, $2t+2 \leq \gamma(W_{4,n}) \leq 2t+3$. We show that $\gamma(W_{4,n}) = 2t+3$. Suppose to the contrary that $\gamma(W_{4,n}) = 2t+2$. Let D be a minimum dominating set of $W_{4,n}$. Then by the Pigeonhole Principal either $|D \cap U| \leq t+1$ or $|D \cap V| \leq t+1$. Without loss of generality, assume that $|D \cap U| = t+1-a$, where $a \geq 0$. Then $|D \cap V| = t+1+a$. Note that $D \cap U$ dominates at most $4t+4-4a$ vertices and therefore D dominates at most $(4t+4-4a)+(t+1+a) = 5t-3a+5$ vertices of V. Since D dominates all vertices of V, we have $5t-3a+5 \geq 5t+3$ and so $a = 0$ and $|D \cap U| = |D \cap V| = t+1$. Also we have $|V - D| = 4t+2 \leq |N(D \cap U)| \leq 4t+4$, and similarly, $|D - U| = 4t+2 \leq |N(D \cap V)| \leq 4t+4$.

Let $D \cap U = \{u_{i_1}, u_{i_2}, \cdots, u_{i_{t+1}}\}$ and $n_1, n_2, \cdots, n_{t+1}$ be the cyclic-sequence of $D \cap U$. By Observation 2.2, we have $\sum_{k=1}^{t+1} n_k = 5t+3$ and, therefore, there exists k' such that $n_{k'} < 5$. Then $n_{k'} \in \mathcal{M}_4$ and by Lemma 2.3, $|N(u_{i_{k'}}) \cap N(u_{i_{k'+1}})| \geq 1$. Hence, $D \cap U$ dominates at most $4t+3$ vertices from V and therefore $4t+2 \leq N(D \cap U) \leq 4t+3$.

If $|N(D \cap U)| = 4t+3$, then for each $k \neq k'$ we have $n_k \notin \mathcal{M}_4$. If there exists $k'' \neq k'$ such that $n_{k''} \geq 8$, then $5t+3 = \sum_{k=1}^{t+1} n_k \geq 1+8+5(t-1) = 5t+4$, a contradiction. By symmetry we have $n_1 = n_2 = \cdots = n_t = 5$, $n_{t+1} = 3$ and $D \cap U = \{u_1, u_6, \cdots, u_{5t+1}\}$. Observe that $D \cap U$ doesn't dominate vertices $v_3, v_5, v_{10}, \cdots, v_{5t}$ and so $\{v_3, v_5, v_{10}, \cdots, v_{5t}\} \subseteq D$. Thus $D = \{u_1, u_6, \cdots, u_{5t+1}, v_3, v_5, v_{10}, \cdots, v_{5t}\}$. But the vertices $u_4, u_5, u_{5t-2}, u_{5t+2}$ are not dominated by D, a contradiction.

Thus, $|N(D \cap U)| = 4t+2$. Then there exists precisely two pairs of vertices in $D \cap U$ with index-distances belonging to $\mathcal{M}_4$. If there exists an integer $1 \leq i' \leq t+1$ such that $n_{i'} + n_{i'+1} \in \mathcal{M}_4$, then $\min\{n_{i'}, n_{i'+1}\} \leq 3$. Then $\min\{n_{i'}, n_{i'+1}\} \in \mathcal{M}_4$ and $\max\{n_{i'}, n_{i'+1}\} \notin \mathcal{M}_4$. Now we have $\max\{n_{i'}, n_{i'+1}\} = 5$, $\min\{n_{i'}, n_{i'+1}\} \in \{1, 2\}$, and $n_i \notin \mathcal{M}_4$, for each $i \notin \{i', i'+1\}$. Now a simple calculation shows that the equality $5t+3 = \sum_{k=1}^{t+1} n_k$ does not hold. (Note that if each n_i is less than 8, then we have $\sum_{k=1}^{t+1} n_k \leq 2+5+5(t-1) = 5t+2$; otherwise we have $\sum_{k=1}^{t+1} n_k \geq 1+5+8+5(t-2) = 5t+4$.) Thus there exist exactly two indices j and k such that $n_j, n_k \in \mathcal{M}_4$ and $\{n_1+n_2, n_2+n_3, \cdots, n_t+n_{t+1}, n_{t+1}+n_1\} \cap \mathcal{M}_4 = \varnothing$. By this hypothesis, the only possible cases for the cyclic-sequence of $D \cap U$ are those demonstrated in Table 1.

Table 1. $n = 10t+6$

case	1	2	3	4	5	6	7	8	⋯
n_1	8	4	3	8	4	4	4	4	⋯
n_2	1	1	2	2	4	5	5	5	⋯
n_3	4	8	8	3	5	4	5	5	⋯
n_4	5	5	5	5	5	5	4	5	⋯
n_5	5	5	5	5	5	5	5	4	⋯
n_6	5	5	5	5	5	5	5	5	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
n_{t+1}	5	5	5	5	5	5	5	5	⋯

Note that each column of Table 1 shows the cyclic sequence of $D \cap U$. We show that each case is impossible. For this purpose, we show that the cyclic-sequence of $D \cap U$ posed in the column i, for $i \geq 1$ is impossible.

$i = 1$**).** If $n_1 = 8$, $n_2 = 1$, $n_3 = 4$, $n_4 = \cdots = n_{t+1} = 5$ and $D \cap U = \{u_1, u_9, u_{10}, u_{14}, u_{19}, \cdots, u_{5t-1}\}$, then $D \cap U$ does not dominate the vertices $v_5, v_6, v_7, v_{18}, v_{23}, v_{28}, \ldots, v_{5t+3}$.

Thus we have $D \cap V = \{v_5, v_6, v_7, v_{18}, v_{23}, v_{28}, \ldots, v_{5t+3}\}$. But D does not dominate three vertices u_8, u_{12}, u_{13}, a contradiction.

$i = 2$). If $n_1 = 4, n_2 = 1, n_3 = 8, n_4 = \cdots = n_{t+1} = 5$ and $D \cap U = \{u_1, u_5, u_6, u_{14}, u_{19}, \cdots, u_{5t-1}\}$. Then $D \cap U$ does not dominate the vertices $v_{10}, v_{11}, v_{16}, v_{18}, v_{23}, v_{28}, \ldots, v_{5t+3}$. Thus we have $D \cap V = \{v_{10}, v_{11}, v_{16}, v_{18}, v_{23}, v_{28}, \ldots, v_{5t+3}\}$. But D does not dominate three vertices u_2, u_{12}, u_{5t+1}, a contradiction.

$i \in \{3, 4\}$). As before, we obtain that $u_8 \notin D$. But $N(u_8) = \{v_8, v_9, v_{11}, v_{15}\} \subseteq N(D \cap U)$ and therefore $N(u_8) \cap D = \varnothing$. Hence, $N[u_8] \cap D = \varnothing$ and D does not dominate u_8, a contradiction.

$i \geq 5$). As before, we obtain that $u_3 \notin D$. But $N(u_3) = \{v_3, v_4, v_6, v_{10}\} \subseteq N(D \cap U)$ and therefore $N(u_3) \cap D = \varnothing$. Hence, $N[u_3] \cap D = \varnothing$ and D does not dominate u_3, a contradiction.

Consequently, $\gamma(W_{4,n}) = 2t + 3$, as desired. □

Lemma 3.2 determines the domination number of $W_{4,n}$ when $n \equiv 6 \pmod{10}$ and $n \geq 46$. The only values of n for $n \equiv 6 \pmod{10}$ are thus 16, 26 and 36. We study these cases in the following lemma.

Lemma 3.3. *For $n \in \{16, 26, 36\}$, we have:*

Table 2. $n = 16, 26, 36$

n	16	26	36
$\gamma(W_{4,n})$	4	7	8

Proof. For $n = 16$, by Theorem 1.2 we have $\gamma(W_{4,16}) \geq \frac{16}{5} > 3$. On the other hand, the set $D = \{u_1, u_2, v_6, v_7\}$ is a dominating set for $W_{4,16}$, and therefore $\gamma(W_{4,16}) = 4$.

For $n = 26$, by Theorem 1.2 we have $\gamma(W_{4,26}) \geq \frac{26}{5} > 5$. On the other hand, the set $D = \{u_1, u_4, u_9, u_{10}, v_1, v_2, v_6\}$ is a dominating set for $W_{4,26}$ and therefore $6 \leq \gamma(W_{4,26}) \leq 7$. We show that $\gamma(W_{4,26}) = 7$. Suppose, on the contrary, that $\gamma(W_{4,26}) = 6$. Let D be a minimum dominating set for $W_{4,26}$. Then by the Pigeonhole Principal either $|D \cap U| \leq 3$ or $|D \cap V| \leq 3$. If $|D \cap U| = 3 - a$, where $a \geq 0$, then $|D \cap V| = 3 + a$. Now, the elements of $D \cap U$ dominate at most $4(3 - a)$ elements of V and D dominates at most $4(3-a)+(3+a) = 15-3a$ vertices of V. Thus $15-3a \geq |V| = 13$, which implies $a = 0$ and $|D \cap U| = |D \cap V| = 3$. Let $|D \cap U| = \{u_1, u_i, u_j\}$, where $1 < i < j \leq 13$ and $n_1 = i - 1, n_2 = j - i, n_3 = 13 + 1 - j$. Since $n_1 + n_2 + n_3 = 13$, we have $\{n_1, n_2, n_3\} \cap \mathcal{M}_4 \neq \varnothing$.

If $\mathcal{M}_4$ includes at least three numbers of $n_1, n_2, n_3, n_1 + n_2, n_2 + n_3, n_3 + n_1$, then by Lemma 2.3, $D \cap U$ dominates at most $4 \times 3 - 3 = 9$ vertices of V and $|D \cap V| \geq 13 - 9 = 4$, a contradiction.

If $\mathcal{M}_4$ includes exactly one number of n_1, n_2, n_3, then we have, by symmetry, $n_1 = 3, n_2 = 5, n_3 = 5$ and $D \cap U = \{u_1, u_4, u_9\}$, $N(D \cap U) = \{v_1, v_2, v_3, v_4, v_5, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$. Then $\{v_6, v_{13}\} \subseteq D$. But $\{u_1, u_4, u_9, v_6, v_{13}\}$ does not dominate the vertices $\{u_2, u_7, u_8, u_{11}\}$ and we need at least 2 other vertices to dominate this four vertices, and hence $|D| \geq 7$, a contradiction.

Thus, we assume that $\mathcal{M}_4$ includes two numbers of n_1, n_2, n_3 and $\{n_1 + n_2, n_2 + n_3, n_3 + n_1\} \cap \mathcal{M}_4 = \varnothing$. We thus have five possibilities for the cyclic-sequence of $D \cap U$ that are demonstrated in Table 3. Note that each column of Table 3 shows the cyclic sequence of $D \cap U$. We show that each case is impossible. For this purpose, we show that the cyclic sequence of $D \cap U$ posed in the column i, for $i \geq 1$ is impossible.

Table 3. $n = 26$

case	1	2	3	4	5
n_1	1	1	2	2	4
n_2	4	8	3	8	4
n_3	8	4	8	3	5

$i = 1$**)** If $n_1 = 1, n_2 = 4, n_3 = 8$, then $D \cap U = \{u_1, u_2, u_6\}$ and $N(D \cap U) = \{v_1, v_2, \cdots, v_9, v_{13}\}$. Thus $\{v_{10}, v_{11}, v_{12}\} \subseteq D$ and $D = \{u_1, u_2, u_6, v_{10}, v_{11}, v_{12}\}$. But D does not dominate the vertex u_{13}, a contradiction.

$i = 2$**)** If $n_1 = 1, n_2 = 8, n_3 = 4$, then $D \cap U = \{u_1, u_2, u_{10}\}$ and $N(D \cap U) = \{v_1, v_2, \cdots, v_{11}, v_{13}\}$. Thus $\{v_6, v_7, v_{12}\} \subseteq D$ and $D = \{u_1, u_2, u_{10}, v_6, v_7, v_{12}\}$. But D does not dominate the vertex u_8, a contradiction.

$i = 3$**)** If $n_1 = 2, n_2 = 3, n_3 = 8$, then $D \cap U = \{u_1, u_3, u_6\}$, $N(D \cap U) = \{v_1, v_2, \cdots, v_{10}, v_{13}\}$.Thus $\{v_5, v_{11}, v_{12}\} \subseteq D$ and $D = \{u_1, u_3, u_6, v_5, v_{11}, v_{12}\}$. But D does not dominate the vertices u_7 and u_{13}.

$i = 4$**)** If $n_1 = 2, n_2 = 8, n_3 = 3$, then $D \cap U = \{u_1, u_3, u_{11}\}$ and $N(D \cap U) = \{v_1, v_2, \cdots, v_6, v_8, v_{10}, v_{11}, v_{12}\}$. Thus $\{v_7, v_9, v_{13}\} \subseteq D$ and $D = \{u_1, u_3, u_{11}, v_7, v_9, v_{13}\}$. But D does not dominate the vertex u_5, a contradiction.

$i = 5$**)** If $n_1 = 4, n_2 = 4, n_3 = 5$, then $D \cap U = \{u_1, u_5, u_9\}$ and $N(D \cap U) = \{v_1, v_2, \cdots, v_{10}, v_{12}\}$. Thus $\{v_7, v_{11}, v_{13}\} \subseteq D$ and $D = \{u_1, u_3, u_{11}, v_7, v_9, v_{13}\}$. But D does not dominate the vertices u_2 and u_3.

Consequently, $\gamma(W_{4,26}) = 7$.

We now consider the case $n = 36$. By Theorem 1.2, $\gamma(W_{4,36}) \geq \frac{36}{5} > 7$. On the other hand, the set $D = \{u_1, u_2, u_{10}, u_{11}, v_6, v_7, v_{15}, v_{16}\}$ is a dominating set for the graph $W_{4,36}$ and, therefore, $\gamma(W_{4,36}) = 8$. □

We now consider the case $n \equiv 8 \pmod{10}$. For $n = 18, 28$ and 38 we have the following lemma.

Lemma 3.4. *For $n \in \{18, 28, 38\}$, we have:*

Table 4. $n = 18, 28, 38$

n	18	28	38
$\gamma(W_{4,n})$	4	7	10

Proof. For $n = 18$, by Theorem 1.2 we have $\gamma(W_{4,18}) \geq \frac{18}{5} > 3$. But the set $D = \{u_1, u_2, v_6, v_7\}$ is a dominating set for $W_{4,18}$ and, therefore, $\gamma(W_{4,18}) = 4$.

For $n = 28$, by Theorem 1.2 we have $\gamma(W_{4,28}) \geq \frac{28}{5} > 5$ and the set $D = \{u_1, u_6, u_{11}, u_{13}, v_3, v_5, v_9\}$ is a dominating set for $W_{4,28}$ and, therefore, $6 \leq \gamma(W_{4,28}) \leq 7$. Suppose on the contradiction that $\gamma(W_{4,28}) = 6$ and D is a minimum dominating set for $\gamma(W_{4,28})$. By the Pigeonhole Principal, either $|D \cap U| \leq 3$ or $|D \cap V| \leq 3$. If $|D \cap U| = 3 - a$ and $a \geq 0$, then $|D \cap V| = 3 + a$. Now, the elements of $D \cap U$ dominate at most $4(3 - a)$ elements of V and D dominates at most $4(3 - a) + (3 + a) = 15 - 3a$ vertices of V. Thus $15 - 3a \geq |V| = 14$. Therefore, $a = 0$ and so $|D \cap U| = |D \cap V| = 3$. Let $|D \cap U| = \{u_1, u_i, u_j\}$ and $1 < i < j \leq 14$ and $n_1 = i - 1, n_2 = j - i, n_3 = 14 + 1 - j$. Since $n_1 + n_2 + n_3 = 14$, we have $\{n_1, n_2, n_3\} \cap \mathcal{M}_4 \neq \emptyset$. If $\mathcal{M}_4$ includes at least two numbers in $\{n_1, n_2, n_3, n_1 + n_2, n_2 + n_3, n_3 + n_1\}$, then $D \cap U$ dominates at most $4 \times 3 - 2 = 10$ vertices of V and $|D \cap V| \geq 14 - 10 = 4$, a contradiction.

The only remaining case is that $\mathcal{M}_4$ includes exactly one of the three numbers n_1, n_2 and n_3 and also $\{n_1 + n_2, n_2 + n_3, n_3 + n_1\} \cap \mathcal{M}_4 = \emptyset$. By symmetry we have $n_1 = 4, n_2 = 5, n_3 = 5$ and $D \cap U = \{u_1, u_5, u_{10}\}$, $N(D \cap U) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_{10}, v_{11}, v_{12}, v_{13}\}$ thus $\{v_7, v_9, v_{14}\} \subseteq D$ and $D = \{u_1, u_5, u_{10}, v_7, v_9, v_{14}\}$. But D does not dominate the vertex u_3. That is a contradiction and therefore $\gamma(W_{4,28}) = 7$. For $n = 38$, by Theorem 1.2 we have $\gamma(W_{4,38}) \geq \frac{38}{5} > 7$ and the set $D = \{u_1, u_6, u_{11}, u_{16}, u_{18}, v_3, v_5, v_{10}, v_{13}, v_{15}\}$ is a dominating set for $W_{4,38}$ and therefore $8 \leq \gamma(W_{4,38}) \leq 10$. Let $\gamma(W_{4,38}) < 10$ and D is a dominating set for $\gamma(W_{4,38})$ with $|D| = 9$. Then by the Pigeonhole Principal either $|D \cap U| \leq 4$ or $|D \cap V| \leq 4$. If $|D \cap U| = 4 - a$ and $a \geq 0$, then $|D \cap V| = 5 + a$. Now, the elements of $D \cap U$ dominate at most $4(4 - a)$ elements of V and D dominates at most $4(4 - a) + (5 + a) = 21 - 3a$ vertices of V. Thus $21 - 3a \geq |V| = 19$ that results $a = 0$ and we have $|D \cap U| = 4$ and $|D \cap V| = 5$. Let $|D \cap U| = \{u_1, u_i, u_j, u_k\}$, where $1 < i < j < k \leq 19$ and n_1, n_2, n_3, n_4 be the cyclic-sequence of $D \cap U$. Since $n_1 + n_2 + n_3 + n_4 = 19$, we have $\{n_1, n_2, n_3, n_4\} \cap \mathcal{M}_4 \neq \emptyset$. If $\mathcal{M}_4$ includes at least three numbers of $\{n_1, n_2, n_3, n_4, n_1 + n_2, n_2 + n_3, n_3 + n_4, n_4 + n_1\}$, then $D \cap U$ dominates at most $4 \times 4 - 3 = 13$ vertices of V and $|D \cap V| \geq 19 - 13 = 6$, a contradiction.

If we wish that $\mathcal{M}_4$ includes exactly one number out of n_1, n_2, n_3, n_4, we have three cases:

Table 5. $n = 38$ with one n_i in $\mathcal{M}_\triangle$

case	1	2	3
n_1	4	1	8
n_2	5	8	1
n_3	5	5	5
n_4	5	5	5

i=1) If $n_1 = 4, n_2 = 5, n_3 = 5, n_4 = 5$ and $D \cap U = \{u_1, u_5, u_{10}, u_{15}\}$, then $\{v_7, v_9, v_{14}, v_{19}\} \subseteq D$ but $\{u_1, u_5, u_{10}, u_{15}, v_7, v_9, v_{14}, v_{19}\}$ does not dominate the vertices u_3 and u_{17}. For dominating u_3 and u_{17}, we need two vertices and therefore $|D| \geq 10$, a contradiction.

i=2) If $n_1 = 1, n_2 = 8, n_3 = 5, n_4 = 5$ and $D \cap U = \{u_1, u_2, u_{10}, u_{15}\}$, then $\{v_6, v_7, v_{12}, v_{14}, v_{19}\} \subseteq D$ and $D = \{u_1, u_2, u_{10}, u_{15}, v_6, v_7, v_{12}, v_{14}, v_{19}\}$ but D does not dominate the vertices u_8 and u_{17}.

i=3) If $n_1 = 8, n_2 = 1, n_3 = 5, n_4 = 5$ and $D \cap U = \{u_1, u_9, u_{10}, u_{15}\}$, then $\{v_5, v_6, v_7, v_{14}, v_{19}\} \subseteq D$ and $D = \{u_1, u_9, u_{10}, u_{15}, v_5, v_6, v_7, v_{14}, v_{19}\}$ but D does not dominate the vertex u_8.

Now we consider the cases that $\mathcal{M}_4$ includes exactly two numbers of the cyclic-sequence n_1, n_2, n_3, n_4 and $\{n_1 + n_2, n_2 + n_3, n_3 + n_4, n_4 + n_1\} \cap \mathcal{M}_4 = \varnothing$. By symmetry we have ten cases:

Table 6. $n = 38$ with two n_i in $\mathcal{M}_\triangle$

case	1	2	3	4	5	6	7	8	9	10
n_1	1	2	4	9	3	9	8	3	3	3
n_2	9	8	1	1	2	2	3	6	5	5
n_3	1	1	9	4	9	3	5	5	6	5
n_4	8	8	5	5	5	5	3	5	5	6

i=1) If $n_1 = 1, n_2 = 9, n_3 = 1, n_4 = 8$ and $D \cap U = \{u_1, u_2, u_{11}, u_{12}\}$, then $\{v_6, v_7, v_{10}, v_{16}, v_{17}\} \subseteq D$ and $D = \{u_1, u_2, u_{11}, u_{12}, v_6, v_7, v_{10}, v_{16}, v_{17}\}$ but D does not dominate the vertex u_8.

i=2) If $n_1 = 2, n_2 = 8, n_3 = 1, n_4 = 8$ and $D \cap U = \{u_1, u_3, u_{11}, u_{12}\}$, then $\{v_5, v_7, v_9, v_{16}, v_{17}\} \subseteq D$ and $D = \{u_1, u_3, u_{11}, u_{12}, v_5, v_7, v_9, v_{16}, v_{17}\}$ but D does not dominate the vertex u_{18}.

i=3) If $n_1 = 4, n_2 = 1, n_3 = 9, n_4 = 5$ and $D \cap U = \{u_1, u_5, u_6, u_{15}\}$, then $\{v_{10}, v_{11}, v_{14}, v_{17}, v_{19}\} \subseteq D$ and $D = \{u_1, u_5, u_6, u_{15}, v_{10}, v_{11}, v_{14}, v_{17}, v_{19}\}$ but D does not dominate the vertex u_2.

i=4) If $n_1 = 9, n_2 = 1, n_3 = 4, n_4 = 5$ and $D \cap U = \{u_1, u_{10}, u_{11}, u_{15}\}$, then $\{v_5, v_6, v_7, v_9, v_{19}\} \subseteq D$ and $D = \{u_1, u_{10}, u_{11}, u_{15}, v_5, v_6, v_7, v_9, v_{19}\}$ but D does not dominate the vertices u_{13} and u_{14}.

i=5) If $n_1 = 3, n_2 = 2, n_3 = 9, n_4 = 5$ and $D \cap U = \{u_1, u_4, u_6, u_{15}\}$, then $\{v_{10}, v_{12}, v_{14}, v_{17}, v_{19}\} \subseteq D$ and $D = \{u_1, u_4, u_6, u_{15}, v_{10}, v_{12}, v_{14}, v_{17}, v_{19}\}$ but D does not dominate the vertices u_2 and u_8.

i=6) If $n_1 = 9, n_2 = 2, n_3 = 3, n_4 = 5$ and $D \cap U = \{u_1, u_{10}, u_{12}, u_{15}\}$, then $\{v_5, v_6, v_7, v_9, v_{14}\} \subseteq D$ and $D = \{u_1, u_{10}, u_{12}, u_{15}, v_5, v_6, v_7, v_9, v_{14}\}$ but D does not dominate the vertex u_{16}.

i=7) If $n_1 = 8, n_2 = 3, n_3 = 5, n_4 = 3$ and $D \cap U = \{u_1, u_9, u_{12}, u_{17}\}$, then $\{v_3, v_6, v_7, v_{11}, v_{14}\} \subseteq D$ and $D = \{u_1, u_9, u_{12}, u_{17}, v_3, v_6, v_7, v_{11}, v_{14}\}$ but D does not dominate the vertex u_{16}.

i=8) If $n_1 = 3, n_2 = 6, n_3 = 5, n_4 = 5$ and $D \cap U = \{u_1, u_4, u_{10}, u_{15}\}$, then $\{v_6, v_9, v_{12}, v_{14}, v_{19}\} \subseteq D$ and $D = \{u_1, u_4, u_{10}, u_{15}, v_6, v_9, v_{12}, v_{14}, v_{19}\}$ but D does not dominate the vertex u_{17}.

i=9) If $n_1 = 3, n_2 = 5, n_3 = 6, n_4 = 5$ and $D \cap U = \{u_1, u_4, u_9, u_{15}\}$, then $\{v_6, v_{13}, v_{14}, v_{17}, v_{19}\} \subseteq D$ and $D = \{u_1, u_4, u_9, u_{15}, v_6, v_{13}, v_{14}, v_{17}, v_{19}\}$ but D does not dominate the vertices u_2 and u_8.

i=10) If $n_1 = 3, n_2 = 5, n_3 = 5, n_4 = 6$ and $D \cap U = \{u_1, u_4, u_9, u_{14}\}$, then $\{v_3, v_6, v_{13}, v_{18}, v_{19}\} \subseteq D$ and $D = \{u_1, u_4, u_9, u_{14}, v_3, v_6, v_{13}, v_{18}, v_{19}\}$ but D does not dominate the vertices u_7 and u_8.

Hence, $W_{4,38}$ has not any dominating set with 9 vertices and $W_{4,38} = 10$ as desired. □

Now for $n \geq 48$ with $n \equiv 8 \pmod{10}$ we determine the domination number of $W_{4,n}$ as follows.

Lemma 3.5. *For each even integer $n \geq 48$, $n \equiv 8 \pmod{10}$, we have $\gamma(W_{4,n}) = 2\lfloor \frac{n}{10} \rfloor + 4$.*

Proof. Let $n = 10t + 8$, where $t \geq 4$. By Theorem 1.2, we have $\gamma(W_{4,n}) \geq \frac{n}{5} > 2t + 1$. The set $D = \{u_1, u_6, \cdots, u_{5t+1}\} \cup \{v_5, v_{10}, \cdots, v_{5t}\} \cup \{v_3, v_{5t-2}, v_{5t+3}\}$ is a dominating set with $2t + 4$ elements, and so, $2t + 2 \leq \gamma(W_{4,n}) \leq 2t + 4$. We show that $\gamma(W_{4,n}) = 2t + 4$.

First, assume that $\gamma(W_{4,n}) = 2t+2$. Let D be a minimum dominating set of $W_{4,n}$. Then by the Pigeonhole Principal either $|D\cap U| \leq t+1$ or $|D\cap V| \leq t+1$. Without loss of generality assume that $|D\cap U| = t+1-a$, where $a \geq 0$. Then $|D\cap V| = t+1+a$. Observe that $D\cap U$ dominates at most $4t+4-4a$ vertices of V, and therefore, D dominates at most $(4t+4-4a)+(t+1+a) = 5t-3a+5$ vertices of V. Since D dominates all vertices in V, we have $5t-3a+5 \geq 5t+4$ and so $a = 0$. Then $|D\cap U| = |D\cap V| = t+1$. Also we have $|V-D| = 4t+3 \leq |N(D\cap U)| \leq 4t+4$ and $|U-D| = 4t+3 \leq |N(D\cap V)| \leq 4t+4$. Let $D\cap U = \{u_{i_1}, u_{i_2}, \cdots, u_{i_{t+1}}\}$ and $n_1, n_2, \cdots, n_{t+1}$. By Observation 2.2, we have $\sum_{k=1}^{t+1} n_k = 5t+4$ and therefore there exist k' such that $n_{k'} \in \mathcal{M}_4$. By Lemma 2.3, $D\cap U$ dominates at most $4t + 3$ vertices from V. Then $|N(D \cap U)| = |N(D \cap V)| = 4t + 3$ and k' is unique. If there exists $1 \leq k'' \leq t + 1$ such that $n_{k''} \geq 8$, then $5t + 4 = \sum_{k=1}^{t+1} n_k \geq n_{k'} + n_{k''} + 5(t - 1) \geq 1 + 8 + 5(t - 1) = 5t + 4$ which implies $n_{k'} = 1$, $n_{k''} = 8$ and for each $k \notin \{k', k''\}$ we have $n_k = 5$. Now in each arrangement of the cyclic sequence of $D\cap U$, we have one adjacency between 1 and 5. Then we have two vertices in $D\cap U$ with index-distance equal to 6, a contradiction. Thus for $k \neq k'$ we have $n_k = 5$ and $n_{k'} = 4$. We have (by symmetry) $n_1 = n_2 = \cdots = n_t = 5$ and $n_{t+1} = 4$ and $D\cap U = \{u_1, u_6, \cdots, u_{5t+1}\}$. Now $D\cap U$ doesn't dominate the vertices $v_3, v_5, v_{10}, \cdots, v_{5t}$ and so $\{v_3, v_5, v_{10}, \cdots, v_{5t}\} \subseteq D$. Thus $D = \{u_1, u_6, \cdots, u_{5t+1}, v_3, v_5, v_{10}, \cdots, v_{5t}\}$, but D does not dominate two vertices u_{5t-2} and u_{5t+3}, a contradiction.

Now, assume that $\gamma(W_{4,n}) = 2t+3$. Let D be a minimum dominating set of $W_{4,n}$. Then by the Pigeonhole Principal either $|D \cap U| \leq t + 1$ or $|D \cap V| \leq t + 1$. Without loss of generality, suppose $|D \cap U| = t + 1 - a$, where $a \geq 0$. Then $|D\cap V| = t+2+a$. Observe that $D\cap U$ dominates at most $4t+4-4a$ vertices of V and, therefore, D dominates at most $(4t+4-4a)+(t+2+a) = 5t-3a+6$ vertices of V. Since D dominates all vertices in V, we have $5t-3a+6 \geq 5t+4$ and $a = 0$, $|D\cap U| = t+1$ and $|D\cap V| = t+2$. Also we have $4t+2 \leq |N(D\cap U)| \leq 4t+4$. Since $4t + 2 \leq |N(D \cap U)| \leq 4t + 4$, at most two elements of $n_1, n_2, \cdots, n_{t+1}$ can be in $\mathcal{M}_4$. If x is the number of 5's in the cyclic-sequence of $D \cap U$, then by Observation 2.2, we have $\sum_{k=1}^{t+1} n_k = 5t+4 \geq 1+1+8(t-x-1)+5x$ and, therefore, $3x \geq 3t - 10$ which implies $x \geq t - 3$. Thus $t - 3$ elements of the cyclic sequence are equal to 5. The sum of the remaining four values of the cyclic sequence is 19, and at most two of them are in $\mathcal{M}_4$. In the last case of Lemma 3.3, for $n = 38$, we identified all such cyclic sequences and placed them in two tables, Table 5 and Table 6. We now continue according to Table 5 and Table 6.

In the case (i=1) in Table 5 we have $n_1 = 4, n_2 = \cdots = n_{t+1} = 5$ and $D \cap U = \{u_1, u_5, u_{10}, \cdots, u_{5t}\}$. Thus $\{v_7, v_9, v_{14}, \cdots, v_{5t+4}\} \subseteq D$. But $\{u_1, u_5, u_{10}, \cdots, u_{5t}, v_7, v_9, v_{14}, \cdots, v_{5t+4}\}$ does not dominate the vertices u_3 and u_{5t+2}. For dominating u_3 and u_{5t+2}, we need two vertices and therefore $|D| \geq 2t + 4$, a contradiction. In the cases (i$\in \{2, 3\}$) in Table 5 we have to add 5's to the end of the cyclic sequence and construct the corresponding set D with $2t + 3$ elements. In both cases we obtain that $N[u_8] \cap D = \varnothing$. Then D is not a dominating set, a contradiction.

In the case (i=1) in Table 6 we cannot add a 5 to the cyclic sequence, since by adding a 5 to the cyclic-sequence we obtain two consecutive values of the cyclic sequence which one is 5 and the other is 1 and their sum is 6 which belongs to $\mathcal{M}_4$, a contradiction.

In the case (i=2) in Table 6 we cannot add a 5 to the cyclic sequence, since by adding a 5 to the cyclic sequence we obtain two consecutive values of the cyclic sequence which one is 5 and the other is 1 or 2, and their sum is 6 or 7, which belongs to $\mathcal{M}_4$, a contradiction.

In the cases (i$\in \{3, 4, 5, 6\}$) in Table 6 we have to add 5's to the end of the cyclic sequence and construct the corresponding set D with $2t + 3$ elements. In (i=3), we obtain that $N[u_2] \cap D = \varnothing$, in (i=4), we obtain that $N[\{u_{13}, u_{14}\}]\cap D = \varnothing$, in (i=5), we obtain that $N[\{u_2, u_8\}]\cap D = \varnothing$, and in (i=6), we obtain that $N[u_{16}]\cap D = \varnothing$. In all four cases, D is not a dominating set, a contradiction.

In the case (i=7) in Table 6, by adding 5's to the cyclic-sequence, we obtain some different new cyclic-sequences. We divide them into three categories.

c1) $n_1 = 8$, $n_2 = 3$ and $n_3 = 5$. In this category, the constructed set, D, does not dominate u_{16}, a contradiction.

c2) $n_1 = 8$, $n_2 = 5$, $n_3 = 3$ and $n_4 = 5$. In this category, the constructed set, D, does not dominate u_{15} a contradiction.

c3) $n_1 = 8$, $n_2 = n_3 = 5$ and if $n_i = n_j = 3$, then $|i - j| \geq 2$. In this category, the constructed set, D, does not dominate u_{5i+1}, a contradiction. (Notice that this category does not appear for $n \leq 5$.) In the cases (i$\in$ $\{8, 9, 10\}$) in Table 6, by adding 5's to the cyclic-sequences, we obtain some different new cyclic-sequences. we divide them into two categories.

c1) $n_1 = 3$ and $n_{t+1} = 5$. In this category, the constructed set, D, does not dominate u_2, a contradiction.

c2) $n_1 = 3$ and $n_{t+1} = 6$. In this category, the constructed set, D, does not dominate u_7 and u_8. Hence, D is not a dominating set, a contradiction.

Hence, $\gamma(W_{4,n}) = 2t + 4 = 2\lfloor \frac{n}{10} \rfloor + 4$ □

Now a consequence of Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 implies the following theorem which is the main result of this section.

Theorem 3.6. *For each integer $n \geq 16$, we have:*

$$\gamma(W_{4,n}) = 2\lfloor \frac{n}{10} \rfloor + \begin{cases} 0 & n \equiv 0 \pmod{10} \\ 2 & n = 16, 18, 36\,;\ n \equiv 2, 4 \pmod{10} \\ 3 & n = 28\,;\ n \equiv 6 \pmod{10},\ n \neq 16, 36 \\ 4 & n \equiv 8 \pmod{10},\ n \neq 18, 28 \end{cases}.$$

4 Conclusion and Suggestion

In this manuscript we studied the domination number of 4-regular Knödel graphs. The following are some open related problems.

Problem 1. *Obtain the domination number of k-regular Knödel graphs for $k \geq 5$.*

Problem 2. *Obtain the total domination number of k-regular Knödel graphs for $k \geq 4$.*

Problem 3. *Obtain the connected domination number of k-regular Knödel graphs for $k \geq 3$.*

Problem 4. *Obtain the independent domination number of k-regular Knödel graphs for $k \geq 3$.*

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