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The existence of solutions to certain type of nonlinear difference-differential equations

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Abstract: In this paper we study the entire solutions to a certain type of difference-differential equations. We also give an affirmative answer to the conjecture of Zhang et al. In addition, our results improve and complement earlier ones due to Yang-Laine, Latreuch, Liu-Lü et al. and references therein.

Keywords: Nevanlinna theory, Entire solution, Difference equation, Differential polynomial

MSC: 34M05, 30D35, 39A10, 39B32

1 Introduction and main results

In studying difference-differential equations in the complex plane $\mathbb{C}$, it is always an interesting and quite difficult problem to prove the existence or uniqueness of the entire or meromorphic solutions to a given difference-differential equation. There have been many studies and results obtained lately that relate to the existence or growth of the entire or meromorphic solutions of various types of difference or differential equations, see, e.g., [1–9] and references therein.

Herein let f denote a non-constant meromorphic function and we assume that the reader is familiar with the standard terminology and results of Nevanlinna theory such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$ and the counting function $N(r, f)$ (see, e.g., [10–12]). However, for the convenience of the reader, we shall repeat some notations needed below.

We call a meromorphic function $\alpha \neq 0, \infty$ a small function with respect to f , if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \rightarrow \infty$, possibly outside a set of r of finite linear measure. The order of f is

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r},$$

and the hyper-order $\rho_2(f)$ is defined as

$$\rho_2(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

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Definition 1.1. A difference polynomial, respectively, a difference-differential polynomial, in f is a finite sum of difference products of f and its shifts, respectively, of products of f , derivatives of f and of their shifts, with all the coefficients of these monomials being small functions of f .

Definition 1.2. Given a nonzero constant c , we define the difference operators by

$$\Delta_c f(z) = f(z+c) - f(z) \text{ and } \Delta_c^n f(z) = \Delta_c(\Delta_c^{n-1} f(z)) \ (n \geq 2).$$

For the sake of simplicity, we let $\Delta f(z) = f(z+1) - f(z)$ and $\Delta^n f(z) = \Delta(\Delta^{n-1} f(z)) \ (n \geq 2)$ for the case $c = 1$ (see, e. g., [2, 13] and [14]).

For the benefit of the readers, we shall give some related results. Yang and Laine considered the following difference equation and proved:

Theorem A ([7]). A nonlinear difference equation

$$f^3(z) + q(z)f(z+1) = c \sin bz,$$

where q is a non-constant polynomial and $b, c \in \mathbb{C} \setminus \{0\}$, does not admit entire solutions of finite order. If q is a nonzero constant, then the above equation possesses three distinct entire solutions of finite order, provided that $b = 3n\pi$ and $q^3 = (-1)^{n+1}c^2 27/4$ for a nonzero integer n .

In 2014, Liu and Lü et al. proved the following result.

Theorem B ([15]). Let $n \geq 4$ be an integer, q be a polynomial, and $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. If there exists some entire solution f of finite order to the following equation

$$f^n(z) + q(z)\Delta f(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

then q is a constant, and one of the following relations holds:

- (1) $f(z) = c_1 e^{\frac{\alpha_1}{n} z}$, and $c_1(e^{\alpha_1/n} - 1)q = p_2$, $\alpha_1 = n\alpha_2$,
 - (2) $f(z) = c_2 e^{\frac{\alpha_2}{n} z}$, and $c_2(e^{\alpha_2/n} - 1)q = p_1$, $\alpha_2 = n\alpha_1$,
- where c_1, c_2 are constants satisfying $c_1^n = p_1, c_2^n = p_2$.

Recently, Zhang et al. obtained the following result.

Theorem C ([16]). Let q be a polynomial, and $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. If f is an entire solution of finite order to the following equation:

$$f^3(z) + q(z)\Delta f(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \quad (1)$$

then q is a constant, and one of the following relations holds:

- (1) $T(r, f) = N_1(r, \frac{1}{f}) + S(r, f)$,
- (2) $f(z) = c_1 \exp(\frac{\alpha_1 z}{3})$, and $c_1(\exp(\frac{\alpha_1}{3}) - 1)q = p_2$, $\alpha_1 = 3\alpha_2$,
- (3) $f(z) = c_2 \exp(\frac{\alpha_2 z}{3})$, and $c_2(\exp(\frac{\alpha_2}{3}) - 1)q = p_1$, $\alpha_2 = 3\alpha_1$,

where $N_1(r, \frac{1}{f})$ denotes the counting function corresponding to simple zeros of f , and c_1, c_2 are constants satisfying $c_1^3 = p_1, c_2^3 = p_2$.

Remark 1.3. In [16], the authors also gave an example to show that the case (1) occurs indeed.

Example 1.4. Let $f(z) = e^{\pi iz} + e^{-\pi iz} = 2i \sin(\pi iz)$. Then f is a solution of the following equation:

$$f^3(z) + 3\Delta f(z) = e^{3\pi iz} + e^{-3\pi iz}.$$

Obviously, $T(r, f) = N_1(r, \frac{1}{f}) + S(r, f)$. Thus, the case (1) occurs indeed.

Since in the above example $\alpha_1 + \alpha_2 = 3\pi i + (-3\pi i) = 0$, consequently, Zhang et al. posed the following conjecture.

Conjecture 1.5 ([16]). *If $\alpha_1 \neq \alpha_2$, $\alpha_1 + \alpha_2 \neq 0$, then the conclusion (1) of Theorem A is impossible. In fact, any entire solution f of (1) must have 0 as its Picard exceptional value.*

In 2017, Latreuch gave an affirmative answer to Conjecture 1.5. In fact, he obtained the following result.

Theorem D ([17]). *Let q be a polynomial, and $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$ and $\alpha_1 + \alpha_2 \neq 0$. If f is an entire solution of finite order of (1), then q is a constant, and one of the following relations holds:*

$$(1) f(z) = c_1 \exp\left(\frac{\alpha_1 z}{3}\right), \text{ and } c_1(\exp\left(\frac{\alpha_1}{3}\right) - 1)q = p_2, \alpha_1 = 3\alpha_2;$$

$$(2) f(z) = c_2 \exp\left(\frac{\alpha_2 z}{3}\right), \text{ and } c_2(\exp\left(\frac{\alpha_2}{3}\right) - 1)q = p_1, \alpha_2 = 3\alpha_1,$$

where c_1, c_2 are constants satisfying $c_1^3 = p_1, c_2^3 = p_2$. Furthermore, (1) does not have any entire solution of infinite order satisfies any one of the following conditions:

$$(3) \rho_2(f) < 1;$$

$$(4) \lambda(f) < \rho(f) = \infty \text{ and } \rho_2(f) < \infty,$$

here $\lambda(f)$ denotes the exponent of convergence of zeros sequence of f .

In the present paper we continue discussing Conjecture 1.5. Moreover, our result will include several known results for difference or differential equations obtained earlier as its special case. In fact, we consider a slightly more general form of (1) and obtain the following result.

Theorem 1.6. *Let $L(z, f)$ denote a difference-differential polynomial in f of degree one with small functions as its coefficients such that $L(z, 0) \equiv 0$, and let $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. If f is an entire solution with $\rho_2(f) < 1$ to the following equation:*

$$f^3 + L(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \quad (2)$$

then one of the following relations holds:

$$(1) f(z) = c_1 \exp\left(\frac{\alpha_1 z}{3}\right) + c_2 \exp\left(\frac{\alpha_2 z}{3}\right), \text{ where } c_1 \text{ and } c_2 \text{ are two nonzero constants satisfying } c_1^3 = p_1, c_2^3 = p_2 \text{ and } \alpha_1 + \alpha_2 = 0;$$

$$(2) f^3(z) = (p_1 - c_1) \exp(\alpha_1 z), \text{ and } L(z, f) = c_1 \exp(\alpha_1 z) + p_2 \exp(\alpha_2 z), \text{ where } c_1 \text{ is a constant};$$

$$(3) f^3(z) = (p_2 - c_2) \exp(\alpha_2 z), \text{ and } L(z, f) = p_1 \exp(\alpha_1 z) + c_2 \exp(\alpha_2 z), \text{ where } c_2 \text{ is a constant}.$$

From Theorem 1.6, we have

Corollary 1.7. *Equation (2) does not have any entire solution f with $\rho(f) = \infty$ and $\rho_2(f) < 1$.*

Remark 1.8. *It is obvious from Theorem 1.6 that the above conjecture is true. We also point out that if an entire function solution f in Theorem 1.6 is replaced by a meromorphic solution with $N(r, f) = S(r, f)$, the conclusion of Theorem 1.6 still holds.*

Remark 1.9. *Lemma 2.2 (in section 2) is crucial to the proofs of our main results. However, it may be false if the condition “ $\rho_2(f) < 1$ ” is violated. There is no difficulty in showing that $f(z) = \exp(\exp(z))$ is a counterexample. Now one may raise the questions: what will happen if we delete the condition $\rho_2(f) < 1$ in Theorem 1.6, Corollary 1.7 and so on?*

2 Some lemmas

In order to prove Theorem 1.6, we need the following results.

Lemma 2.1 ([18]). Let m, n be positive integers satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Then there are no transcendental entire solutions of f and g satisfy the following equation

$$a(z)f(z)^n + b(z)g(z)^m = 1, \quad (3)$$

with a, b being small functions of f , and g , respectively.

Lemma 2.2 ([19]). Let f be a transcendental meromorphic function of hyper-order $\rho_2(f) < 1$. Then for $c \in \mathbb{C}$, we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f), \quad (4)$$

outside of a possible exceptional set with finite logarithmic measure.

Remark 2.3. The following result is the analogue of the logarithmic derivatives lemma [10, 11] for the difference-differential polynomials of a meromorphic function f . It can be proved by applying Lemma 2.2 and the logarithmic derivatives lemma with a similar reasoning as in [19–21] and stated as follows.

Lemma 2.4. Let f be a transcendental meromorphic function with $\rho_2(f) < 1$. Given $L(z, f)$ as to Theorem 1.6, then for any positive integer k , we have

$$m\left(r, \frac{L(z, f)}{f(z)}\right) + m\left(r, \frac{L^{(k)}(z, f)}{f(z)}\right) = S(r, f), \quad (5)$$

outside of a possible exceptional set with finite logarithmic measure.

Lemma 2.5 ([12], Theorem 1.55). Let $g_1, g_2, \dots, g_p$ be transcendental meromorphic functions satisfying $\Theta(\infty, g_j) = 1$ ($j = 1, 2, \dots, p$). If $\sum_{j=1}^p a_j g_j = 1$, then for $a_j \in \mathbb{C} \setminus \{0\}$ ($j = 1, 2, \dots, p$), we have $\sum_{j=1}^p \delta(0, g_j) \leq p - 1$.

Remark 2.6. By the same methods as in the proof of Theorem 1.55 used in [12], we also point out that if nonzero constants $a_1, a_2, \dots, a_p$ are replaced by small functions of $g_1, g_2, \dots, g_p$, the conclusion of Lemma 2.5 still holds.

Lemma 2.7 ([22]). Suppose that f is a transcendental meromorphic function, a, b, c and d are small functions of f such that $acd \neq 0$. If

$$af^2 + bff' + c(f')^2 = d,$$

then

$$c(b^2 - 4ac)\frac{d'}{d} + b(b^2 - 4ac) - c(b^2 - 4ac)' + (b^2 - 4ac)c' = 0.$$

Lemma 2.8 ([23]). Assume that $c \in \mathbb{C}$ is a nonzero constant, α is a non-constant meromorphic function. Then the differential equation $f^2 + (cf^{(n)})^2 = \alpha$ has no transcendental meromorphic solutions satisfying $T(r, \alpha) = S(r, f)$.

Lemma 2.9 ([12]). Assume that f is a meromorphic function. Then for all irreducible rational functions in f ,

$$R(z, f) = \frac{\sum_{i=0}^p a_i f^i}{\sum_{j=0}^q b_j f^j},$$

with meromorphic coefficients a_i, b_j satisfying

$$T(r, a_i) = S(r, f), \quad i = 0, \dots, p, \quad T(r, b_j) = S(r, f), \quad j = 0, \dots, q,$$

the characteristic function of $R(z, f)$ satisfies

$$T(r, R(z, f)) = dT(r, f) + S(r, f),$$

where $d = \max(p, q)$.

3 Proof of Theorem 1.6

Suppose that f is an entire solution with $\rho_2(f) < 1$ to (2). Obviously, f is a transcendental function. For the simplicity, we replace $f(z)$, $f'(z)$ and $L(z, f)$ by f , f' and L , respectively.

By differentiating both sides of (2), we obtain

$$3f^2f' + L' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \quad (6)$$

Combining (2) and (6) yields

$$\alpha_2 f^3 + \alpha_2 L - 3f^2f' - L' = (\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \quad (7)$$

By differentiating (7) again, we derive that

$$3\alpha_2 f^2f' + \alpha_2 L' - 6f(f')^2 - 3f^2f'' - L'' = \alpha_1(\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \quad (8)$$

It follows from (7) and (8) that

$$f\varphi = T(z, f), \quad (9)$$

where

$$\begin{aligned} \varphi &= \alpha_1 \alpha_2 f^2 - 3(\alpha_1 + \alpha_2)ff' + 6(f')^2 + 3ff'', \\ T(z, f) &= -\alpha_1 \alpha_2 L + (\alpha_1 + \alpha_2)L' - L''. \end{aligned} \quad (10)$$

Two cases will now be considered below, depending on whether or not φ vanishes identically.

If $\varphi \equiv 0$, then (9) shows that $T(z, f) \equiv 0$, namely

$$L'' - (\alpha_1 + \alpha_2)L' + \alpha_1 \alpha_2 L = 0. \quad (11)$$

Further, the general solution of (11) is given by

$$L = c_1 e^{\alpha_1 z} + c_2 e^{\alpha_2 z}, \quad (12)$$

where c_1 and c_2 are constants. Thus, (2) and (12) would give

$$f^3 = (p_1 - c_1) e^{\alpha_1 z} + (p_2 - c_2) e^{\alpha_2 z}. \quad (13)$$

We claim that $p_1 = c_1$ or $p_2 = c_2$. Assume now, contrary on the assertion, that $p_1 \neq c_1$ and $p_2 \neq c_2$. We rewrite (13) as

$$\left(\frac{e^{-\alpha_2 z/3} f}{\sqrt[3]{p_2 - c_2}} \right)^3 - \left(\sqrt[3]{\frac{p_1 - c_1}{p_2 - c_2}} e^{(\alpha_1 - \alpha_2)z/3} \right)^3 = 1,$$

which contradicts Lemma 2.1. Hence, $p_1 = c_1$ or $p_2 = c_2$. In this case, we can derive the conclusions (2) and (3).

In the following, we will consider the case $\varphi \not\equiv 0$. In this case, (9) gives

$$\varphi(z) = \frac{T(z, f)}{f(z)}. \quad (14)$$

Since $T(z, f)$ is a difference-differential polynomial in f of degree 1, and $L(z, 0) \equiv 0$, it follows from (14), Lemma 2.4 and the lemma on the logarithmic derivatives that $m(r, \varphi) = S(r, f)$. Note that φ is an entire function, so $T(r, \varphi) = m(r, \varphi) = S(r, f)$, which means that φ is a small function of f .

Now, we rewrite (10) as

$$\frac{\varphi}{f^2} = \alpha_1 \alpha_2 - 3(\alpha_1 + \alpha_2) \frac{f'}{f} + 6 \left(\frac{f'}{f} \right)^2 + 3 \frac{f''}{f}. \quad (15)$$

Applying the lemma on the logarithmic derivatives to (15), we find $m(r, \frac{\varphi}{f^2}) = S(r, f)$. Since φ is a small function of f , one can get $m(r, \frac{1}{f}) = S(r, f)$. Thus, the first fundamental theorem implies $T(r, f) = N(r, \frac{1}{f}) + S(r, f)$.

On the other hand, by (10) again, we have $N_{(2)}(r, 1/f) = S(r, f)$, and

$$T(r, f) = N_1(r, \frac{1}{f}) + S(r, f), \quad (16)$$

where $N_1(r, \frac{1}{f})$ denotes the counting function corresponding to simple zeros of f . Differentiating (10) yields

$$\varphi' = 2\alpha_1\alpha_2 f f'' - 3(\alpha_1 + \alpha_2)[f f'' + (f')^2] + 15f' f'' + 3f f'''. \quad (17)$$

For brevity, in the following, we assume that z_0 is a simple zero of f , and we can assume, without loss of generality, by (16) that $\psi(z_0) \neq 0, \infty$, where ψ is any non-vanishing small function of f . Thus, (10) enables us to deduce the following fact

$$\varphi(z_0) = 6(f'(z_0))^2. \quad (18)$$

Now, we are ready to present $\varphi' \neq 0$. Suppose, contrary to our assertion, that $\varphi' \equiv 0$, namely, φ is a constant, say A .

If z_0 is a zero of $f'(z) - \sqrt{A/6}$, then we set

$$h(z) = \frac{f'(z) - \sqrt{A/6}}{f(z)}. \quad (19)$$

Trivially, $h \neq 0$. Then by the lemma on the logarithmic derivatives, the facts $m(r, 1/f) = S(r, f)$, $N_{(2)}(r, 1/f) = S(r, f)$ and (18), we have $T(r, h) = S(r, f)$.

By (19), we therefore have

$$f' = hf + \sqrt{A/6}, \quad f'' = (h' + h^2)f + h\sqrt{\frac{A}{6}}. \quad (20)$$

Substituting (20) into (10) yields

$$[\alpha_1\alpha_2 - 3(\alpha_1 + \alpha_2)h + 3h' + 9h^2]f = 3[(\alpha_1 + \alpha_2) - 5h]\sqrt{\frac{A}{6}},$$

which implies

$$\alpha_1 + \alpha_2 \equiv 5h, \quad \alpha_1\alpha_2 - 3(\alpha_1 + \alpha_2)h + 9h^2 \equiv 0.$$

Thereby we have

$$h = \frac{\alpha_1 + \alpha_2}{5} = \frac{\alpha_1}{3}, \quad \text{or} \quad h = \frac{\alpha_1 + \alpha_2}{5} = \frac{\alpha_2}{3}. \quad (21)$$

Thus, (19) and (21) would give

$$f(z) = \frac{B}{h} e^{hz} - \frac{1}{h} \sqrt{\frac{A}{6}}, \quad (22)$$

where B is a nonzero constant.

On the other hand, substituting (22) into (2), it follows by Lemma 2.5 that $\frac{1}{h} \sqrt{\frac{A}{6}} = 0$. This, however, contradicts (16), and thus $\varphi' \neq 0$.

Using the same way as above, $\varphi' \neq 0$ is also obtained by setting

$$h(z) = \frac{f'(z) + \sqrt{A/6}}{f(z)}$$

assuming that $f'(z_0) + \sqrt{A/6} = 0$.

Moreover, (17) gives

$$\varphi'(z_0) = [-3(\alpha_1 + \alpha_2)(f')^2 + 15f' f''](z_0) = 0. \quad (23)$$

In order to prove Theorem 1.6, we discuss two cases below:

Case 1. $[2\varphi' + (\alpha_1 + \alpha_2)\varphi]f' - 5\varphi f'' \equiv 0$.

In this case, let us write it in the following form

$$f'' = \left[\frac{2}{5} \frac{\varphi'}{\varphi} + \frac{1}{5} (\alpha_1 + \alpha_2) \right] f' := sf', \quad (24)$$

and consequently

$$f''' = (s' + s^2)f'. \quad (25)$$

Substituting (24) and (25) into (17), we then immediately derive

$$\alpha_1 \alpha_2 \varphi' f = [2\alpha_1 \alpha_2 \varphi - 3(\alpha_1 + \alpha_2)s\varphi + 3(s' + s^2)\varphi + 3(\alpha_1 + \alpha_2)\varphi' - 3s\varphi']f',$$

which is impossible by (16) and the facts that the coefficients $\varphi' (\neq 0)$, $2\alpha_1 \alpha_2 \varphi - 3(\alpha_1 + \alpha_2)s\varphi + 3(s' + s^2)\varphi + 3(\alpha_1 + \alpha_2)\varphi' - 3s\varphi'$ are small functions of f . So, this case can not occur.

Case 2. $[2\varphi' + (\alpha_1 + \alpha_2)\varphi]f' - 5\varphi f'' \neq 0$.

Obviously, in this case, by (18), (23) and $f'(z_0) \neq 0$, we then see that z_0 is a zero of the function $[2\varphi' + (\alpha_1 + \alpha_2)\varphi]f' - 5\varphi f''$.

Accordingly, we set

$$\phi(z) = \frac{[2\varphi'(z) + (\alpha_1 + \alpha_2)\varphi(z)]f'(z) - 5\varphi f''(z)}{f(z)}. \quad (26)$$

Then by the lemma on the logarithmic derivatives, the facts $N_{(2)}(r, 1/f) = S(r, f)$, $m(r, 1/f) = S(r, f)$, and (17), we have $T(r, \phi) = S(r, f)$. Thereby, from (26), we obtain

$$f'' = \left[\frac{2}{5} \frac{\varphi'}{\varphi} + \frac{1}{5} (\alpha_1 + \alpha_2) \right] f' - \frac{\phi}{5\varphi} f := sf' + tf. \quad (27)$$

Trivially, s, t are small functions of f .

By (10) and (27), we have

$$af^2 + bff' + 6(f')^2 = \varphi, \quad (28)$$

where $a = \alpha_1 \alpha_2 + 3t$, $b = 3[s - (\alpha_1 + \alpha_2)]$.

In the following, we consider two subcases.

Subcase 2.1 Suppose that $a \equiv 0$. In this case, (28) becomes

$$(bf + 6f')f' = \varphi,$$

which gives

$$f' = \varphi_1 e^\beta, \quad bf + 6f' = \varphi_2 e^{-\beta}, \quad (29)$$

where β, φ_1 and φ_2 are entire functions such that $\varphi_1 \varphi_2 = \varphi$.

Trivially, in this case, $b \neq 0$, and it follows by (29) that

$$f = \frac{\varphi_2 e^{-\beta} - 6\varphi_1 e^\beta}{b}. \quad (30)$$

Thus, by (29) and (30), we have

$$\left[\varphi_1 + 6\left(\frac{\varphi_1}{b}\right)' + 6\frac{\varphi_1}{b}\beta' \right] e^{2\beta} = \left(\frac{\varphi_2}{b}\right)' - \frac{\varphi_2}{b}\beta',$$

which shows that

$$\left(\frac{\varphi_2}{b}\right)' - \frac{\varphi_2}{b}\beta' \equiv 0. \quad (31)$$

Obviously, (31) gives $\log \frac{\varphi_2}{b} = \beta + C$, where C is a constant. Therefore, e^β is a small function of f , this shows that f' is also a small function of f . The contradiction $T(r, f) = S(r, f)$ now follows by (30).

Subcase 2.2 $a \neq 0$. In this case, applying Lemma 2.7 to (28), we immediately get the following equation

$$6(b^2 - 24a) \frac{\varphi'}{\varphi} + b(b^2 - 24a) - 6(b^2 - 24a)' \equiv 0. \quad (32)$$

Now, we consider two cases.

Firstly, assume that $b^2 - 24a \neq 0$.

Note that $b = \frac{6}{5} \frac{\varphi'}{\varphi} - \frac{12}{5}(\alpha_1 + \alpha_2)$, and we then rewrite (32) as

$$\frac{11}{5} \frac{\varphi'}{\varphi} - \frac{12}{5}(\alpha_1 + \alpha_2) = \frac{(b^2 - 24a)'}{b^2 - 24a}.$$

By integrating the above equation, we have

$$11 \log \varphi - 12(\alpha_1 + \alpha_2)z = 5 \log(b^2 - 24a) + \log D, \quad (33)$$

where D is a constant. Obviously, (33) gives $\varphi^{11} = D(b^2 - 24a)^5 e^{12(\alpha_1 + \alpha_2)z}$. If $\alpha_1 + \alpha_2 \neq 0$, then $e^{\alpha_1 z}$, $e^{\alpha_2 z}$ are also two small functions of f because φ and $b^2 - 24a$ are small functions of f . Rewrite (2) as

$$f^2 = -\frac{L}{f} + \frac{p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}}{f}.$$

Then

$$\begin{aligned} 2m(r, f) &= m(r, f^2) = m\left(r, -\frac{L}{f} + \frac{p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}}{f}\right) \\ &\leq m\left(r, \frac{1}{f}\right) + S(r, f) = S(r, f), \end{aligned}$$

which is a contradiction, since f is an entire function. Therefore, $\alpha_1 + \alpha_2 = 0$.

Combining (2) and (6) yields

$$\alpha_1 f^3 + \alpha_1 L - 3f^2 f' - L' = (\alpha_1 - \alpha_2) p_2 e^{\alpha_2 z}. \quad (34)$$

It follows by (7), (34) and $\alpha_1 + \alpha_2 = 0$ that

$$f^4 [-\alpha_1^2 f^2 + 9(f')^2] = 2\alpha_1^2 L f^3 - 6L' f^2 f' + (\alpha_1 L)^2 - (L')^2 - 4\alpha_1^2 p_1 p_2. \quad (35)$$

Obviously, $P_4(f) := 2\alpha_1^2 L f^3 - 6L' f^2 f' + (\alpha_1 L)^2 - (L')^2 - 4\alpha_1^2 p_1 p_2$ is a difference-differential polynomial of f , and its total degree at most 4.

If $P_4(f) \equiv 0$, it follows from (35) that $9(f')^2 - \alpha_1^2 f^2 \equiv 0$, and then $f' = \pm \frac{\alpha_1}{3} f$. Substituting the above expression into (10), we arrive at $\varphi \equiv 0$, a contradiction. Therefore, $P_4(f) \not\equiv 0$. Set $\beta = 9(f')^2 - \alpha_1^2 f^2$. In this case, we rewrite (35) as

$$\beta = \frac{P_4(f)}{f^4},$$

which, Lemma 2.4 and the fact that we have proved $m(r, \frac{1}{f}) = S(r, f)$ must show that $m(r, \beta) = S(r, f)$, i.e. β is a small function of f . Moreover, from Lemma 2.8, it is easy to see that β is a constant. By differentiating both sides of $\beta = 9(f')^2 - \alpha_1^2 f^2$, we get

$$f'' - \left(\frac{\alpha_1}{3}\right)^2 f = 0. \quad (36)$$

It follows from (36) that

$$f(z) = c_1 e^{\frac{\alpha_1}{3} z} + c_2 e^{-\frac{\alpha_1}{3} z}, \quad (37)$$

where c_1, c_2 are constants. By (37) and (2), we have $c_1^3 = p_1, c_2^3 = p_2$. Conclusion (1) has consequently been proved.

Now, we assume that $b^2 - 24a \equiv 0$.

By making use of (28), we have $6(f' + \frac{b}{12}f)^2 = \varphi$, which shows that $\gamma := f' + \frac{b}{12}f$ is a small function of f . Thus, $f'' = (\frac{b^2}{144} - \frac{b'}{12})f + \gamma' - \frac{b}{12}\gamma$.

Substituting two above expressions into (10), we obtain

$$\left[\alpha_1\alpha_2 + \frac{b}{4}(\alpha_1 + \alpha_2) + \frac{b^2}{144} - \frac{b'}{4}\right]f^2 + [-3(\alpha_1 + \alpha_2)\gamma - b\gamma + 3(\gamma' - \frac{b'}{12}\gamma)]f = \varphi. \quad (38)$$

Using (38) and Lemma 2.9, we therefore have

$$\begin{aligned} \alpha_1\alpha_2 + \frac{b}{4}(\alpha_1 + \alpha_2) + \frac{b^2}{144} - \frac{b'}{4} &\equiv 0, \\ -3(\alpha_1 + \alpha_2)\gamma - b\gamma + 3(\gamma' - \frac{b'}{12}\gamma) &\equiv 0 \quad \text{and} \quad \varphi \equiv 0. \end{aligned}$$

This contradicts $\varphi \neq 0$.

Thus, we finish the proof of Theorem 1.6.

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