Open Mathematics

Research Article

Alexander V. Osipov*

On the different kinds of separability of the space of Borel functions

https://doi.org/10.1515/math-2018-0070
Received June 18, 2017; accepted May 24, 2018.

Abstract: In paper we prove that:

- a space of Borel functions $B(X)$ on a set of reals X, with pointwise topology, to be countably selective sequentially separable if and only if X has the property $S_1(B_\Gamma, B_\Gamma)$;
- there exists a consistent example of sequentially separable selectively separable space which is not selective sequentially separable. This is an answer to the question of A. Bella, M. Bonanzinga and M. Matveev;
- there is a consistent example of a compact T_2 sequentially separable space which is not selective sequentially separable. This is an answer to the question of A. Bella and C. Costantini;
- $\min\{\mathfrak{b}, \mathfrak{q}\} = \{\kappa : 2^\kappa$ is not selective sequentially separable$\}$. This is a partial answer to the question of A. Bella, M. Bonanzinga and M. Matveev.

Keywords: $S_1(\mathcal{D}, \mathcal{D})$, $S_1(\mathcal{S}, \mathcal{S})$, $S_{fin}(\mathcal{S}, \mathcal{S})$, Function spaces, Selection principles, Borel function, σ-set, $S_1(B_\Omega, B_\Omega)$, $S_1(B_\Gamma, B_\Gamma)$, $S_1(B_\Omega, B_\Gamma)$, Sequentially separable, Selectively separable, Selective sequentially separable, Countably selective sequentially separable

MSC: 54C35, 54C05, 54C65, 54A20

1 Introduction

In [12], Osipov and Pytkeev gave necessary and sufficient conditions for the space $B_1(X)$ of the Baire class 1 functions on a Tychonoff space X, with pointwise topology, to be (strongly) sequentially separable. In this paper, we consider some properties of a space $B(X)$ of Borel functions on a set of reals X, with pointwise topology, that are stronger than (sequential) separability.

2 Main definitions and notation

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

***Corresponding Author: Alexander V. Osipov:** Krasovskii Institute of Mathematics and Mechanics, Ural Federal University, Ural State University of Economics, 620219, Yekaterinburg, Russia, E-mail: OAB@list.ru

$U_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

An open cover $\mathcal{U}$ of a space X is:
- an *ω-cover* if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$;
- a *γ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

For a topological space X we denote:
- Ω — the family of all countable open ω-covers of X;
- Γ — the family of all countable open γ-covers of X;
- B_Ω — the family of all countable Borel ω-covers of X;
- B_Γ — the family of all countable Borel γ-covers of X;
- F_Γ — the family of all countable closed γ-covers of X;
- $\mathcal{D}$ — the family of all countable dense subsets of X;
- $\mathcal{S}$ — the family of all countable sequentially dense subsets of X.

A γ-cover $\mathcal{U}$ of co-zero sets of X is *γ_F-shrinkable* if there exists a γ-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of X with $F(U) \subset U$ for every $U \in \mathcal{U}$.

For a topological space X we denote Γ_F, the family of all countable γ_F-shrinkable γ-covers of X.

We will use the following notations.
- $C_p(X)$ is the set of all real-valued continuous functions $C(X)$ defined on a space X, with pointwise topology.
- $B_1(X)$ is the set of all first Baire class 1 functions $B_1(X)$ i.e., pointwise limits of continuous functions, defined on a space X, with pointwise topology.
- $B(X)$ is the set of all Borel functions, defined on a space X, with pointwise topology.

If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. If D is a countable, sequentially dense subset of X then X call *sequentially separable* space.

Call a space X *strongly sequentially separable* if X is separable and every countable dense subset of X is sequentially dense.

A space X is (*countably*) *selectively separable* (or M-separable, [3]) if for every sequence $(D_n : n \in \mathbb{N})$ of (countable) dense subsets of X one can pick finite $F_n \subset D_n$, $n \in \mathbb{N}$, so that $\bigcup\{F_n : n \in \mathbb{N}\}$ is dense in X.

In [3], the authors started to investigate a selective version of sequential separability.

A space X is (*countably*) *selectively sequentially separable* (or M-sequentially separable, [3]) if for every sequence $(D_n : n \in \mathbb{N})$ of (countable) sequentially dense subsets of X, one can pick finite $F_n \subset D_n$, $n \in \mathbb{N}$, so that $\bigcup\{F_n : n \in \mathbb{N}\}$ is sequentially dense in X.

In Scheeper's terminology [16], countably selectively separability equivalently to the selection principle $S_{fin}(\mathcal{D}, \mathcal{D})$, and countably selective sequentially separability equivalently to the $S_{fin}(\mathcal{S}, \mathcal{S})$.

Recall that the cardinal $\mathfrak{p}$ is the smallest cardinal so that there is a collection of $\mathfrak{p}$ many subsets of the natural numbers with the strong finite intersection property but no infinite pseudo-intersection. Note that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$.

For $f, g \in \mathbb{N}^{\mathbb{N}}$, let $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n. $\mathfrak{b}$ is the minimal cardinality of a $\leq^*$-unbounded subset of $\mathbb{N}^{\mathbb{N}}$. A set $B \subset [\mathbb{N}]^\infty$ is unbounded if the set of all increasing enumerations of elements of B is unbounded in $\mathbb{N}^{\mathbb{N}}$, with respect to $\leq^*$. It follows that $|B| \geq \mathfrak{b}$. A subset S of the real line is called a Q-set if each one of its subsets is a G_δ. The cardinal $\mathfrak{q}$ is the smallest cardinal so that for any $\kappa < \mathfrak{q}$ there is a Q-set of size κ. (See [7] for more on small cardinals including $\mathfrak{p}$).

3 Properties of a space of Borel functions

Theorem 3.1. *For a set of reals X, the following statements are equivalent:*

1. $B(X)$ *satisfies* $S_1(\mathcal{S},\mathcal{S})$ *and* $B(X)$ *is sequentially separable;*
2. X *satisfies* $S_1(B_\Gamma, B_\Gamma)$*;*
3. $B(X) \in S_{fin}(\mathcal{S},\mathcal{S})$ *and* $B(X)$ *is sequentially separable;*
4. X *satisfies* $S_{fin}(B_\Gamma, B_\Gamma)$*;*
5. $B_1(X)$ *satisfies* $S_1(\mathcal{S},\mathcal{S})$*;*
6. X *satisfies* $S_1(F_\Gamma, F_\Gamma)$*;*
7. $B_1(X)$ *satisfies* $S_{fin}(\mathcal{S},\mathcal{S})$*.*

Proof. It is obvious that $(1) \Rightarrow (3)$.

$(2) \Leftrightarrow (4)$. By Theorem 1 in [15], $U_{fin}(B_\Gamma, B_\Gamma) = S_1(B_\Gamma, B_\Gamma) = S_{fin}(B_\Gamma, B_\Gamma)$.

$(3) \Rightarrow (2)$. Let $\{\mathcal{F}_i\} \subset B_\Gamma$ and $\mathcal{S} = \{h_m\}_{m\in\mathbb{N}}$ be a countable sequentially dense subset of $B(X)$. For each $i \in \mathbb{N}$ we consider a countable sequentially dense subset $\mathcal{S}_i$ of $B(X)$ and $\mathcal{F}_i = \{F_i^m\}_{m\in\mathbb{N}}$ where

$\mathcal{S}_i = \{f_i^m\} := \{f_i^m \in B(X) : f_i^m \upharpoonright F_i^m = h_m$ and $f_i^m \upharpoonright (X \setminus F_i^m) = 1$ for $m \in \mathbb{N}\}$.

Since $\mathcal{F}_i = \{F_i^m\}_{m\in\mathbb{N}}$ is a Borel γ-cover of X and $\mathcal{S}$ is a countable sequentially dense subset of $B(X)$, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $B(X)$ for each $i \in \mathbb{N}$. Indeed, let $h \in B(X)$, there is a sequence $\{h_s\}_{s\in\mathbb{N}} \subset \mathcal{S}$ such that $\{h_s\}_{s\in\mathbb{N}}$ converges to h. We claim that $\{f_i^s\}_{s\in\mathbb{N}}$ converges to h. Let $K = \{x_1, \dots, x_k\}$ be a finite subset of X, $\epsilon > 0$ and let $W = \langle h, K, \epsilon\rangle := \{g \in B(X) : |g(x_j) - h(x_j)| < \epsilon$ for $j = 1, \dots, k\}$ be a base neighborhood of h, then there is $m_0 \in \mathbb{N}$ such that $K \subset F_i^m$ for each $m > m_0$ and $h_s \in W$ for each $s > m_0$. Since $f_i^s \upharpoonright K = h_s \upharpoonright K$ for every $s > m_0$, $f_i^s \in W$ for every $s > m_0$. It follows that $\{f_i^s\}_{s\in\mathbb{N}}$ converges to h.

Since $B(X)$ satisfies $S_{fin}(\mathcal{S},\mathcal{S})$, there is a sequence $(F_i = \{f_i^{m_1}, \dots, f_i^{m_{s(i)}}\} : i \in \mathbb{N})$ such that for each i, $F_i \subset \mathcal{S}_i$, and $\bigcup_{i\in\mathbb{N}} F_i$ is a countable sequentially dense subset of $B(X)$.

For $0 \in B(X)$ there is a sequence $\{f_{i_j}^{m_{s(i_j)}}\}_{j\in\mathbb{N}} \subset \bigcup_{i\in\mathbb{N}} F_i$ such that $\{f_{i_j}^{m_{s(i_j)}}\}_{j\in\mathbb{N}}$ converges to 0. Consider a sequence $(F_{i_j}^{m_{s(i_j)}} : j \in \mathbb{N})$. Then

(1) $F_{i_j}^{m_{s(i_j)}} \in \mathcal{F}_{i_j}$;

(2) $\{F_{i_j}^{m_{s(i_j)}} : j \in \mathbb{N}\}$ is a γ-cover of X.

Indeed, let K be a finite subset of X and $U = \langle 0, K, \frac{1}{2}\rangle$ be a base neighborhood of 0, then there is $j_0 \in \mathbb{N}$ such that $f_{i_j}^{m_{s(i_j)}} \in U$ for every $j > j_0$. It follows that $K \subset F_{i_j}^{m_{s(i_j)}}$ for every $j > j_0$. We thus get that X satisfies $U_{fin}(B_\Gamma, B_\Gamma)$, and, hence, by Theorem 1 in [15], X satisfies $S_1(B_\Gamma, B_\Gamma)$.

$(2) \Rightarrow (1)$. Let $\{S_i\} \subset \mathcal{S}$ and $S = \{d_n : n \in \mathbb{N}\} \in \mathcal{S}$. Consider the topology τ generated by the family $\mathcal{P} = \{f^{-1}(G) : G$ is an open set of $\mathbb{R}$ and $f \in S \cup \bigcup\limits_{i\in\mathbb{N}} S_i\}$. Since $P = S \cup \bigcup\limits_{i\in\mathbb{N}} S_i$ is a countable dense subset of $B(X)$ and X is Tychonoff, we have that the space $Y = (X, \tau)$ is a separable metrizable space. Note that a function $f \in P$, considered as mapping from Y to $\mathbb{R}$, is a continuous function i.e. $f \in C(Y)$ for each $f \in P$. Note also that an identity map φ from X on Y, is a Borel bijection. By Corollary 12 in [6], Y is a QN-space and, hence, by Corollary 20 in [17], Y has the property $S_1(B_\Gamma, B_\Gamma)$. By Corollary 21 in [17], $B(Y)$ is an α_2 space.

Let $q : \mathbb{N} \mapsto \mathbb{N} \times \mathbb{N}$ be a bijection. Then we enumerate $\{S_i\}_{i\in\mathbb{N}}$ as $\{S_{q(i)}\}_{q(i)\in\mathbb{N}\times\mathbb{N}}$. For each $d_n \in S$ there are sequences $s_{n,m} \subset S_{n,m}$ such that $s_{n,m}$ converges to d_n for each $m \in \mathbb{N}$. Since $B(Y)$ is an α_2 space, there is $\{b_{n,m} : m \in \mathbb{N}\}$ such that for each m, $b_{n,m} \in s_{n,m}$, and, $b_{n,m} \to d_n$ $(m \to \infty)$. Let $B = \{b_{n,m} : n, m \in \mathbb{N}\}$. Note that $S \subset [B]_{seq}$.

Since X is a σ-set (that is, each Borel subset of X is F_σ)(see [17]), $B_1(X) = B(X)$ and $\varphi(B(Y)) = \varphi(B_1(Y)) \subseteq B(X)$ where $\varphi(B(Y)) := \{p \circ \varphi : p \in B(Y)\}$ and $\varphi(B_1(Y)) := \{p \circ \varphi : p \in B_1(Y)\}$.

Since S is a countable, sequentially dense subset of $B(X)$, for any $g \in B(X)$ there is a sequence $\{g_n\}_{n\in\mathbb{N}} \subset S$ such that $\{g_n\}_{n\in\mathbb{N}}$ converges to g. But g we can consider as a mapping from Y into $\mathbb{R}$ and a set $\{g_n : n \in \mathbb{N}\}$ as subset of $C(Y)$. It follows that $g \in B_1(Y)$. We get that $\varphi(B(Y)) = B(X)$.

We claim that $B \in \mathcal{S}$, i.e. that $[B]_{seq} = B(X)$. Let $f \in B(Y)$ and $\{f_k : k \in \mathbb{N}\} \subset S$ such that $f_k \to f$ $(k \to \infty)$. For each $k \in \mathbb{N}$ there is $\{f_k^n : n \in \mathbb{N}\} \subset B$ such that $f_k^n \to f_k$ $(n \to \infty)$. Since Y is a QN-space (Theorem 16 in

[6]), there exists an unbounded $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $\{f_k^{\beta(k)}\}$ converges to f on Y. It follows that $\{f_k^{\beta(k)} : k \in \mathbb{N}\}$ converge to f on X and $[B]_{seq} = B(X)$.

$(5) \Rightarrow (6)$. By Velichko's Theorem ([18]), a space $B_1(X)$ is sequentially separable for any separable metric space X.

Let $\{\mathcal{F}_i\} \subset F_\Gamma$ and $\mathcal{S} = \{h_m\}_{m\in\mathbb{N}}$ be a countable sequentially dense subset of $B_1(X)$.

Similarly implication $(3) \Rightarrow (2)$ we get X satisfies $U_{fin}(F_\Gamma, F_\Gamma)$, and, hence, by Lemma 13 in [17], X satisfies $S_1(F_\Gamma, F_\Gamma)$.

$(6) \Rightarrow (5)$. By Corollary 20 in [17], X satisfies $S_1(B_\Gamma, B_\Gamma)$. Since X is a σ-set (see [17]), $B_1(X) = B(X)$ and, by implication $(2) \Rightarrow (1)$, we get $B_1(X)$ satisfies $S_1(\mathcal{S}, \mathcal{S})$. □

In [16], (Theorem 13) M. Scheepers proved the following result.

Theorem 3.2 (Scheepers). *For X a separable metric space, the following are equivalent:*
1. *$C_p(X)$ satisfies $S_1(\mathcal{D}, \mathcal{D})$;*
2. *X satisfies $S_1(\Omega, \Omega)$.*

We claim the theorem for a space $B(X)$ of Borel functions.

Theorem 3.3. *For a set of reals X, the following are equivalent:*
1. *$B(X)$ satisfies $S_1(\mathcal{D}, \mathcal{D})$;*
2. *X satisfies $S_1(B_\Omega, B_\Omega)$.*

Proof. $(1) \Rightarrow (2)$. Let X be a set of reals satisfying the hypotheses and β be a countable base of X. Consider a sequence $\{\mathcal{B}_i\}_{i\in\mathbb{N}}$ of countable Borel ω-covers of X where $\mathcal{B}_i = \{W_i^j\}_{j\in\mathbb{N}}$ for each $i \in \mathbb{N}$.

Consider a topology τ generated by the family $\mathcal{P} = \{W_i^j \cap A : i, j \in \mathbb{N}$ and $A \in \beta\} \bigcup \{(X \setminus W_i^j) \cap A : i, j \in \mathbb{N}$ and $A \in \beta\}$.

Note that if χ_P is a characteristic function of P for each $P \in \mathcal{P}$, then a diagonal mapping $\varphi = \Delta_{P\in\mathcal{P}}\chi_P : X \mapsto 2^\omega$ is a Borel bijection. Let $Z = \varphi(X)$.

Note that $\{\mathcal{B}_i\}$ is countable open ω-cover of Z for each $i \in \mathbb{N}$. Since $B(Z)$ is a dense subset of $B(X)$, then $B(Z)$ also has the property $S_1(\mathcal{D}, \mathcal{D})$. Since $C_p(Z)$ is a dense subset of $B(Z)$, $C_p(Z)$ has the property $S_1(\mathcal{D}, \mathcal{D})$, too.

By Theorem 3.2, the space Z has the property $S_1(\Omega, \Omega)$. It follows that there is a sequence $\{W_i^{j(i)}\}_{i\in\mathbb{N}}$ such that $W_i^{j(i)} \in \mathcal{B}_i$ and $\{W_i^{j(i)} : i \in \mathbb{N}\}$ is an open ω-cover of Z. It follows that $\{W_i^{j(i)} : i \in \mathbb{N}\}$ is Borel ω-cover of X.

$(2) \Rightarrow (1)$. Assume that X has the property $S_1(B_\Omega, B_\Omega)$. Let $\{D_k\}_{k\in\mathbb{N}}$ be a sequence countable dense subsets of $B(X)$ and $D_k = \{f_i^k : i \in \mathbb{N}\}$ for each $k \in \mathbb{N}$. We claim that for any $f \in B(X)$ there is a sequence $\{f_k\} \subset B(X)$ such that $f_k \in D_k$ for each $k \in \mathbb{N}$ and $f \in \overline{\{f_k : k \in \mathbb{N}\}}$. Without loss of generality we can assume $f = 0$. For each $f_i^k \in D_k$ let $W_i^k = \{x \in X : -\frac{1}{k} < f_i^k(x) < \frac{1}{k}\}$.

If for each $j \in \mathbb{N}$ there is $k(j)$ such that $W_{i(j)}^{k(j)} = X$, then a sequence $f_{k(j)} = f_{i(j)}^{k(j)}$ uniformly converges to f and, hence, $f \in \overline{\{f_{k(j)} : j \in \mathbb{N}\}}$.

We can assume that $W_i^k \neq X$ for any $k, i \in \mathbb{N}$.

(a). $\{W_i^k\}_{i\in\mathbb{N}}$ a sequence of Borel sets of X.

(b). For each $k \in \mathbb{N}$, $\{W_i^k : i \in \mathbb{N}\}$ is a ω-cover of X.

By (2), X has the property $S_1(B_\Omega, B_\Omega)$, hence, there is a sequence $\{W_{i(k)}^k\}_{k\in\mathbb{N}}$ such that $W_{i(k)}^k \in \{W_i^k\}_{i\in\mathbb{N}}$ for each $k \in \mathbb{N}$ and $\{W_{i(k)}^k\}_{k\in\mathbb{N}}$ is a ω-cover of X.

Consider $\{f_{i(k)}^k\}$. We claim that $f \in \overline{\{f_{i(k)}^k : k \in \mathbb{N}\}}$. Let K be a finite subset of X, $\epsilon > 0$ and $U = \langle f, K, \epsilon\rangle$ be a base neighborhood of f, then there is $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \epsilon$ and $K \subset W_{i(k_0)}^{k_0}$. It follows that $f_{i(k_0)}^{k_0} \in U$.

Let $D = \{d_n : n \in \mathbb{N}\}$ be a dense subspace of $B(X)$. Given a sequence $\{D_i\}_{i\in\mathbb{N}}$ of dense subspace of $B(X)$, enumerate it as $\{D_{n,m} : n, m \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, pick $d_{n,m} \in D_{n,m}$ so that $d_n \in \overline{\{d_{n,m} : m \in \mathbb{N}\}}$. Then $\{d_{n,m} : m, n \in \mathbb{N}\}$ is dense in $B(X)$. □

In [16], (Theorem 35) and [4] (Corollary 2.10) proved the following result.

Theorem 3.4 (Scheepers). *For X a separable metric space, the following are equivalent:*

1. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}, \mathcal{D})$;*
2. *X satisfies $S_{fin}(\Omega, \Omega)$.*

Then for the space $B(X)$ we have an analogous result.

Theorem 3.5. *For a set of reals X, the following are equivalent:*

1. *$B(X)$ satisfies $S_{fin}(\mathcal{D}, \mathcal{D})$;*
2. *X satisfies $S_{fin}(B_\Omega, B_\Omega)$.*

Proof. It is proved similarly to the proof of Theorem 3.3. □

4 Question of A. Bella, M. Bonanzinga and M. Matveev

In [3], Question 4.3, it is asked to find a sequentially separable selectively separable space which is not selective sequentially separable.

The following theorem answers this question.

Theorem 4.1 (CH). *There is a consistent example of a space Z, such that Z is sequentially separable, selectively separable, not selective sequentially separable.*

Proof. By Theorem 40 and Corollary 41 in [15], there is a $\mathfrak{c}$-Lusin set X which has the property $S_1(B_\Omega, B_\Omega)$, but X does not have the property $U_{fin}(\Gamma, \Gamma)$.

Consider a space $Z = C_p(X)$. By Velichko's Theorem ([18]), a space $C_p(X)$ is sequentially separable for any separable metric space X.

(a). Z is sequentially separable. Since X is Lindelöf and X satisfies $S_1(B_\Omega, B_\Omega)$, X has the property $S_1(\Omega, \Omega)$.

By Theorem 3.2, $C_p(X)$ satisfies $S_1(\mathcal{D}, \mathcal{D})$, and, hence, $C_p(X)$ satisfies $S_{fin}(\mathcal{D}, \mathcal{D})$.

(b). Z is selectively separable. By Theorem 4.1 in [11], $U_{fin}(\Gamma, \Gamma) = U_{fin}(\Gamma_F, \Gamma)$ for Lindelöf spaces.

Since X does not have the property $U_{fin}(\Gamma, \Gamma)$, X does not have the property $S_{fin}(\Gamma_F, \Gamma)$. By Theorem 8.11 in [9], $C_p(X)$ does not have the property $S_{fin}(\mathcal{S}, \mathcal{S})$.

(c). Z is not selective sequentially separable. □

Theorem 4.2 (CH). *There is a consistent example of a space Z, such that Z is sequentially separable, countably selectively separable, countably selectively separable, not countably selective sequentially separable.*

Proof. Consider the $\mathfrak{c}$-Lusin set X (see Theorem 40 and Corollary 41 in [15]), then X has the property $S_1(B_\Omega, B_\Omega)$, but X does not have the property $U_{fin}(\Gamma, \Gamma)$ and, hence, X does not have the property $S_{fin}(B_\Gamma, B_\Gamma)$.

Consider a space $Z = B_1(X)$. By Velichko's Theorem in [18], a space $B_1(X)$ is sequentially separable for any separable metric space X.

(a). Z is sequentially separable. By Theorem 3.3, $B(X)$ satisfies $S_1(\mathcal{D}, \mathcal{D})$. Since Z is dense subset of $B(X)$ we have that Z satisfies $S_1(\mathcal{D}, \mathcal{D})$ and, hence, Z satisfies $S_{fin}(\mathcal{D}, \mathcal{D})$.

(b). Z is countably selectively separable. Since X does not have the property $S_{fin}(B_\Gamma, B_\Gamma)$, by Theorem 3.1, $B_1(X)$ does not have the property $S_{fin}(\mathcal{S}, \mathcal{S})$.

(c). Z is not countably selective sequentially separable. □

5 Question of A. Bella and C. Costantini

In [5], Question 2.7, it is asked to find a compact T_2 sequentially separable space which is not selective sequentially separable.

The following theorem answers this question.

Theorem 5.1. $(\mathfrak{b} < \mathfrak{q})$ *There is a consistent example of a compact T_2 sequentially separable space which is not selective sequentially separable.*

Proof. Let D be a discrete space of size $\mathfrak{b}$. Since $\mathfrak{b} < \mathfrak{q}$, a space $2^{\mathfrak{b}}$ is sequentially separable (see Proposition 3 in [13]).

We claim that $2^{\mathfrak{b}}$ is not selective sequentially separable.

On the contrary, suppose that $2^{\mathfrak{b}}$ is selective sequentially separable. Since $non(S_{fin}(B_\Gamma, B_\Gamma)) = \mathfrak{b}$ (see Theorem 1 and Theorem 27 in [15]), there is a set of reals X such that $|X| = \mathfrak{b}$ and X does not have the property $S_{fin}(B_\Gamma, B_\Gamma)$. Hence there exists sequence $(A_n : n \in \mathbb{N})$ of elements of B_Γ that for any sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, we have that $\bigcup_{n\in\mathbb{N}} B_n \notin B_\Gamma$.

Consider an identity mapping $id : D \mapsto X$ from the space D onto the space X. Denote $C_n^i = id^{-1}(A_n^i)$ for each $A_n^i \in A_n$ and $n, i \in \mathbb{N}$. Let $C_n = \{C_n^i\}_{i\in\mathbb{N}}$ (i.e. $C_n = id^{-1}(A_n)$) and let $\mathcal{S} = \{h_i\}_{i\in\mathbb{N}}$ be a countable sequentially dense subset of $B(D, \{0, 1\}) = 2^{\mathfrak{b}}$.

For each $n \in \mathbb{N}$ we consider a countable sequentially dense subset $\mathcal{S}_n$ of $B(D, \{0, 1\})$ where $\mathcal{S}_n = \{f_n^i\} := \{f_n^i \in B(D, 2) : f_n^i \upharpoonright C_n^i = h_i \text{ and } f_n^i \upharpoonright (X \setminus C_n^i) = 1 \text{ for } i \in \mathbb{N}\}$.

Since $C_n = \{C_n^i\}_{i\in\mathbb{N}}$ is a Borel γ-cover of D and $\mathcal{S}$ is a countable sequentially dense subset of $B(D, \{0, 1\})$, we have that $\mathcal{S}_n$ is a countable sequentially dense subset of $B(D, \{0, 1\})$ for each $n \in \mathbb{N}$.

Indeed, let $h \in B(D, \{0, 1\})$, there is a sequence $\{h_s\}_{s\in\mathbb{N}} \subset \mathcal{S}$ such that $\{h_s\}_{s\in\mathbb{N}}$ converges to h. We claim that $\{f_n^s\}_{s\in\mathbb{N}}$ converges to h. Let $K = \{x_1, \dots, x_k\}$ be a finite subset of D, $\epsilon = \{\epsilon_1, \dots, \epsilon_k\}$ where $\epsilon_j \in \{0, 1\}$ for $j = 1, \dots, k$, and $W = \langle h, K, \epsilon\rangle := \{g \in B(D, \{0, 1\}) : |g(x_j) - h(x_j)| \in \epsilon_j \text{ for } j = 1, \dots, k\}$ be a base neighborhood of h, then there is a number m_0 such that $K \subset C_n^i$ for $i > m_0$ and $h_s \in W$ for $s > m_0$. Since $f_n^s \upharpoonright K = h_s \upharpoonright K$ for each $s > m_0$, $f_n^s \in W$ for each $s > m_0$. It follows that a sequence $\{f_n^s\}_{s\in\mathbb{N}}$ converges to h.

Since $B(D, \{0, 1\})$ is selective sequentially separable, there is a sequence $\{F_n = \{f_n^{i_1}, \dots, f_n^{i_{s(n)}}\} : n \in \mathbb{N}\}$ such that for each n, $F_n \subset \mathcal{S}_n$, and $\bigcup_{n\in\mathbb{N}} F_n$ is a countable sequentially dense subset of $B(D, \{0, 1\})$.

For $0 \in B(D, \{0, 1\})$ there is a sequence $\{f_{n_j}^{i_j}\}_{j\in\mathbb{N}} \subset \bigcup_{n\in\mathbb{N}} F_n$ such that $\{f_{n_j}^{i_j}\}_{j\in\mathbb{N}}$ converges to 0. Consider a sequence $\{C_{n_j}^{i_j} : j \in \mathbb{N}\}$. Then

(1) $C_{n_j}^{i_j} \in C_{n_j}$;

(2) $\{C_{n_j}^{i_j} : j \in \mathbb{N}\}$ is a γ-cover of D.

Indeed, let K be a finite subset of D and $U = \langle 0, K, \{0\}\rangle$ be a base neighborhood of 0, then there is a number j_0 such that $f_{n_j}^{i_j} \in U$ for every $j > j_0$. It follows that $K \subset C_{n_j}^{i_j}$ for every $j > j_0$. Hence, $\{A_{n_j}^{i_j} = id(C_{n_j}^{i_j}) : j \in \mathbb{N}\} \in B_\Gamma$ in the space X, a contradiction. □

Let $\mu = \min\{\kappa : 2^\kappa \text{ is not selective sequentially separable}\}$. It is well-known that $\mathfrak{p} \le \mu \le \mathfrak{q}$ (see [3]).

Theorem 5.2. $\mu = \min\{\mathfrak{b}, \mathfrak{q}\}$.

Proof. Let $\kappa < \min\{\mathfrak{b}, \mathfrak{q}\}$. Then, by Proposition 3 in [13], 2^κ is a sequentially separable space.

Let X be a set of reals such that $|X| = \kappa$ and X be a Q-set.

Analogous to the proof of implication $(2) \Rightarrow (1)$ in Theorem 3.1, we can claim that $B(X, \{0, 1\}) = 2^X = 2^\kappa$ is selective sequentially separable.

It follows that $\mu \ge \min\{\mathfrak{b}, \mathfrak{q}\}$.

Since $\mu \le \mathfrak{q}$, we suppose that $\mu > \mathfrak{b}$ and $\mathfrak{b} < \mathfrak{q}$. Then, by Theorem 5.1, $2^{\mathfrak{b}}$ is not selective sequentially separable. It follows that $\mu = \min\{\mathfrak{b}, \mathfrak{q}\}$. □

In [3], Question 4.12 : is it the case $\mu \in \{\mathfrak{p}, \mathfrak{q}\}$?

A partial positive answer to this question is the existence of the following models of set theory (Theorem 8 in [1]):
1. $\mu = \mathfrak{p} = \mathfrak{b} < \mathfrak{q}$;
2. $\mathfrak{p} < \mu = \mathfrak{b} = \mathfrak{q}$;
and
3. $\mu = \mathfrak{p} = \mathfrak{q} < \mathfrak{b}$.

The author does not know whether, in general, the answer can be negative. In this regard, the following question is of interest.

Question. *Is there a model of set theory in which* $\mathfrak{p} < \mathfrak{b} < \mathfrak{q}$ *?*

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