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Research Article

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Curves in the Lorentz-Minkowski plane: elasticae, catenaries and grim-reapers

Abstract: This article is motivated by a problem posed by David A. Singer in 1999 and by the classical Euler elastic curves. We study spacelike and timelike curves in the Lorentz-Minkowski plane $\mathbb{L}^2$ whose curvature is expressed in terms of the Lorentzian pseudodistance to fixed geodesics. In this way, we get a complete description of all the elastic curves in $\mathbb{L}^2$ and provide the Lorentzian versions of catenaries and grim-reaper curves. We show several uniqueness results for them in terms of their geometric linear momentum. In addition, we are able to get arc-length parametrizations of all the aforementioned curves and they are depicted graphically.

Keywords: Lorentzian curves, Curvature, Elastic curves, Catenaries, Grim-reaper curves

MSC: 53A35, 14H50, 53B30, 74H05

1 Introduction

The motivation of this article is the following problem posed by David A. Singer in [1]:

Can a plane curve be determined if its curvature is given in terms of its position?

Probably, the most interesting solution to this question corresponds to the classical Euler elastic curves (cf. [2] for instance), whose curvature is proportional to one of the coordinate functions. Singer himself proved (see Theorem 3.1 in [1]) that the problem of determining a curve α whose curvature is $\kappa(r)$, where r is the distance from the origin, is solvable by quadratures when $r\kappa(r)$ is a continuous function. But even the simple case $\kappa(r) = r$ studied in [1], where elliptic integrals appear, illustrated the following fact: although the corresponding differential equation is integrable by quadratures, it does not imply that the integrations are easy to perform. There are many interesting papers devoted to studying particular cases on the proposed problem of determining $\alpha = (x, y)$ given $\kappa = \kappa(r)$: for example [3–9]. In addition, the authors studied recently the cases $\kappa = \kappa(y)$ and $\kappa = \kappa(r)$ in [10] and [11] respectively, for a large number of prescribed curvature functions.

The aim of this paper is the study of Singer's problem in the Lorentz-Minkowski plane; that is, to try to determine those curves $\gamma = (x, y)$ in $\mathbb{L}^2$ whose curvature κ depends on some given function $\kappa = \kappa(x, y)$. We must focus on spacelike and timelike curves, since the curvature κ is in general not well defined on lightlike

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points, and because lightlike curves in $\mathbb{L}^2$ are segments parallel to the straight lines determining the light cone. When the ambient space is $\mathbb{L}^2$, our knowledge is more restricted in comparison with the Euclidean case. Even though the fundamental existence and uniqueness theorem, which states that a spacelike or timelike curve is uniquely determined (up to Lorentzian transformations) by its curvature given as a function of its arc-length, is still valid. Eventually, it is very difficult to find the curve explicitly in practice and most cases become elusive. In fact, we can only mention the articles [12, 13] in this line, both devoted to Sturmian spiral curves. We should remark that although some families of space curves in the Lorentz-Minkowski space $\mathbb{L}^3$ (helices, Bertrand and Mannheim curves) are well studied as in Euclidean case, the papers in the pseudo-Euclidean plane are limited, to our knowledge (see [14, 15] for example).

From a geometric-analytic point of view, we deal with the following case of Singer's problem in the Lorentzian setting (see Section 2 for details): For a unit speed parametrization of a spacelike or timelike curve $\gamma = (x, y)$ in $\mathbb{L}^2$, we prescribe its curvature with the analytic extrinsic condition $\kappa = \kappa(y)$ or $\kappa = \kappa(x)$. Obviously, if one writes the curve γ as the graphs $x = f(y)$ or $y = f(x)$ locally, the above condition is satisfied. But we aim to study the problem from a different point of view: We offer a geometric interpretation of these conditions on κ in terms of the Lorentzian pseudodistance to spacelike or timelike fixed geodesics, and would like to determine the analytic representation of the arc-length parametrization $\gamma(s)$ explicitly and, consequently, its intrinsic equation $\kappa = \kappa(s)$.

Singer's proof of the aforementioned Theorem 3.1 in [1] is based on giving such a curvature $\kappa = \kappa(r)$ an interpretation of a central potential in the plane and finding the trajectories by the standard methods in classical mechanics. On the other hand, since the curvature κ may be also interpreted as the tension that γ receives at each point as a consequence of the way it is immersed in $\mathbb{L}^2$, we make use of the notion of *geometric linear momentum* of γ when $\kappa = \kappa(y)$ or $\kappa = \kappa(x)$ in order to get two abstract integrability results (Theorems 2.1 and 2.6) in the same spirit of Theorem 3.1 in [1]. We show that the problem of determining such a curve is solvable by three quadratures if $\kappa = \kappa(y)$ or $\kappa = \kappa(x)$ is a continuous function. In addition, the geometric linear momentum turns to be a primitive function of the curvature and determines uniquely such a curve (up to translations in x-direction or in y-direction respectively). In general, one finds great difficulties (see Remark 2.5) in carrying out the computations in most cases. Hence we focus on finding interesting curves for which the required computations can be achieved *explicitly*, in terms of standard functions, and we pay attention to identifying, computing and plotting such examples.

In this way, we are first successful in the complete description of all the spacelike and timelike *elastic* curves in the Lorentz-Minkowski plane. Elastic curves in Euclidean plane were already classified by Euler in 1743. The classification problem of elastic curves and its generalizations in real space forms has been considered through different approaches (see [16–20], etc.) But in $\mathbb{L}^2$ only the elastic Sturmian spirals recently studied in [13] were known to us. In Section 3, we characterize most of the spacelike and timelike elastic curves in $\mathbb{L}^2$ —according to Singer's problem— by the condition $\kappa(y) = 2\lambda y + \mu$, $\lambda > 0$, $\mu \in \mathbb{R}$, and this allows us their explicit description by arc-length parametrizations in terms of Jacobi elliptic functions.

Moreover, we find out the Lorentzian versions of some interesting classical curves in the Euclidean context. Specifically, we study the generatrix curves of the maximal catenoids of the first and the second kind described in [21] in Section 4, which we will call *Lorentzian catenaries*. We also consider curves that satisfy a translating-type soliton equation in Section 5, which we will call *Lorentzian grim-reapers* (see [22]). We provide uniqueness results for both of them in terms of their geometric linear momentum (Corollaries 4.1, 5.1, 6.1, 6.2 and 6.3). We also generalize them by describing all the spacelike and timelike curves in $\mathbb{L}^2$ whose curvature satisfies $\kappa(y) = \lambda/y^2$, $\lambda > 0$, and $\kappa(y) = \lambda e^y$, $\lambda > 0$.

In [23], we afford two other cases of Singer's problem in the Lorentz-Minkowski plane: the spacelike and timelike curves in $\mathbb{L}^2$ whose curvature depends on the Lorentzian pseudodistance from the origin, and those ones whose curvature depends on the Lorentzian pseudodistance through the horizontal or vertical geodesic to a fixed lightlike geodesic.

2 Spacelike and timelike curves in Lorentz-Minkowski plane

We denote by $\mathbb{L}^2 := (\mathbb{R}^2, g = -dx^2 + dy^2)$ the Lorentz-Minkowski plane, where (x, y) are the rectangular coordinates on $\mathbb{L}^2$. We say that a non-zero vector $v \in \mathbb{L}^2$ is spacelike if $g(v, v) > 0$, lightlike if $g(v, v) = 0$, and timelike if $g(v, v) < 0$.

Let $\gamma = (x, y) : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a curve. We say that $\gamma = \gamma(t)$ is spacelike (resp. timelike) if the tangent vector $\gamma'(t)$ is spacelike (resp. timelike) for all $t \in I$. A point $\gamma(t)$ is called a lightlike point if $\gamma'(t)$ is a lightlike vector. We study geometric properties of curves that have no lightlike points in this section, because the curvature is not in general well defined at the lightlike points.

Hence, let $\gamma = (x, y)$ be a spacelike (resp. timelike) curve parametrized by arc-length; that is, $g(\dot{\gamma}(s), \dot{\gamma}(s)) = 1$ (resp. $g(\dot{\gamma}(s), \dot{\gamma}(s)) = -1$) $\forall s \in I$, where I is some interval in $\mathbb{R}$. Here $\dot{}$ means derivation with respect to s. We will say that $\gamma = \gamma(s)$ is a unit-speed curve in both cases.

We define the Frenet dihedron in such a way that the curvature κ has a sign and then it is only preserved by direct rigid motions (see [14]): Let $T = \dot{\gamma} = (\dot{x}, \dot{y})$ be the tangent vector to the curve γ and we choose $N = \dot{\gamma}^\perp = (\dot{y}, \dot{x})$ as the corresponding normal vector. We remark that T and N have different causal character. Let $g(T, T) = \epsilon$, with $\epsilon = 1$ if γ is spacelike, and $\epsilon = -1$ if γ is timelike. Then $g(N, N) = -\epsilon$. The (signed) curvature of γ is the function $\kappa = \kappa(s)$ such that

$$\dot{T}(s) = \kappa(s)N(s), \tag{1}$$

where

$$\kappa(s) = -\epsilon g(\dot{T}(s), N(s)) = \epsilon(\ddot{x}\dot{y} - \dot{x}\ddot{y}). \tag{2}$$

The Frenet equations of γ are given by (1) and

$$\dot{N}(s) = \kappa(s)T(s). \tag{3}$$

It is possible, as it happens in the Euclidean case, to obtain a parametrization by arc-length of the curve γ in terms of integrals of the curvature. Specifically, any spacelike curve $\alpha(s)$ in $\mathbb{L}^2$ can be represented by

$$\alpha(s) = \left(\int \sinh \varphi(s) ds, \int \cosh \varphi(s) ds \right), \quad \frac{d\varphi(s)}{ds} = \kappa(s), \tag{4}$$

and any timelike curve $\beta(s)$ in $\mathbb{L}^2$ can be represented by

$$\beta(s) = \left(\int \cosh \phi(s) ds, \int \sinh \phi(s) ds \right), \quad \frac{d\phi(s)}{ds} = \kappa(s). \tag{5}$$

For example, up to a translation, any spacelike geodesic can be written as

$$\alpha_{\varphi_0}(s) = (\sinh \varphi_0\, s, \cosh \varphi_0\, s), \quad s \in \mathbb{R}, \quad \varphi_0 \in \mathbb{R}, \tag{6}$$

and any timelike geodesic can be written as

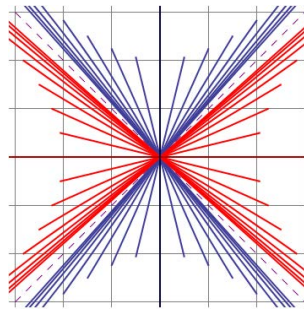$$\beta_{\phi_0}(s) = (\cosh \phi_0\, s, \sinh \phi_0\, s), \quad s \in \mathbb{R}, \quad \phi_0 \in \mathbb{R}. \tag{7}$$

On the other hand, the transformation $R_\nu : \mathbb{L}^2 \to \mathbb{L}^2$, $\nu \in \mathbb{R}$, given by

$$R_\nu(x, y) = (\cosh \nu\, x + \sinh \nu\, y, \sinh \nu\, x + \cosh \nu\, y)$$

is an isometry of $\mathbb{L}^2$ that preserves the curvature of a curve γ and satisfies

$$R_\nu \circ \alpha_{\varphi_0} = \alpha_{\varphi_0 + \nu}, \quad R_\nu \circ \beta_{\phi_0} = \beta_{\phi_0 + \nu}.$$

In this way, any spacelike geodesic is congruent to α_0, i.e. the y-axis, and any timelike geodesic is congruent to β_0, i.e. the x-axis (see Figure 1).

Fig. 1. Spacelike (blue) and timelike (red) geodesics in $\mathbb{L}^2$.

## 2.1 Curves in $\mathbb{L}^2$ such that $\kappa = \kappa(y)$

Given a spacelike or timelike curve $\gamma = (x, y)$ in $\mathbb{L}^2$, we are first interested in the analytical condition $\kappa = \kappa(y)$. We look for its geometric interpretation. For this purpose, we define the *Lorentzian pseudodistance* by

$$\delta : \mathbb{L}^2 \times \mathbb{L}^2 \to [0, +\infty), \;\; \delta(P, Q) = \sqrt{|g(\overrightarrow{PQ}, \overrightarrow{PQ})|}.$$

We fix the timelike geodesic β_0, i.e. the x-axis. Given an arbitrary point $P = (x, y) \in \mathbb{L}^2$, $y \neq 0$, we consider all the spacelike geodesics α_m with slope $m = \coth \varphi_0$, $|m| > 1$, passing through P, and let $P' = (x - y/m, 0)$ the crossing point of α_m and the x-axis (see Figure 2). Then:

$$0 < \delta(P, P')^2 = \left(1 - \frac{1}{m^2}\right) y^2 = \frac{y^2}{\cosh^2 \varphi_0} \leq y^2$$

and the equality holds if and only if $\varphi_0 = 0$, that is, α_m is a vertical geodesic. Thus $|y|$ is the maximum Lorentzian pseudodistance through spacelike geodesics from $P = (x, y) \in \mathbb{L}^2$, $y \neq 0$, to the timelike geodesic given by the x-axis.

Fig. 2. Spacelike geodesics in $\mathbb{L}^2$ passing through P.
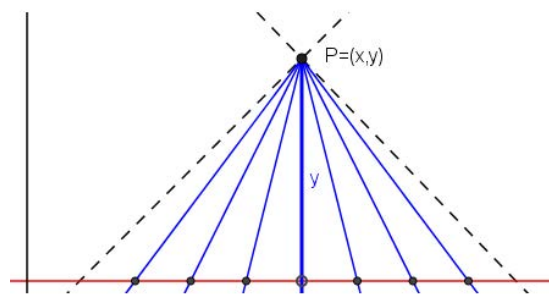
At a given point $\gamma(s) = (x(s), y(s))$ on the curve, the *geometric linear momentum (with respect to the x-axis)* $\mathcal{K}$ is given by

$$\mathcal{K}(s) = \dot{x}(s). \tag{8}$$

In physical terms, using Noether's Theorem, $\mathcal{K}$ may be interpreted as the linear momentum with respect to the x-axis of a particle of unit mass with unit-speed and trajectory γ.

Given that γ is unit-speed, that is, $-\dot{x}^2 + \dot{y}^2 = \epsilon$, and (8), we easily obtain that

$$ds = \frac{dy}{\sqrt{\epsilon + \dot{x}^2}} = \frac{dy}{\sqrt{\mathcal{K}^2 + \epsilon}}, \quad dx = \mathcal{K}ds. \tag{9}$$

Thus, given $\mathcal{K} = \mathcal{K}(y)$ as an explicit function, looking at (9) one may attempt to compute $y(s)$ and $x(s)$ in three steps: integrate to get $s = s(y)$, invert to get $y = y(s)$ and integrate to get $x = x(s)$.

In addition, we have that the curvature κ satisfies (1) and (3), i.e. $\ddot{x} = \kappa \dot{y}$. Taking into account (8), we deduce that $\frac{d\mathcal{K}}{ds} = \kappa \dot{y}$ and, since we are assuming that $\kappa = \kappa(y)$, we finally arrive at

$$d\mathcal{K} = \kappa(y)dy, \tag{10}$$

that is, $\mathcal{K} = \mathcal{K}(y)$ can be interpreted as an anti-derivative of $\kappa(y)$.

As a summary, we have proved the following result in the spirit of Theorem 3.1 in [1].

Theorem 2.1. *Let $\kappa = \kappa(y)$ be a continuous function. Then the problem of determining locally a spacelike or timelike curve in $\mathbb{L}^2$ whose curvature is $\kappa(y)$ with geometric linear momentum $\mathcal{K}(y)$ satisfying (10) —$|y|$ being the (non constant) maximum Lorentzian pseudodistance through spacelike geodesics to the x-axis— is solvable by quadratures considering the unit speed curve $(x(s), y(s))$, where $y(s)$ and $x(s)$ are obtained through (9) after inverting $s = s(y)$. Such a curve is uniquely determined by $\mathcal{K}(y)$ up to a translation in the x-direction (and a translation of the arc parameter s).*

Remark 2.2. *If we prescribe $\kappa = \kappa(y)$, the method described in Theorem 2.1 clearly implies the computation of three quadratures, following the sequence:*
 (i) *Anti-derivative of $\kappa(y)$:*
$$\int \kappa(y)dy = \mathcal{K}(y).$$

(ii) *Arc-length parameter s of $(x(s), y(s))$ in terms of y:*

$$s = s(y) = \int \frac{dy}{\sqrt{\mathcal{K}(y)^2 + \epsilon}},$$

 where $\mathcal{K}(y)^2 + \epsilon > 0$, and inverting $s = s(y)$ to get $y = y(s)$.
(iii) *First coordinate of $(x(s), y(s))$ in terms of s:*

$$x(s) = \int K(y(s))ds.$$

We note that we get a one-parameter family of curves in $\mathbb{L}^2$ satisfying $\kappa = \kappa(y)$ according to the geometric linear momentum chosen in (i). It will distinguish geometrically the curves inside the same family by their relative position with respect to the x-axis. We remark that we can recover κ from $\mathcal{K}$ simply by means of $\kappa(y) = d\mathcal{K}/dy$.

We show two illustrative examples applying steps (i)-(iii) in Remark 2.2:

Example 2.3 ($\kappa \equiv 0$). *Then $\mathcal{K} \equiv c \in \mathbb{R}$, $s = \int dy/\sqrt{c^2 + \epsilon} = y/\sqrt{c^2 + \epsilon}$, $c^2 + \epsilon > 0$. So $y(s) = \sqrt{c^2 + \epsilon}\, s$ and $x(s) = c\, s$, $s \in \mathbb{R}$. If $\epsilon = 1$, we write $K \equiv c := \sinh \varphi_0$ and then we arrive at the spacelike geodesics α_{φ_0}. We observe that $c = 0 = \varphi_0$ corresponds to the y-axis. If $\epsilon = -1$, we write $K \equiv c := \cosh \phi_0$ and then we arrive at the timelike geodesics β_{ϕ_0}. We note that $c = 1 \Leftrightarrow \phi_0 = 0$ corresponds to the x-axis. See Figure 1.*

Example 2.4 ($\kappa \equiv k_0 > 0$). *Now $\mathcal{K}(y) = k_0 y + c$, $c \in \mathbb{R}$. Thus $s = \int dy/\sqrt{(k_0 y + c)^2 + \epsilon}$. If $\epsilon = 1$, then $s = \operatorname{arcsinh}(k_0 y + c)/k_0$, $y(s) = (\sinh(k_0 s) - c)/k_0$ and $x(s) = \cosh(k_0 s)/k_0$. If $\epsilon = -1$, then $s = \operatorname{arccosh}(k_0 y + c)/k_0$, $y(s) = (\cosh(k_0 s) - c)/k_0$ and $x(s) = \sinh(k_0 s)/k_0$. They correspond respectively to spacelike and timelike pseudocircles in $\mathbb{L}^2$ of radius $1/k_0$ (see Figure 3). When $c = 0$, we obtain the branches of $x^2 - y^2 = \epsilon/k_0^2$ with positive curvature k_0, that are asymptotic to the light cone of $\mathbb{L}^2$.*

Remark 2.5. *The main difficulties one can find carrying on the strategy described in Theorem 2.1 (or in Remark 2.2) to determine a curve (x, y) in $\mathbb{L}^2$ whose curvature is $\kappa = \kappa(y)$ are the following:*
 1. *The integration of $s = s(y)$: Even in the case when $\mathcal{K}(y)$ were polynomial, the integral is not necessarily elementary. For example, when $\mathcal{K}(y)$ is a quadratic polynomial, it can be solved using Jacobian elliptic functions (see [24]). We will study such curves in Section 3.*
 2. *The previous integration gives us $s = s(y)$; it is not always possible to obtain explicitly $y = y(s)$, what is necessary to determine the curve.*

Fig. 3. Spacelike (blue) and timelike (red) pseudocircles in $\mathbb{L}^2$ with constant positive curvature.
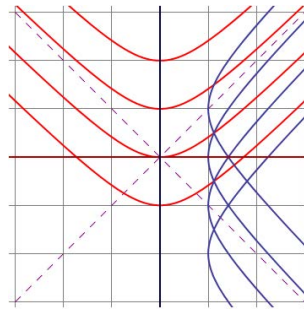
3. *Even knowing explicitly $y = y(s)$, the integration to get $x(s)$ may be impossible to perform using elementary or known functions.*

2.2 Curves in $\mathbb{L}^2$ such that $\kappa = \kappa(x)$

Given a spacelike or timelike curve $\gamma = (x, y)$ in $\mathbb{L}^2$, we are now interested in the analytical condition $\kappa = \kappa(x)$ and we search for its geometric interpretation using again the Lorentzian pseudodistance δ. We fix the spacelike geodesic α_0, i.e. the y-axis. Given an arbitrary point $P = (x, y) \in \mathbb{L}^2$, $x \neq 0$, we consider all the timelike geodesics β_m with slope $m = \tanh \phi_0$, $|m| < 1$, passing through P, and let $P' = (0, y - mx)$ the crossing point of β_m and the y-axis (see Figure 4). Then:

$$0 > -\delta(P, P')^2 = (m^2 - 1)x^2 = -\frac{x^2}{\cosh^2 \phi_0} \geq -x^2$$

and the equality holds if and only if $\phi_0 = 0$, that is, β_m is a horizontal geodesic. Thus $|x|$ is now the maximum Lorentzian pseudodistance through timelike geodesics from $P = (x, y) \in \mathbb{L}^2$, $x \neq 0$, to the spacelike geodesic given by the y-axis.

Fig. 4. Timelike geodesics in $\mathbb{L}^2$ passing through P.

$P=(x,y)$

x

Fig. 4. Timelike geodesics in $\mathbb{L}^2$ passing through P.
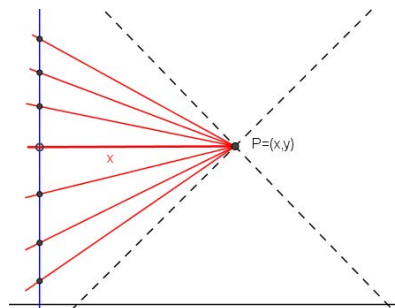
We now make a similar study to the one made in the preceding section. At a given point $\gamma(s) = (x(s), y(s))$ on the curve, the *geometric linear momentum (respect to the y-axis)* $\mathcal{K}$ is given by

$$\mathcal{K}(s) = \dot{y}(s). \tag{11}$$

In physical terms, using Noether's Theorem, $\mathcal{K}$ may be interpreted as the linear momentum with respect to the y-axis of a particle of unit mass with unit-speed and trajectory γ.

Using that γ is unit-speed, that is, $-\dot{x}^2 + \dot{y}^2 = \epsilon$, and (11), we easily obtain that

$$ds = \frac{dx}{\sqrt{\dot{y}^2 - \epsilon}} = \frac{dx}{\sqrt{\mathcal{K}^2 - \epsilon}}, \quad dy = \mathcal{K}ds. \tag{12}$$

Thus, given $\mathcal{K} = \mathcal{K}(x)$ as an explicit function, looking at (12) one may attempt to compute $x(s)$ and $y(s)$ in three steps: integrate to get $s = s(x)$, invert to get $x = x(s)$ and integrate to get $y = y(s)$.

In addition, we have that the curvature κ satisfies (1) and (3), i.e. $\ddot{y} = \kappa\dot{x}$. Taking into account (11), we deduce that $\frac{d\mathcal{K}}{ds} = \kappa\dot{x}$ and, since we are assuming that $\kappa = \kappa(x)$, we finally arrive at

$$d\mathcal{K} = \kappa(x)dx, \tag{13}$$

that is, $\mathcal{K} = \mathcal{K}(x)$ can be interpreted as an anti-derivative of $\kappa(x)$.

As a summary, we have proved the following result, dual in a certain sense to Theorem 2.1

Theorem 2.6. *Let $\kappa = \kappa(x)$ be a continuous function. Then the problem of determining locally a spacelike or timelike curve in $\mathbb{L}^2$ whose curvature is $\kappa(x)$ with geometric linear momentum $\mathcal{K}(x)$ satisfying (13) $-|x|$ being the (non constant) maximum Lorentzian pseudodistance through timelike geodesics to the y-axis— is solvable by quadratures considering the unit speed curve $(x(s), y(s))$, where $x(s)$ and $y(s)$ are obtained through (12) after inverting $s = s(x)$. Such a curve is uniquely determined by $\mathcal{K}(x)$ up to a translation in the y-direction (and a translation of the arc parameter s).*

Remark 2.7. *The duality between Theorems 2.1 and 2.6 is explained thanks to the following observation: If $\gamma = (x, y)$ is a spacelike (resp. timelike) curve in $\mathbb{L}^2$ such that $\kappa = \kappa(y)$, then $\hat{\gamma} = (y, x)$ is a timelike (resp. spacelike) curve in $\mathbb{L}^2$ such that $\kappa = \kappa(x)$.*

3 Elastic curves in the Lorentz-Minkowski plane

A unit speed spacelike or timelike curve γ in $\mathbb{L}^2$ is said to be an *elastica under tension σ* (see [25]) if it satisfies the differential equation

$$2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0, \tag{14}$$

for some value of $\sigma \in \mathbb{R}$. They are critical points of the elastic energy functional $\int_\gamma (\kappa^2 + \sigma)ds$ acting on curves in $\mathbb{L}^2$ with suitable boundary conditions. If $\sigma = 0$, then γ is called a *free elastica*. The possible constant solutions of (14) are the trivial solution $\kappa \equiv 0$ and $\kappa \equiv \sqrt{-\sigma}$, $\sigma < 0$.

Multiplying (14) by $2\dot{\kappa}$ and integration allow us to introduce the *energy $E \in \mathbb{R}$* of an elastica:

$$E := \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2. \tag{15}$$

If $E = \sigma^2/4$, (15) reduces to $\dot{\kappa}^2 = (\kappa^2/2 + \sigma/2)^2$, whose solutions are given by $\kappa(s) = \pm\sqrt{\sigma}\tan(\sqrt{\sigma}s/2)$, if $\sigma > 0$; $\kappa(s) = \pm 2/s$, if $\sigma = 0$; and $\kappa(s) = \pm\sqrt{-\sigma}\coth(\sqrt{-\sigma}s/2)$, if $\sigma < 0$. These special elasticae (see Figure 5) can be easily integrated using (4) and (5). They are studied in Section 6 of [23].

Fig. 5. Special elastic curves $\sigma = 1, 0, -1$ (blue, spacelike; red, timelike).
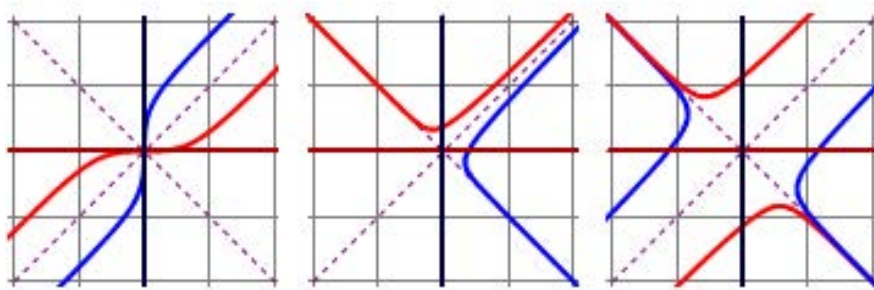
In this section we will study those spacelike and timelike curves in $\mathbb{L}^2$ satisfying

$$\kappa(y) = 2ay + b, \ a \neq 0, \ b \in \mathbb{R}, \tag{16}$$

and we will show its close relationship with the elastic curves of $\mathbb{L}^2$. Following Theorem 2.1, we must consider the geometric linear momentum $\mathcal{K}(y) = ay^2 + by + c$, $c \in \mathbb{R}$. In the following result, we show that we are studying precisely elastic curves.

Proposition 3.1. *Let γ be a spacelike or timelike curve in $\mathbb{L}^2$.*
(i) *If the curvature of γ is given by (16) with geometric linear momentum $\mathcal{K}(y) = ay^2 + by + c$, $a \neq 0$, $b, c \in \mathbb{R}$, then γ is an elastica under tension $\sigma = 4ac - b^2$ and energy $E = 4\epsilon a^2 + \sigma^2/4$ (where $\epsilon = 1$ if γ is spacelike and $\epsilon = -1$ if γ is timelike).*
(ii) *If γ is an elastica under tension σ and energy E, with $E \neq \sigma^2/4$, then the curvature of γ is given by (16).*

Proof. Without restriction we can consider γ parametrized by arc-length. Assume first that $\mathcal{K}(y) = ay^2 + by + c$, $a \neq 0$, $b, c \in \mathbb{R}$. In order γ to be an elastica, we must check that κ given by (16) satisfies (14) or (15). We have that $\dot{\kappa}^2 = 4a^2\dot{y}^2$ and (9) implies that $\dot{y}^2 = \mathcal{K}^2 + \epsilon = (ay^2 + bc + c)^2 + \epsilon$. By substituting $y = (\kappa - b)/2a$, it is a long exercise to check (15) taking $\sigma = 4ac - b^2$ and $E = 4\epsilon a^2 + \sigma^2/4$. We observe that $E - \sigma^2/4 \neq 0$.

Conversely, assume that γ is an elastica under tension σ and energy E. We write locally γ as a graph $x = x(y)$ and then $\kappa = \kappa(y)$ obviously. Using Theorem 2.1, $\dot{\kappa}^2 = \kappa'(y)^2\dot{y}^2 = \mathcal{K}''(y)^2(\mathcal{K}(y)^2 + \epsilon)$. Then (15) translates into

$$(\mathcal{K}(y)^2 + \epsilon)\mathcal{K}''(y)^2 = \mathcal{K}'(y)^4/4 + \sigma\mathcal{K}'(y)^2/2 + E. \tag{17}$$

If $E \neq \sigma^2/4$, we can define $a^2 = \frac{E - \sigma^2/4}{4\epsilon}$ by considering $\epsilon = \pm 1$ according to the sign of $E - \sigma^2/4$. In addition, we can take $b, c \in \mathbb{R}$ such that $\sigma = 4ac - b^2$. After a long straightforward computation, it is not difficult to check now that $\mathcal{K}(y) = ay^2 + by + c$ satisfies (17), what finishes the proof. □

Given $\gamma = (x, y)$ satisfying (16) with $a > 0$ without restriction, we take $\hat{\gamma} = \sqrt{a}(x, y + b/2a)$ and then, up to a translation in the y-direction and a dilation, we can only afford the condition

$$\kappa(y) = 2y. \tag{18}$$

The trivial solution $\kappa \equiv 0$ to (14) corresponds in (18) to the x-axis $y \equiv 0$.

Following the strategy described in Remark 2.2, we can control the spacelike or timelike curves $(x(s), y(s))$ in $\mathbb{L}^2$ satisfying (18) with geometric linear momentum $\mathcal{K}(y) = y^2 + c$, $c \in \mathbb{R}$, by means of

$$s = s(y) = \int \frac{dy}{\sqrt{y^4 + 2cy^2 + c^2 + \epsilon}} \tag{19}$$

and

$$x(s) = \int (y(s)^2 + c) ds \tag{20}$$

with $c \in \mathbb{R}$. The integrations of (19) and (20) involve Jacobi elliptic functions and elliptic integrals.

3.1 Spacelike elastic curves in the Lorentz-Minkowski plane

We take $\epsilon = 1$ and put $c = \sinh\eta$, $\eta \in \mathbb{R}$. Then these spacelike elastic curves will have energy $E = 4\cosh^2\eta$ (see Proposition 3.1). Now (19) is rewritten as

$$s = s(y) = \int \frac{dy}{\sqrt{(y^2 + \sinh\eta + i)(y^2 + \sinh\eta - i)}}. \tag{21}$$

After a long computation, using formulas 225.00 and 129.04 of [24] and abbreviating $c_\eta := \cosh\eta$, we finally get that

$$y_\eta(s) = \sqrt{c_\eta}\,\mathrm{cs}(\sqrt{c_\eta}\,s, k_\eta)\,\mathrm{nd}(\sqrt{c_\eta}\,s, k_\eta), \ k_\eta^2 = \frac{1 - \tanh\eta}{2} \tag{22}$$

with $s \in (2mK(k_\eta)/\sqrt{c_\eta}, 2(m + 1)K(k_\eta)/\sqrt{c_\eta})$, $m \in \mathbb{N}$, that is expressed in terms of the Jacobian elliptic functions $cs(\cdot, k_\eta)$ and $nd(\cdot, k_\eta)$ of modulus k_η, where $K(k_\eta)$ denotes the complete elliptic integral of the first kind of modulus k_η (see [24]). Using (22) in (20), formula 361.20 of [24] leads to

$$x_\eta(s) = (s_\eta + c_\eta)s + \tag{23}$$
$$\sqrt{c_\eta}\left(cn(\sqrt{c_\eta}\,s, k_\eta)\left(k_\eta^2\,sd(\sqrt{c_\eta}\,s, k_\eta) - ds(\sqrt{c_\eta}\,s, k_\eta)\right) - 2E(\sqrt{c_\eta}\,s, k_\eta)\right),$$

where $s_\eta := \sinh\eta$ and $cn(\cdot, k_\eta)$, $sd(\cdot, k_\eta)$ and $ds(\cdot, k_\eta)$ are Jacobian elliptic functions and $E(\cdot, k_\eta)$ denotes the elliptic integral of the second kind of modulus k_η (see [24]). We remark that, using (18), the intrinsic equations of the curves $\alpha_\eta = (x_\eta, y_\eta)$, $\eta \in \mathbb{R}$ (see Figure 6), are given by

$$\kappa_\eta(s) = 2\sqrt{c_\eta}\,cs(\sqrt{c_\eta}\,s, k_\eta)\,nd(\sqrt{c_\eta}\,s, k_\eta).$$

Fig. 6. Spacelike elastic curves α_η ($\eta = 0, 1.5, -1.5$).
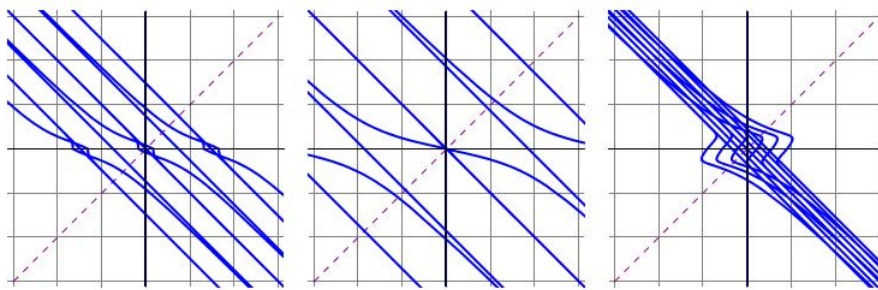
3.2 Timelike elastic curves in the Lorentz-Minkowski plane

When we take $\epsilon = -1$ in (19), we have that

$$s = s(y) = \int \frac{dy}{\sqrt{y^4 + 2cy^2 + c^2 - 1}} \tag{24}$$

and then we must distinguish five cases:

1. $c = 1$, i.e. $\mathcal{K}(y) = y^2 + 1$.
2. $c = -1$, i.e. $\mathcal{K}(y) = y^2 - 1$.
3. $c > 1$; put $c := \cosh^2\delta$, $\delta > 0$, and so $\mathcal{K}(y) = y^2 + \cosh^2\delta$.
4. $|c| < 1$: put $c := \sin\psi$, $-\pi/2 < \psi < \pi/2$, and so $\mathcal{K}(y) = y^2 + \sin\psi$.
5. $c < -1$: put $c := -\cosh^2\tau$, $\tau > 0$, and so $\mathcal{K}(y) = y^2 - \cosh^2\tau$.

3.2.1 Timelike elastic curves in $\mathbb{L}^2$ with $\mathcal{K}(y) = y^2 \pm 1$

In these cases, (24) becomes elementary and both of them produce timelike elastic curves with null energy (see Proposition 3.1). A straightforward computation, using (24) and (20), provides us the only (up to translations in the x-direction) timelike elastic curve $(x_1(s)), y_1(s)$ with geometric linear momentum $\mathcal{K}(y) = y^2 + 1$ (see Figure 7, left), given by

$$y_1(s) = -\frac{\sqrt{2}}{\sinh(\sqrt{2}s)}, \quad x_1(s) = s - \sqrt{2}\coth(\sqrt{2}s), \quad s \neq 0,$$

and the only (up to translations in the x-direction) timelike elastic curve $(x_{-1}(s)), y_{-1}(s)$ with geometric linear momentum $\mathcal{K}(y) = y^2 - 1$ (see Figure 7, right), given by
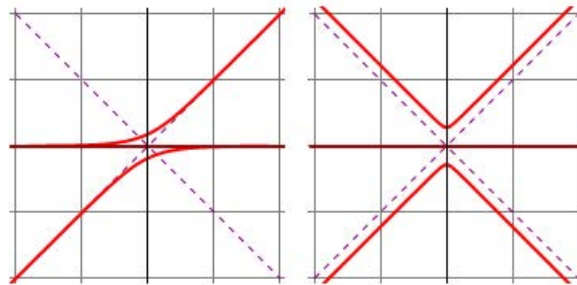
$$y_{-1}(s) = \pm \frac{\sqrt{2}}{\cos(\sqrt{2}s)}, \; x_{-1}(s) = \sqrt{2}\tan(\sqrt{2}s) - s, \; |s| < \frac{\pi}{2\sqrt{2}}.$$

Using (18), the intrinsic equations of these curves are given by

$$\kappa_1(s) = -\frac{2\sqrt{2}}{\sinh(\sqrt{2}s)}, \; \kappa_{-1}(s) = \frac{\mp 2\sqrt{2}}{\cos(\sqrt{2}s)},$$

respectively.

Fig. 7. Timelike elastic curves with $\mathcal{K}(y) = y^2 \pm 1$.

3.2.2 Timelike elastic curves in $\mathbb{L}^2$ with $\mathcal{K}(y) = y^2 + \cosh^2 \delta, \delta > 0$

Since $c = \cosh^2 \delta$ in this case, these timelike elastic curves will have energy $E = 4\sinh^2 \delta(\cosh^2 \delta + 1) > 0$ (see Proposition 3.1) and we can write (24) simply as

$$s = s(y) = \int \frac{dy}{\sqrt{(y^2 + \sinh^2 \delta)(y^2 + \cosh^2 \delta + 1)}}. \tag{25}$$

Using formula 221.00 of [24] and abbreviating $c_\delta := \cosh \delta$ and $s_\delta := \sinh \delta$, we obtain that

$$y_\delta(s) = s_\delta \, \mathrm{tn}(\sqrt{c_\delta^2 + 1}\, s, k_\delta), \; k_\delta^2 = \frac{2}{1 + \cosh^2 \delta}, \tag{26}$$

with $s \in \left((2m - 1)K(k_\delta)/\sqrt{c_\delta^2 + 1}, (2m + 1)K(k_\delta)/\sqrt{c_\delta^2 + 1}\right)$, $m \in \mathbb{N}$, that is expressed in terms of the Jacobian elliptic function $\mathrm{tn}(\cdot, k_\delta)$ of modulus k_δ, where $K(k_\delta)$ denotes the complete elliptic integral of the first kind of modulus k_δ (see [24]). Using (26) in (20), formula 316.02 of [24] leads to

$$x_\delta(s) = c_\delta^2 s + \sqrt{c_\delta^2 + 1}\left(\mathrm{dn}(\sqrt{c_\delta^2 + 1}\, s, k_\delta)\right. \tag{27}$$

$$\left.\mathrm{tn}(\sqrt{c_\delta^2 + 1}\, s, k_\delta) - E(\sqrt{c_\delta^2 + 1}\, s, k_\delta)\right),$$

where $\mathrm{dn}(\cdot, k_\delta)$ is a Jacobian elliptic function and $E(\cdot, k_\delta)$ denotes the elliptic integral of the second kind of modulus k_δ (see [24]). We point out that, using (18), the intrinsic equations of the curves $\beta_\delta = (x_\delta, y_\delta), \delta > 0$ (see Figure 8), are given by

$$\kappa_\delta(s) = 2s_\delta \, \mathrm{tn}(\sqrt{c_\delta^2 + 1}\, s, k_\delta).$$

Fig. 8. Timelike elastic curves β_δ ($\delta = 0.5, 1, 1.5$).
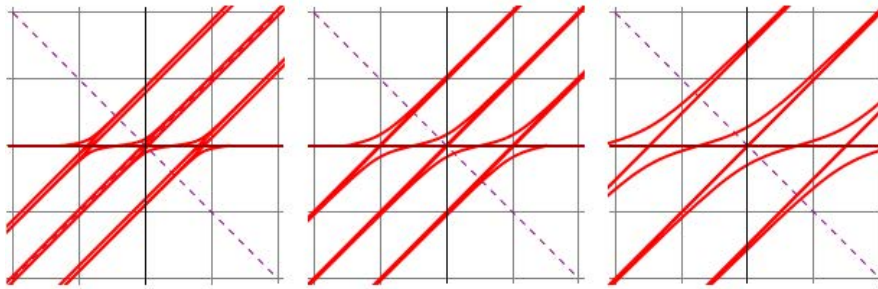
3.2.3 Timelike elastic curves in $\mathbb{L}^2$ with $\mathcal{K}(y) = y^2 + \sin\psi$, $|\psi| < \pi/2$

As $c = \sin\psi$ in this case, these timelike elastic curves will have energy $E = -4\cos^2\psi < 0$ (see Proposition 3.1) and (24) can be written as

$$s = s(y) = \int \frac{dy}{\sqrt{(y^2 + \sin\psi + 1)(y^2 + \sin\psi - 1)}}. \tag{28}$$

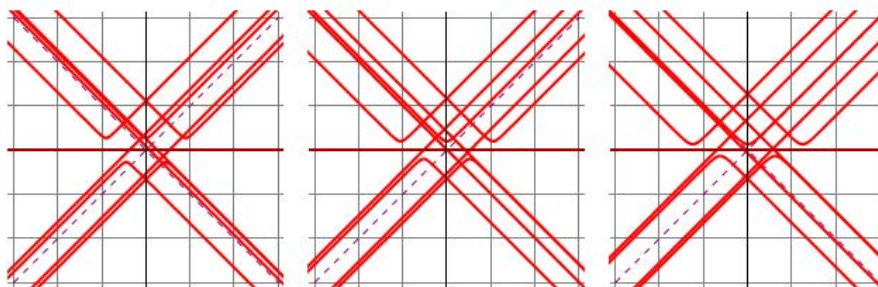Using formula 211.00 of [24] and abbreviating $s_\psi := \sin\psi$, we deduce that

$$y_\psi(s) = \sqrt{1 - s_\psi}\, \mathrm{nc}(\sqrt{2}\,s, k_\psi), \quad k_\psi^2 = \frac{1 + \sin\psi}{2}, \tag{29}$$

with $s \in \left((2m - 1)K(k_\psi)/\sqrt{2}, (2m + 1)K(k_\psi)/\sqrt{2}\right)$, $m \in \mathbb{N}$, that is expressed in terms of the Jacobian elliptic function $\mathrm{nc}(\cdot, k_\psi)$ of modulus k_ψ, where $K(k_\psi)$ denotes the complete elliptic integral of the first kind of modulus k_ψ (see [24]). Using (29) in (20), formula 313.02 of [24] leads to

$$x_\psi(s) = s + \sqrt{2}\left(\mathrm{dn}(\sqrt{2}\,s, k_\psi)\,\mathrm{tn}(\sqrt{2}\,s, k_\psi) - E(\sqrt{2}\,s, k_\psi)\right), \tag{30}$$

where $\mathrm{dn}(\cdot, k_\psi)$ and $\mathrm{tn}(\cdot, k_\psi)$ are Jacobian elliptic function and $E(\cdot, k_\psi)$ denotes the elliptic integral of the second kind of modulus k_ψ (see [24]). We point out that, using (18), the intrinsic equations of the curves $\beta_\psi = (x_\psi, y_\psi)$, $|\psi| < \pi/2$ (see Figure 9), are given by

$$\kappa_\psi(s) = 2\sqrt{1 - s_\psi}\, \mathrm{nc}(\sqrt{2}\,s, k_\psi).$$

Fig. 9. Timelike elastic curves β_ψ ($\psi = -\pi/4, 0, \pi/6$).

3.2.4 Timelike elastic curves in $\mathbb{L}^2$ with $\mathcal{K}(y) = y^2 - \cosh^2 \tau$, $\tau > 0$

When $c = -\cosh^2 \tau$, these timelike elastic curves will have energy $E = 4\sinh^2 \tau(\cosh^2 \tau + 1) > 0$ (see Proposition 3.1) and we can write (24) simply as

$$s = s(y) = \int \frac{dy}{\sqrt{(y^2 - \sinh^2 \tau)(y^2 - (\cosh^2 \delta + 1))}}. \tag{31}$$

Using formula 216.00 of [24] and abbreviating $c_\tau := \cosh \tau$, after a long straightforward computation we arrive at

$$y_\tau(s) = \sqrt{1 + c_\tau^2}\ \mathrm{dc}(\sqrt{1 + c_\tau^2}\ s, k_\tau), \quad k_\tau^2 = \frac{\sinh^2 \tau}{1 + \cosh^2 \tau} \tag{32}$$

with $s \in \left((2m - 1)K(k_\tau)/\sqrt{1 + c_\tau^2}, (2m + 1)K(k_\tau)/\sqrt{1 + c_\tau^2}\right)$, $m \in \mathbb{N}$, that is expressed in terms of the Jacobian elliptic function $\mathrm{dc}(\cdot, k_\tau)$ of modulus k_τ, where $K(k_\tau)$ denotes the complete elliptic integral of the first kind of modulus k_τ (see [24]). Using (32) in (20), formula 321.02 of [24] leads to

$$x_\tau(s) = s + \sqrt{1 + c_\tau^2}\left(\mathrm{dn}(\sqrt{1 + c_\tau^2}\ s, k_\tau)\ \mathrm{tn}(\sqrt{1 + c_\tau^2}\ s, k_\tau) - E(\sqrt{1 + c_\tau^2}\ s, k_\tau)\right), \tag{33}$$

where $\mathrm{dn}(\cdot, k_\tau)$ and $\mathrm{tn}(\cdot, k_\tau)$ are Jacobian elliptic function and $E(\cdot, k_\tau)$ denotes the elliptic integral of the second kind of modulus k_τ (see [24]). We point out that, using (18), the intrinsic equations of the curves $\beta_\tau = (x_\tau, y_\tau)$, $\tau > 0$ (see Figure 10), are given by

$$\kappa_\tau(s) = 2\sqrt{1 + c_\tau^2}\ \mathrm{dc}(\sqrt{1 + c_\tau^2}\ s, k_\tau).$$

Fig. 10. Timelike elastic curves β_τ ($\tau = 1, 2, 3$).
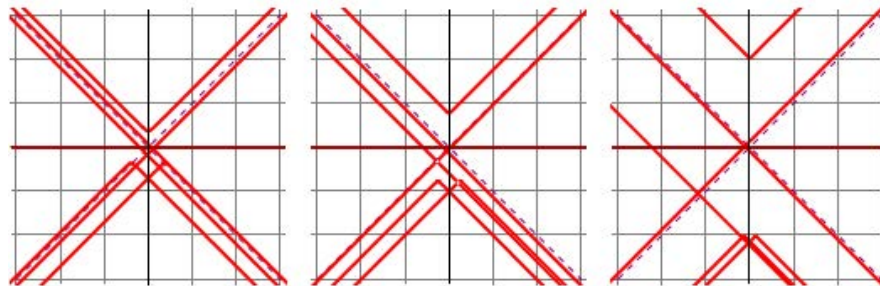
4 Curves in $\mathbb{L}^2$ such that $\kappa(y) = \lambda/y^2$, $\lambda > 0$

In this section we will study those spacelike and timelike curves in $\mathbb{L}^2$ satisfying

$$\kappa(y) = \lambda/y^2, \lambda > 0, \tag{34}$$

and we will show that some of them can be considered as Lorentzian versions of catenaries in $\mathbb{L}^2$ (see Section 4 in [10]). Given $\gamma = (x, y)$ satisfying (34), if we take $\hat{\gamma} = \frac{1}{\lambda}(x, y)$ then, up to a dilation, we can only afford the condition

$$\kappa(y) = 1/y^2, \tag{35}$$

with $y \neq 0$. Following Theorem 2.1, we must consider the geometric linear momentum $\mathcal{K}(y) = c - 1/y$, $c \in \mathbb{R}$.

4.1 Case $\mathcal{K}(y) = -1/y$. *Lorentzian catenaries*

We follow the steps described in Remark 2.2 and so

$$s = \int \frac{dy}{\sqrt{1/y^2 + \epsilon}} = \int \frac{y\,dy}{\sqrt{1 + \epsilon y^2}} = \epsilon\sqrt{1 + \epsilon y^2}.$$
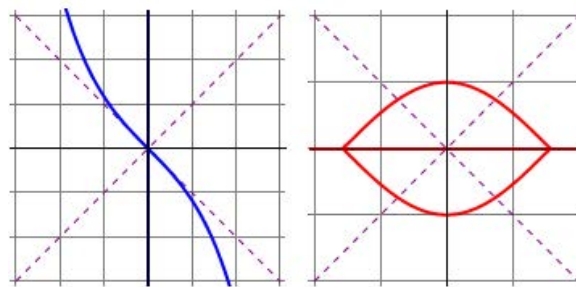
Then $s^2 = 1 + \epsilon y^2$, and hence

$$\epsilon = 1 : y(s) = \pm\sqrt{s^2 - 1}, \, |s| > 1; \quad \epsilon = -1 : y(s) = \pm\sqrt{1 - s^2}, \, |s| < 1.$$

Consequently, recalling that $\mathcal{K}(y) = -1/y$, we get:

$$\epsilon = 1 : x(s) = \mp\operatorname{arccosh} s, \, s > 1; \quad \epsilon = -1 : x(s) = \mp\arcsin s, \, |s| < 1.$$

We arrive at the graphs $y = -\sinh x$, $x \in \mathbb{R}$, at the spacelike case $\epsilon = 1$, and $y = \pm\cos x$, $|x| < \pi/2$, at the timelike case $\epsilon = -1$ (see Figure 11). Using (35), their intrinsic equations are given by $\kappa(s) = \frac{1}{s^2-1}$, $s > 1$, and $\kappa(s) = \frac{1}{1-s^2}$, $|s| < 1$, respectively.

Fig. 11. Curves with $\mathcal{K}(y) = -1/y$; spacelike (left), timelike (right).

On the other hand, Kobayashi introduced in [21], by studying maximal rotation surfaces in $\mathbb{L}^3 := (\mathbb{R}^3, -dx^2 + dy^2 + dz^2)$, a couple of catenoids. Specifically, Example 2.5 in [21] presents (up to dilations) the catenoid of the first kind by the equation $y^2 + z^2 - \sinh^2 x = 0$ and we can deduce the equation $x^2 - z^2 = \cos^2 y$ for the catenoid of the second kind given (up to dilations) in Example 2.6 in [21] (see Figure 12).

Fig. 12. Catenoid of the first kind (left) and the second kind (right) in $\mathbb{L}^3$.
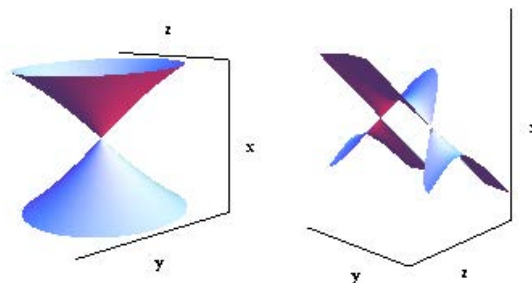
The generatrix curves (in a certain sense) of both catenoids will be referred as *Lorentzian catenaries*. Specifically, we call the graph $y = -\sinh x$, $x \in \mathbb{R}$, the Lorentzian catenary of the first kind, and the bigraph $x = \pm\cos y$, $|y| < \pi/2$, the Lorentzian catenary of the second kind. As a summary of this section, taking into account Remark 2.7, we conclude with the following geometric characterization of them.

Corollary 4.1.

(i) *The Lorentzian catenary of the first kind $y = -\sinh x$, $x \in \mathbb{R}$, is the only spacelike curve (up to translations in the x-direction) in $\mathbb{L}^2$ with geometric linear momentum $\mathcal{K}(y) = -1/y$.*

(ii) *The Lorentzian catenary of the second kind $x = \pm\cos y$, $|y| < \pi/2$, is the only spacelike curve (up to translations in the y-direction) in $\mathbb{L}^2$ with geometric linear momentum $\mathcal{K}(x) = -1/x$.*

4.2 Case $\mathcal{K}(y) = c - 1/y$, $c \neq 0$

When $c \neq 0$, it is difficult to get the arc parameter s as a function of y; however, we can eliminate ds using parts (ii) and (iii) in Remark 2.2, obtaining

$$x = x(y) = \int \frac{(cy - 1)dy}{\sqrt{P(y)}}, \quad P(y) = (c^2 + \epsilon)y^2 - 2cy + 1.$$
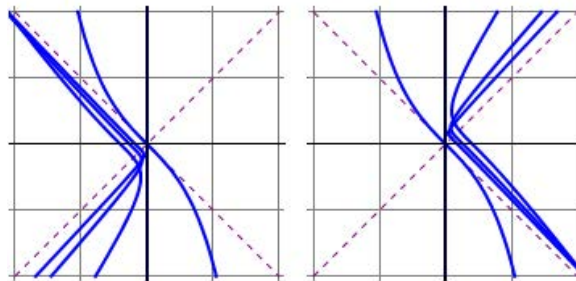
If $c = 0$, we easily recover the Lorentzian catenaries studied in the previous section. If $c \neq 0$, we distinguish the following cases according to the expression of polynomial $P(y)$:

1. Spacelike case ($\epsilon = 1$), $\mathcal{K}(y) = c - 1/y$:

$$x = \frac{1}{c^2 + 1}\left(c\sqrt{(c^2 + 1)y^2 - 2cy + 1} - \frac{1}{\sqrt{c^2 + 1}}\operatorname{arcsinh}((c^2 + 1)y - c)\right).$$

We notice that if $c = 0$ above, we recover $x = -\operatorname{arcsinh} y$ (see Figure 13).

Fig. 13. Curves with $\mathcal{K}(y) = c - 1/y$; spacelike $c \leq 0$ (left), spacelike $c \geq 0$ (right).

2. Timelike case ($\epsilon = -1$):
   (a) $\mathcal{K}(y) = 1 - 1/y$ (see Figure 14):

$$x = \frac{(2 - y)\sqrt{1 - 2y}}{3}, \quad y < 1/2.$$

   (b) $\mathcal{K}(y) = -1 - 1/y$ (see Figure 14):

$$x = -\frac{(2 + y)\sqrt{1 + 2y}}{3}, \quad y > -1/2.$$
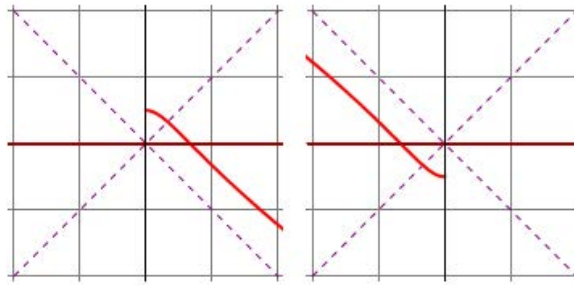
   (c) $\mathcal{K}(y) = c - 1/y$, $|c| > 1$:

$$x = \frac{1}{c^2 - 1}\left(c\sqrt{(c^2 - 1)y^2 - 2cy + 1} + \right.$$

$$\left. + \frac{1}{\sqrt{c^2 - 1}}\log\left(2(\sqrt{c^2 - 1}\sqrt{(c^2 - 1)y^2 - 2cy + 1} + (c^2 - 1)y - c)\right)\right).$$

   (d) $\mathcal{K}(y) = c - 1/y$, $|c| < 1$:

$$x = \frac{1}{c^2 - 1}\left(c\sqrt{(c^2 - 1)y^2 - 2cy + 1} - \frac{1}{\sqrt{1 - c^2}}\arcsin((c^2 - 1)y - c)\right).$$

Fig. 14. Timelike curves with $\mathcal{K}(y) = 1 - 1/y$ (left) and $\mathcal{K}(y) = -1 - 1/y$ (right).

5 Curves in $\mathbb{L}^2$ such that $\kappa(y) = \lambda e^y$, $\lambda > 0$

In this section we will study those spacelike and timelike curves in $\mathbb{L}^2$ satisfying

$$\kappa(y) = \lambda e^y, \lambda > 0, \tag{36}$$

and we will introduce what can be considered the Lorentzian versions of grim-reaper curves in $\mathbb{L}^2$ (see Section 7 in [10]). Given $\gamma = (x, y)$ satisfying (36), if we take $\hat{\gamma} = (x, y + \log \lambda)$ then, up to a translation, we can only consider the condition

$$\kappa(y) = e^y. \tag{37}$$

Following Theorem 2.1, we deal with the geometric linear momentum $\mathcal{K}(y) = e^y + c$, $c \in \mathbb{R}$.

5.1 Case $\mathcal{K}(y) = e^y$. *Lorentzian grim-reapers*

Following the steps in Remark 2.2, putting $u = e^y$, we have that

$$s = \int \frac{dy}{\sqrt{e^{2y} + \epsilon}} = \int \frac{du}{u\sqrt{u^2 + \epsilon}}.$$

Hence:

$$\epsilon = 1 : \ y(s) = \log(-\operatorname{csch} s), \ s < 0; \quad \epsilon = -1 : \ y(s) = \log \sec s, \ |s| < \pi/2.$$

Consequently, recalling that $\mathcal{K}(y) = e^y$, we obtain:

$$\epsilon = 1 : \ x(s) = -\log \tanh(-s/2), \ s < 0;$$

$$\epsilon = -1 : \ x(s) = \log(\sec s + \tan s), \ |s| < \pi/2.$$

A straightforward computation leads us to the graphs $y = \log(\sinh x)$, $x > 0$, at the spacelike case $\epsilon = 1$, and $y = \log(\cosh x)$, $x \in \mathbb{R}$, at the timelike case $\epsilon = -1$ (see Figure 15). Using (37), their intrinsic equations are given by $\kappa(s) = -\operatorname{csch} s$, $s < 0$, and $\kappa(s) = \sec s$, $|s| < \pi/2$, respectively.

It is straightforward to check that both curves satisfy the translating-type soliton equation $\kappa = g((0, 1), N)$. Hence we have obtained in this section (see also Section 7.1 in [10]) Lorentzian versions of the grim-reaper curves of Euclidean plane. We will simply call them *Lorentzian grim-reapers*. As a summary, we conclude with the following geometric characterization of them.

Corollary 5.1.
 (i) *The Lorentzian grim-reaper $y = \log(\sinh x)$, $x > 0$, is the only spacelike curve (up to translations in the x-direction) in $\mathbb{L}^2$ with geometric linear momentum $\mathcal{K}(y) = e^y$.*
(ii) *The Lorentzian grim-reaper $y = \log(\cosh x)$, $x \in \mathbb{R}$, is the only timelike curve (up to translations in the x-direction) in $\mathbb{L}^2$ with geometric linear momentum $\mathcal{K}(y) = e^y$.*

Fig. 15. Curves with $\mathcal{K}(y) = e^y$; spacelike (left), timelike (right).
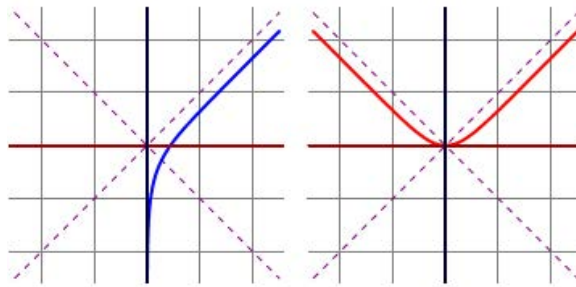
5.2 Case $\mathcal{K}(y) = e^y + c$, $c \neq 0$

When $c \neq 0$, it is longer and more difficult to get the arc parameter s as a function of y; however, we can eliminate ds using parts (ii) and (iii) in Remark 2.2. Putting $u = e^y$, we obtain:

$$x = x(y) = \int \frac{(u+c)du}{u\sqrt{P(u)}}, \quad P(u) = u^2 + 2cu + c^2 + \epsilon.$$

If $c = 0$, we recover the Lorentzian grim-reaper curves studied in the previous section. If $c \neq 0$, we distinguish the following cases according to the expression of polynomial $P(u)$:

1. Spacelike case ($\epsilon = 1$), $\mathcal{K}(y) = e^y + c$:

$$x = \operatorname{arcsinh}(e^y + c) - \frac{c}{\sqrt{c^2 + 1}} \operatorname{arcsinh}\left(c + (c^2 + 1)e^{-y}\right).$$

   We notice that if $c = 0$ above, we recover the graph $y = \log(\sinh x)$ (see Figure 16).

Fig. 16. Curves with $\mathcal{K}(y) = e^y + c$; spacelike $c \leq 0$ (left), spacelike $c \geq 0$ (right).
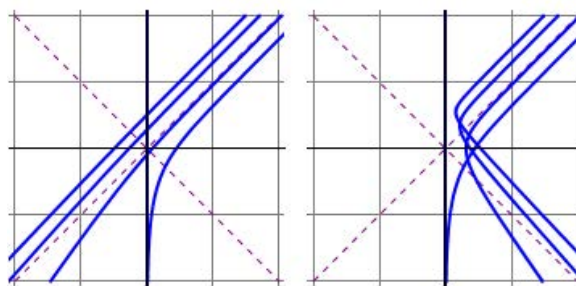
2. Timelike case ($\epsilon = -1$):
   (a) $\mathcal{K}(y) = e^y + 1$: (see Figure 17)

$$x = 2\log(\sqrt{e^y} + \sqrt{e^y + 2}) - \sqrt{1 + 2e^{-y}}, \quad y \in \mathbb{R}.$$

   (b) $\mathcal{K}(y) = e^y - 1$: (see Figure 17):

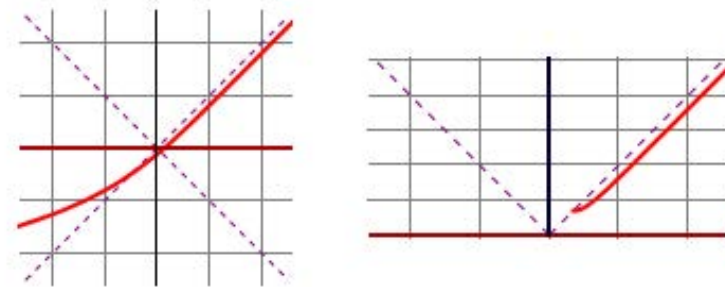$$x = 2\log(\sqrt{e^y} + \sqrt{e^y - 2}) - \sqrt{1 - 2e^{-y}}, \quad y > \log 2.$$

   (c) $\mathcal{K}(y) = e^y + c$, $|c| > 1$:

$$x = \log\left(2(\sqrt{P(e^y)} + e^y + c)\right) - \frac{c}{\sqrt{c^2 - 1}} \log\left(2e^{-y}(\sqrt{c^2 - 1}\sqrt{P(e^y)} + ce^y + c^2 - 1)\right)$$

(d) $\mathcal{K}(y) = e^y + c$, $|c| < 1$:

$$x = \log\left(2(\sqrt{P(e^y)} + e^y + c)\right) + \frac{c}{\sqrt{1 - c^2}} \arcsin\left(c + (c^2 - 1)e^{-y}\right).$$

Fig. 17. Timelike curves with $\mathcal{K}(y) = e^y + 1$ (left) and $\mathcal{K}(y) = e^y - 1$ (right).

6 Other integrable curves in $\mathbb{L}^2$

The aim of this section is to collect some interesting curves in $\mathbb{L}^2$ that can be easily determined by their geometric linear momentum, following the strategy described in Theorem 2.1 and Remark 2.2.

6.1 Timelike curves in $\mathbb{L}^2$ such that $\kappa(y) = \operatorname{csch}^2 y$

In this case, being $y \neq 0$, we only consider the geometric linear momentum $\mathcal{K}(y) = -\coth y$. Then:

$$s = \int \frac{dy}{\sqrt{\coth^2 y - 1}} = \pm \int \sinh y \, dy = \pm \cosh y.$$

Thus:

$$y(s) = \pm \operatorname{arccosh} s, \ s > 1,$$

and

$$x(s) = -\int \coth y(s) ds = \mp \int \frac{s \, ds}{\sqrt{s^2 - 1}} = \mp \sqrt{s^2 - 1}, \ s > 1.$$

We arrive at the graph $x = -\sinh y$, $y \in \mathbb{R}$, whose intrinsic equation is (using that $\kappa(y) = \operatorname{csch}^2 y$) given by $\kappa(s) = \frac{1}{s^2 - 1}$, $s > 1$. We get a similar expression to the Lorentzian catenary of the first kind (see Section 4.1) and, taking into account Remark 2.7, we conclude this new characterization.

Corollary 6.1. *The Lorentzian catenary of the first kind $y = -\sinh x$, $x \in \mathbb{R}$, is the only spacelike curve (up to translations in the y-direction) in $\mathbb{L}^2$ with geometric linear momentum $\mathcal{K}(x) = -\coth x$.*

6.2 Spacelike curves in $\mathbb{L}^2$ such that $\kappa(y) = \sec^2 y$

Considering $|y| < \pi/2$, we only afford the case $\mathcal{K}(y) = \tan y$, since then:

$$s = \int \frac{dy}{\sqrt{\tan^2 y + 1}} = \pm \int \cos y \, dy = \pm \sin y.$$

Thus:

$$y(s) = \pm \arcsin s, \ |s| < 1,$$

and

$$x(s) = \int \tan y(s) ds = \pm \int \frac{s \, ds}{\sqrt{1-s^2}} = \mp\sqrt{1-s^2}, \ |s| < 1.$$

We obtain the graph $x = \mp \cos y$, $|y| < \pi/2$, whose intrinsic equation is (using that $\kappa(y) = \sec^2 y$) given by $\kappa(s) = \frac{1}{1-s^2}$, $|s| < 1$. We arrive at the same expression as for the Lorentzian catenary of the second kind (see Section 4.1) and we deduce this new uniqueness result.

Corollary 6.2. *The Lorentzian catenary of the second kind $x = \pm \cos y$, $|y| < \pi/2$, is the only spacelike curve (up to translations in the x-direction) in $\mathbb{L}^2$ with geometric linear momentum $\mathcal{K}(y) = \tan y$.*

6.3 Timelike curves in $\mathbb{L}^2$ such that $\kappa(y) = \sinh y$

If we take $\mathcal{K}(y) = \cosh y$, we have:

$$s = \int \frac{dy}{\sqrt{\cosh^2 y - 1}} = \int \frac{dy}{\sinh y} \, dy = \log(\tanh(y/2)).$$

Thus:

$$y(s) = 2 \operatorname{arctanh} e^s, \ s < 0,$$

and

$$x(s) = \int \cosh y(s) ds = -\int \coth s \, ds = -\log(\sinh(-s)), \ s < 0.$$

After a straightforward computation, we arrive at the graph $x = \log \sinh y$, $y > 0$, whose intrinsic equation is (using that $\kappa(y) = \sinh y$) given by $\kappa(s) = -\operatorname{csch} s$, $s < 0$. We get a similar expression to the spacelike Lorentzian grim-reaper (see Section 5.1) and, making use of Remark 2.7, we deduce this new characterization.

Corollary 6.3. *The Lorentzian grim-reaper $y = \log(\sinh x)$, $x > 0$, is the only spacelike curve (up to translations in the y-direction) in $\mathbb{L}^2$ with geometric linear momentum $\mathcal{K}(x) = \cosh x$.*

6.4 Spacelike curves in $\mathbb{L}^2$ such that $\kappa(y) = \cosh y$

We only consider the case $\mathcal{K}(y) = \sinh y$. Then:

$$s = \int \frac{dy}{\sinh^2 + 1} = \int \frac{dy}{\cosh y} = 2 \arctan e^y.$$

Thus it is easy to obtain

$$y(s) = \log(\tan(s/2)), \ |s| < \pi$$

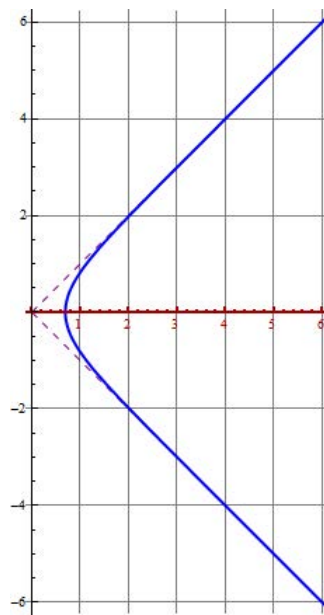and

$$x(s) = \log(2 \csc s), \ |s| < \pi.$$

Using that $\kappa(y) = \cosh y$, its intrinsic equation is given by $\kappa(s) = \csc s$, $|s| < \pi$ (see Figure 18).

Fig. 18. Spacelike curve with $\mathcal{K}(y) = \sinh y$.

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