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## Research Article

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# Existence of asymptotically periodic solutions for semilinear evolution equations with nonlocal initial conditions

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**Abstract:** In this paper we study a class of semilinear evolution equations with nonlocal initial conditions and give some new results on the existence of asymptotically periodic mild solutions. As one would expect, the results presented here would generalize and improve some results in this area.

**Keywords:** Semilinear evolution equation, Nonlocal initial condition, Asymptotic periodicity, Compact semi-group

**MSC:** 35B10, 35K90

## 1 Introduction

It is well known that periodic oscillations are natural and important phenomena in the real world, which can be observed frequently in many fields, such as celestial mechanics, nonlinear vibration, electromagnetic theory, plasma physics, engineering, ecosphere and so on. However, real concrete systems usually exhibit internal variations and external perturbations which are only approximately periodic. Asymptotically periodic function is one of the concepts of approximately periodic function. From an applied perspective asymptotically periodic systems describe our world more realistically and more accurately than periodic ones, one can see [1–3] for more details.

In recent years, the theory of asymptotic periodicity and its various extensions have attracted a great deal of attention of many mathematicians due to both their mathematical interest and significance as well as applications in physics, mathematical biology, control theory and so forth. Some contributions have been made. For instance, Wei and Wang [2] investigated the asymptotically periodic Lotka-Volterra cooperative systems. Cushing [3] examined the forced asymptotically periodic solutions of predator-prey systems with or without hereditary effects. Wei and Wang [4] studied the existence and uniqueness of asymptotically periodic solution of some ecosystems with asymptotically periodic coefficients. Henriquez, Pierri and Táboas [5] gave a relationship between  $S$ -asymptotically  $\omega$ -periodic functions and the class of asymptotically  $\omega$ -periodic functions. Pierri [6] established some conditions under which an  $S$ -asymptotically  $\omega$ -periodic function is asymptotically  $\omega$ -periodic and discussed the existence of asymptotically  $\omega$ -periodic solutions to an abstract integral equation. de Andrade and Cuevas [7] studied the existence and uniqueness of asymptotically  $\omega$ -periodic solution for control systems and partial differential equations with linear part dominated by a Hille-

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Yosida operator with non-dense domain. Agarwal, Cuevas, Soto and El-Gebeily [8] examined the asymptotically  $\omega$ -periodic solutions to an abstract neutral integro-differential equation with infinite delay etc. The study of asymptotically periodic solutions has been one of the most attracting topics in qualitative theory of various kinds of partial differential equations (see [9–11]), fractional differential equations (see [12, 13]), difference equations (see [14]), integro-differential equations (see [15–17]) and so forth. For more on these studies and related issues, we refer the reader to the references cited therein.

In this paper, we are concerned with the existence of asymptotically  $\omega$ -periodic mild solutions for the following nonlocal problem

$$(P) \quad \begin{cases} x'(t) = Ax(t) + F(t, x(t)), & t > 0, \\ x(0) + g(x(t)) = x_0, \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

where  $A : D(A) \subseteq X \rightarrow X$  is a closed bounded linear operator on a Banach space  $X$ ,  $F : \mathbb{R}^+ \times C(\mathbb{R}, X) \rightarrow X$  is a given  $X$ -valued function,  $g : C(\mathbb{R}, X) \rightarrow X$  is a given  $X$ -valued function and  $x_0 \in X$ .

The nonlocal problem (P) is motivated by physical problems. Indeed, the nonlocal initial conditions (2) can be applied in physics with better effect than the classical initial condition  $x(0) = x_0$ . For example, they are used to determine the unknown physical parameters in some inverse heat conduction problems [18, 19]. The study of atomic reactors also gives rise to the nonlocal problem (P) [20, 21]. For this reason, the nonlocal problem (P) has got a considerable attention in recent years (see for instance [22–29] and the references therein). See also [30–34] and the references cited there for recent generalizations of nonlocal problem (P) to various kinds of differential equations and differential inclusions.

In this paper we are interested in the case that  $A$  generates a compact  $C_0$ -semigroup. In [24] the Leray-Schauder alternative was used to study the existence of solutions for the nonlocal problem (P). However, as was shown in [25], the proof of the main results in [24] does not work because the most important place at  $t = 0$  was neglected when checking the compactness of the solution operator. To fill this gap, some authors added conditions on the compactness of  $g$ , see e.g. [25, 26, 35–37] and the references therein. However, in application to physics, these conditions are too strong. For example, as presented by Deng [27], the nonlocal problem (P) with the mapping  $g$  given by

$$g(x) = \sum_{i=1}^p c_i x(t_i), \quad (3)$$

where  $0 < t_1 < t_2 < \dots < t_p < +\infty$ ,  $c_1, c_2, \dots, c_p$  are given constants, is used to describe the diffusion phenomenon of a small amount of gas in a transparent tube, and in many references on nonlocal Cauchy problems (see e.g. [27–29, 38]), the mapping  $g$  is also given by (3). Obviously, compactness condition is not valid for  $g$  in this case. Without assumptions on the compactness of  $g$ , Liang, Liu and Xiao [29] developed a method to deal with the case that  $F(t, x)$  is Lipschitz continuous in  $x$ . By Schauder's fixed point theorem, the authors obtained the existence of mild solutions for the nonlocal problem (P).

To the best of our knowledge, much less is known about the existence of asymptotically periodic solutions to the nonlocal problem (P) when the nonlinearity  $F(t, x)$  as a whole loses the Lipschitz continuity with respect to  $x$ . In this work we will discuss the existence of asymptotically periodic mild solutions for the nonlocal problem (P). Some new existence theorems of asymptotically periodic mild solutions are established. In our results, the nonlinearity  $F(t, x)$  does not have to satisfy a (locally) Lipschitz condition with respect to  $x$  (see Remark 3.1). However, in many papers (for instance [5, 6, 9, 12, 13, 39–41]) on asymptotic periodicity, to be able to apply the well known Banach contraction principle, a (locally) Lipschitz condition for the nonlinearity of corresponding differential equations is needed. As can be seen, the hypotheses in our results are reasonably weak (see Remark 3.3), and our results generalize those as well as related research and have more broad applications.

The rest of this paper is organized as follows. In Section 2, some concepts, the related notations and some useful lemmas are introduced. In Section 3, we present some criteria ensuring the existence of asymptotically periodic mild solutions. An example is given to illustrate our results in Section 4.

## 2 Preliminaries

This section is concerned with some notations, definitions, lemmas and preliminary facts which are used in what follows.

From now on,  $\mathbb{R}$  and  $\mathbb{R}^+$  stand for the set of real numbers and nonnegative real numbers respectively. Denote by  $X$  a Banach space with norm  $\|\cdot\|$ .  $C(\mathbb{R}, X)$  stands for the Banach space of all continuous functions from  $\mathbb{R}$  to  $X$ .  $C_b(\mathbb{R}^+, X)$  stands for the Banach space of all bounded and continuous functions  $x$  from  $\mathbb{R}^+$  to  $X$ . Furthermore, let  $C_0(\mathbb{R}^+, X)$  and  $C_\omega(\mathbb{R}, X)$  be the spaces of functions

$$C_0(\mathbb{R}^+, X) := \left\{ x \in C_b(\mathbb{R}^+, X) : \lim_{t \rightarrow +\infty} \|x(t)\| = 0 \right\},$$

$$C_\omega(\mathbb{R}, X) := \left\{ x \in C(\mathbb{R}, X) : x(t) \text{ is } \omega\text{-periodic} \right\}.$$

It is easy to see that  $C_0(\mathbb{R}^+, X)$  and  $C_\omega(\mathbb{R}, X)$  endowed with the norm

$$\|x\|_\infty = \sup \{ \|x(t)\| : t \in \mathbb{R} \}$$

are both Banach spaces.

We abbreviate  $C_0(\mathbb{R}^+, X)$  to  $C_0(\mathbb{R}^+)$  when  $X = \mathbb{R}^+$ .

$C(\mathbb{R}^+ \times X, X)$  stands for the set of all jointly continuous functions  $F(t, x)$  from  $\mathbb{R}^+ \times X$  to  $X$ , and let the notation  $C_0(\mathbb{R}^+ \times X, X)$  be the set of functions

$$C_0(\mathbb{R}^+ \times X, X) := \left\{ F \in C(\mathbb{R}^+ \times X, X) : \lim_{t \rightarrow +\infty} \|F(t, x)\| = 0 \right. \\ \left. \text{uniformly for } x \text{ in any bounded subset of } X \right\}.$$

For some  $r > 0$ , write

$$\Omega_r := \left\{ x \in C_0(\mathbb{R}^+, X) : \|x\|_\infty \leq r \right\}.$$

Now, we recall some basic definitions and results on asymptotically periodic functions.

**Definition 2.1** ([5]). A function  $U(t) \in C_b(\mathbb{R}^+, X)$  is said to be asymptotically  $\omega$ -periodic if it can be decomposed as  $U(t) = V(t) + W(t)$ , where

$$V(t) \in C_\omega(\mathbb{R}, X), \quad W(t) \in C_0(\mathbb{R}^+, X).$$

Denote by  $AP_\omega(\mathbb{R}^+, X)$  the set of all such functions.

**Lemma 2.2** ([5]).  $AP_\omega(\mathbb{R}^+, X)$  turns out to be a Banach space with the norm  $\|\cdot\|_\infty$ .

For some  $r > 0$ , let

$$S_r := \left\{ x \in AP_\omega(\mathbb{R}^+, X) : \|x\|_0 \leq r \right\}.$$

**Remark 2.3.** Take  $U(t) \in AP_\omega(\mathbb{R}^+, X)$ . Let us note that the decomposition  $U(t) = V(t) + W(t)$  is unique.

Indeed, if there exist

$$V_1(t), V_2(t) \in C_\omega(\mathbb{R}, X), \quad W_1(t), W_2(t) \in C_0(\mathbb{R}^+, X)$$

such that

$$U(t) = V_1(t) + W_1(t) = V_2(t) + W_2(t),$$

then one can find that for fixed  $t \in \mathbb{R}$ ,

$$V_1(t) - V_2(t) = V_1(t + n\omega) - V_2(t + n\omega) = W_2(t + n\omega) - W_1(t + n\omega), \quad n\omega \geq -t.$$

Taking the limit as  $n \rightarrow +\infty$ , one has that  $V_1(t) = V_2(t)$ ,  $t \in \mathbb{R}$  as required.

**Definition 2.4** ([5]). A jointly continuous function  $F(t, x)$  from  $\mathbb{R} \times X$  to  $X$  is said to be  $\omega$ -periodic if

$$F(t + \omega, x) = F(t, x) \text{ for all } t \in \mathbb{R} \text{ and } x \in X.$$

The set of such functions will be denoted by  $C_\omega(\mathbb{R} \times X, X)$ .

**Definition 2.5** ([5]). A function  $F : \mathbb{R}^+ \times X \rightarrow X$  is said to be asymptotically  $\omega$ -periodic if it can be decomposed as  $F(t, x) = G(t, x) + \Phi(t, x)$ , where

$$G(t, x) \in C_\omega(\mathbb{R} \times X, X), \quad \Phi(t, x) \in C_0(\mathbb{R}^+ \times X, X).$$

Denote by  $AP_\omega(\mathbb{R}^+ \times X, X)$  the set of all such functions.

In the following, we present the following compactness criterion, which is a special case of the general compactness result of Theorem 2.1 in [42].

**Lemma 2.6** ([42]). A set  $D \subset C_0(\mathbb{R}^+, X)$  is relatively compact if

- (1)  $D$  is equicontinuous;
- (2)  $\lim_{t \rightarrow +\infty} x(t) = 0$  uniformly for  $x \in D$ ;
- (3) the set  $D(t) := \{x(t) : x \in D\}$  is relatively compact in  $X$  for every  $t \in \mathbb{R}^+$ .

The following Krasnoselskii's fixed point theorem plays a key role in the proofs of our main results, which can be found in many books.

**Lemma 2.7** ([43]). Let  $B$  be a bounded closed and convex subset of  $X$ , and  $J_1, J_2$  be maps of  $B$  into  $X$  such that  $J_1x + J_2y \in B$  for every pair  $x, y \in B$ . If  $J_1$  is a contraction and  $J_2$  is completely continuous, then the equation  $J_1x + J_2x = x$  has a solution on  $B$ .

### 3 Main results

In this section, we study the existence of asymptotically periodic mild solutions for the following nonlocal problem

$$(P) \quad \begin{cases} x'(t) = Ax(t) + F(t, x(t)), & t > 0, \\ x(0) + g(x(t)) = x_0 \end{cases} \quad (4)$$

in the Banach space  $X$ . Here,  $x_0 \in X$ , the operator  $A : D(A) \subset X \rightarrow X$  is the infinitesimal generator of a compact and uniformly exponentially stable  $C_0$ -semigroup  $\{T(t)\}_{t \geq 0}$ , i.e. there exist two constants  $M, \delta > 0$  such that

$$\|T(t)\| \leq Me^{-\delta t} \text{ for all } t > 0.$$

$F : \mathbb{R}^+ \times C(\mathbb{R}, X) \rightarrow X$  is a given function satisfying the following assumption:

(H<sub>1</sub>)  $F(t, x) = F_1(t, x) + F_2(t, x) \in AP_\omega(\mathbb{R}^+ \times X, X)$  with

$$F_1(t, x) \in C_\omega(\mathbb{R} \times X, X), \quad F_2(t, x) \in C_0(\mathbb{R}^+ \times X, X),$$

and there exists a constant  $L_1 > 0$  such that

$$\|F_1(t, x) - F_1(t, y)\| \leq L_1 \|x - y\| \text{ for all } t \in \mathbb{R}, x, y \in X. \quad (6)$$

Moreover, there exist a function  $\beta(t) \in C_0(\mathbb{R}^+)$  and a nondecreasing function  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for all  $t \in \mathbb{R}^+$  and  $x \in X$  with  $\|x\| \leq r$ ,

$$\|F_2(t, x)\| \leq \beta(t)\Phi(r) \text{ and } \liminf_{r \rightarrow +\infty} \frac{\Phi(r)}{r} = \rho_1. \quad (7)$$

The function  $g : AP_\omega(\mathbb{R}^+, X) \rightarrow X$  satisfies the following assumption:

(H<sub>2</sub>) There exists a constant  $L_2 > 0$  such that

$$\|g(x) - g(y)\| \leq L_2 \|x - y\| \text{ for all } x, y \in X. \quad (8)$$

Moreover, there exists a nondecreasing function  $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for all  $x \in S_r$ ,

$$\|g(x)\| \leq \Psi(r) \text{ and } \liminf_{r \rightarrow +\infty} \frac{\Psi(r)}{r} = \rho_2. \quad (9)$$

**Remark 3.1.** Assume that  $F(t, x)$  satisfies the assumption (H<sub>1</sub>), it is noted that  $F(t, x)$  does not have to meet the Lipschitz continuity with respect to  $x$ . Such class of asymptotically  $\omega$ -periodic functions  $F(t, x)$  are more complicated than those satisfying Lipschitz continuity and little is known about them.

**Lemma 3.2.** Given  $F(t, x) = F_1(t, x) + F_2(t, x) \in AP_\omega(\mathbb{R}^+ \times X, X)$  with

$$F_1(t, x) \in C_\omega(\mathbb{R} \times X, X), F_2(t, x) \in C_0(\mathbb{R}^+ \times X, X).$$

Then it yields that

$$\sup_{t \in \mathbb{R}} \|F_1(t, x) - F_1(t, y)\| \leq \sup_{t \in \mathbb{R}^+} \|F(t, x) - F(t, y)\|, \quad x, y \in X. \quad (10)$$

*Proof.* To show our result, it suffices to verify that

$$\{F_1(t, x) - F_1(t, y) : t \in \mathbb{R}\} \subset \overline{\{F(t, x) - F(t, y) : t \in \mathbb{R}^+\}}, \quad x, y \in X.$$

In fact, if this is not the case, then for fixed  $x, y \in X$ , there exist some  $t_0 \in \mathbb{R}$  and  $\varepsilon > 0$  such that

$$\|(F_1(t_0, x) - F_1(t_0, y)) - (F(t, x) - F(t, y))\| \geq \varepsilon \text{ for all } t \in \mathbb{R}^+.$$

It is clear that

$$\lim_{t \rightarrow +\infty} \|F_2(t, x) - F_2(t, y)\| = 0,$$

which implies that there exists a positive number  $T$  such that for all  $t \geq T$ ,

$$\|F_2(t, x) - F_2(t, y)\| < \varepsilon. \quad (11)$$

Since  $F_1(t, x) \in C_\omega(\mathbb{R} \times X, X)$ , then for  $t_0 + n\omega \geq T$ ,

$$\begin{aligned} & \|F_2(t_0 + n\omega, x) - F_2(t_0 + n\omega, y)\| \\ & \geq \|F(t_0 + n\omega, x) - F(t_0 + n\omega, y) - F_1(t_0, x) + F_1(t_0, y)\| - \|F_1(t_0 + n\omega, x) - F_1(t_0, x)\| \\ & \quad - \|F_1(t_0 + n\omega, y) - F_1(t_0, y)\| \\ & = \|F(t_0 + n\omega, x) - F(t_0 + n\omega, y) - F_1(t_0, x) + F_1(t_0, y)\| \geq \varepsilon, \end{aligned}$$

which contradicts (11), completing the proof.  $\square$

**Remark 3.3.** In Lemma 3.2, (10) implies that when  $F(t, x)$  meets the Lipschitz continuity with respect to  $x$  with the Lipschitz constant  $L_1$ , then  $F_1(t, x)$  satisfies (6). Note that in many papers (for instance [5, 6, 9, 12, 13, 39–41]) on asymptotic periodicity, to be able to apply the well known Banach contraction principle, a (locally) Lipschitz condition for the nonlinearities of corresponding differential equations is needed. Thus our conditions in the assumption (H<sub>1</sub>) are weaker than those of [5, 6, 9, 12, 13, 39–41].

In the proofs of our results, we need the following auxiliary result.

**Lemma 3.4.** Given  $y \in X$ ,  $U(t) \in C_\omega(\mathbb{R}, X)$  and  $V(t) \in C_0(\mathbb{R}^+, X)$ . Let

$$\gamma(t) := T(t)y - \int_{-\infty}^0 T(t-s)U(s)ds + \int_0^t T(t-s)V(s)ds, \quad t \in \mathbb{R}^+.$$

Then  $\gamma(t) \in C_0(\mathbb{R}^+, X)$ .

*Proof.* From the exponential stability of  $\{T(t)\}_{t \geq 0}$  it is clear that for all  $t \in \mathbb{R}^+$ ,

$$\begin{aligned} & \left\| T(t)y - \int_{-\infty}^0 T(t-s)U(s)ds + \int_0^t T(t-s)V(s)ds \right\| \\ & \leq Me^{-\delta t} \|y\| + M \int_{-\infty}^0 e^{-\delta(t-s)} \|U(s)\|ds + M \int_0^t e^{-\delta(t-s)} \|V(s)\|ds \\ & \leq M \|y\| + \frac{M}{\delta} [\|U\|_{\infty} + \|V\|_{\infty}], \end{aligned}$$

which implies that  $\mathcal{T}(t)$  is well-defined and continuous on  $\mathbb{R}^+$ . Furthermore

$$\begin{aligned} \left\| T(t)y - \int_{-\infty}^0 T(t-s)U(s)ds \right\| & \leq Me^{-\delta t} \|y\| + M \int_{-\infty}^0 e^{-\delta(t-s)} \|U(s)\|ds \\ & \leq Me^{-\delta t} \|y\| + \frac{Me^{-\delta t}}{\delta} \|U\|_{\infty} \rightarrow 0 \text{ as } t \rightarrow +\infty. \end{aligned}$$

On the other hand, since  $V(t) \in C_0(\mathbb{R}^+, X)$ , given  $\varepsilon > 0$ , one can choose a  $T > 0$  such that

$$\|V(t)\| < \varepsilon \text{ for all } t > T.$$

This, together with the exponential stability of  $\{T(t)\}_{t \geq 0}$ , enables us to conclude that for all  $t \geq T$ ,

$$\left\| \int_T^t T(t-s)V(s)ds \right\| \leq M \int_T^t e^{-\delta(t-s)} \|V(s)\|ds \leq \frac{M}{\delta} \varepsilon.$$

Also, we derive, by the exponential stability of  $\{T(t)\}_{t \geq 0}$

$$\left\| \int_0^T T(t-s)V(s)ds \right\| \leq M \int_0^T e^{-\delta(t-s)} \|V(s)\|ds \leq \frac{Me^{-\delta t}}{\delta} \|V\|_{\infty} \rightarrow 0 \text{ as } t \rightarrow +\infty.$$

We thus gain from the arguments above that

$$\begin{aligned} \|\mathcal{T}(t)\| & \leq \left\| T(t)y - \int_{-\infty}^0 T(t-s)U(s)ds \right\| + \left\| \int_0^t T(t-s)V(s)ds \right\| \\ & \leq \left\| T(t)y - \int_{-\infty}^0 T(t-s)U(s)ds \right\| + \left\| \int_0^T T(t-s)V(s)ds \right\| + \left\| \int_T^t T(t-s)V(s)ds \right\| \\ & \rightarrow 0 \text{ as } t \rightarrow +\infty, \end{aligned}$$

which implies  $\mathcal{T}(t) \in C_0(\mathbb{R}^+, X)$ . □

We give the following definition of mild solution to nonlocal problem (P).

**Definition 3.5.** A continuous function  $x : \mathbb{R}^+ \rightarrow X$  is called a mild solution to nonlocal problem (P) on  $\mathbb{R}^+$  if  $x$  satisfies the integral equation of the form

$$x(t) = T(t)[x_0 - g(x)] + \int_0^t T(t-s)F(s, x(s))ds \text{ for all } t \geq 0.$$

Let  $\beta(t)$  be the function involved in the assumption  $(H_1)$ . Define

$$\sigma(t) := \int_0^t e^{-\delta(t-s)} \beta(s)ds, \quad t \in \mathbb{R}^+.$$

Then  $\sigma(t) \in C_0(\mathbb{R}^+)$ . Put  $\rho_3 := \sup_{t \in \mathbb{R}^+} \sigma(t)$ .

Now we are in a position to present our results.

**Theorem 3.6.** Under the assumptions  $(H_1)$  and  $(H_2)$ , the nonlocal problem  $(P)$  has at least one asymptotically  $\omega$ -periodic mild solution provided that

$$M \max\{L_2, \rho_2\} + ML_1\delta^{-1} + M\rho_1\rho_3 < 1. \quad (12)$$

*Proof.* The proof is divided into the following five steps.

Step 1. Define a mapping  $\Lambda$  on  $C_\omega(\mathbb{R}, X)$  by

$$(\Lambda v)(t) = \int_{-\infty}^t T(t-s)F_1(s, v(s))ds, \quad v(s) \in C_\omega(\mathbb{R}, X), \quad t \in \mathbb{R},$$

and prove  $\Lambda$  has a unique fixed point  $v(t) \in C_\omega(\mathbb{R}, X)$ .

Firstly, from the exponential stability of  $\{T(t)\}_{t \geq 0}$  it is clear that

$$\left\| \int_{-\infty}^t T(t-s)F_1(s, v(s))ds \right\| \leq M \int_{-\infty}^t e^{-\delta(t-s)} \|F_1(s, v(s))\| ds \leq \frac{M}{\delta} \|F_1(\cdot, v(\cdot))\|_\infty,$$

which implies that  $\Lambda$  is well-defined and continuous on  $\mathbb{R}^+$ . Moreover, one easily calculates, by  $v(t) \in C_\omega(\mathbb{R}, X)$  and  $F_1(t, x) \in C_\omega(\mathbb{R} \times X, X)$ ,

$$\begin{aligned} (\Lambda v)(t+\omega) &= \int_{-\infty}^{t+\omega} T(t+\omega-s)F_1(s, v(s))ds = \int_{-\infty}^t T(t-s)F_1(s+\omega, v(s+\omega))ds \\ &= \int_{-\infty}^t T(t-s)F_1(s+\omega, v(s))ds = \int_{-\infty}^t T(t-s)F_1(s, v(s))ds = (\Lambda v)(t), \end{aligned}$$

this implies that  $\Lambda$  maps  $C_\omega(\mathbb{R}, X)$  into itself.

On the other hand, for any  $v_1(t), v_2(t) \in C_\omega(\mathbb{R}, X)$ , by (6) one has

$$\|(\Lambda v_1)(t) - (\Lambda v_2)(t)\| \leq ML_1 \int_{-\infty}^t e^{-\delta(t-s)} \|v_1(s) - v_2(s)\| ds \leq \frac{ML_1}{\delta} \|v_1 - v_2\|_\infty.$$

As a result, one has

$$\|(\Lambda v_1) - (\Lambda v_2)\|_\infty \leq \frac{ML_1}{\delta} \|v_1 - v_2\|_\infty.$$

This, together with (12), proves that  $\Lambda$  is a contraction on  $C_\omega(\mathbb{R}, X)$ . Thus, the Banach's fixed point theorem implies that  $\Lambda$  has a unique fixed point  $v(t) \in C_\omega(\mathbb{R}, X)$ .

Step 2. For the above  $v(t)$ , define a mapping  $\Gamma := \Gamma^1 + \Gamma^2$  as

$$(\Gamma^1 \omega)(t) = T(t)[x_0 - g(v|_{\mathbb{R}^+} + \omega)] + \int_0^t T(t-s)[F_1(s, v(s) + \omega(s)) - F_1(s, v(s))]ds,$$

$$(\Gamma^2 \omega)(t) = - \int_{-\infty}^0 T(t-s)F_1(s, v(s))ds + \int_0^t T(t-s)F_2(s, v(s) + \omega(s))ds$$

for all  $\omega(s) \in C_0(\mathbb{R}^+, X)$ ,  $t \in \mathbb{R}^+$ , and prove that  $\Gamma$  maps  $\Omega_{k_0}$  into itself, where  $k_0$  is a given constant.

Firstly, from (6) it follows that

$$\|F_1(s, v(s) + \omega(s)) - F_1(s, v(s))\| \leq L_1 \|\omega(s)\| \text{ for all } s \in \mathbb{R}, \omega(s) \in X,$$

which implies that

$$F_1(\cdot, v(\cdot) + \omega(\cdot)) - F_1(\cdot, v(\cdot)) \in C_0(\mathbb{R}^+, X) \text{ for each } \omega(\cdot) \in C_0(\mathbb{R}^+, X).$$

By (7), one has for all  $s \in \mathbb{R}^+$  and  $\omega(s) \in X$  with  $\|\omega(s)\| \leq r$ ,

$$\|F_2(s, v(s) + \omega(s))\| \leq \beta(s)\Phi\left(r + \sup_{s \in \mathbb{R}^+} \|v(s)\|\right), \quad (13)$$

which implies that

$$F_2(\cdot, v(\cdot) + \omega(\cdot)) \in C_0(\mathbb{R}^+, X) \text{ as } \beta(s) \in C_0(\mathbb{R}^+).$$

Those, together with Lemma 3.4, yield that  $\Gamma$  is well-defined and maps  $C_0(\mathbb{R}^+, X)$  into itself.

On the other hand, in view of (7), (9) and (12) it is not difficult to see that there exists a constant  $k_0 > 0$  such that

$$M\left[\|x_0\| + \Psi(k_0 + \|v\|_\infty)\right] + \frac{ML_1k_0}{\delta} + \frac{M}{\delta} \sup_{s \in \mathbb{R}} \|F_1(s, v(s))\| + M\Phi(k_0 + \|v\|_\infty)\rho_3 \leq k_0.$$

This enables us to conclude that for any  $t \in \mathbb{R}^+$  and  $\omega_1(t), \omega_2(t) \in \Omega_{k_0}$ ,

$$\begin{aligned} & \|(\Gamma^1\omega_1)(t) + (\Gamma^2\omega_2)(t)\| \\ & \leq \|T(t)[x_0 - g(v|_{\mathbb{R}^+} + \omega_1)]\| + \left\| \int_0^t T(t-s)[F_1(s, v(s) + \omega_1(s)) - F_1(s, v(s))]ds \right\| \\ & \quad + \left\| \int_{-\infty}^0 T(t-s)F_1(s, v(s))ds \right\| + \left\| \int_0^t T(t-s)F_2(s, v(s) + \omega_2(s))ds \right\| \\ & \leq M(\|x_0\| + \|g(v|_{\mathbb{R}^+} + \omega_1)\|) + M \int_0^t e^{-\delta(t-s)} \|F_1(s, v(s) + \omega_1(s)) - F_1(s, v(s))\|ds \\ & \quad + M \int_{-\infty}^0 e^{-\delta(t-s)} \|F_1(s, v(s))\|ds + M \int_0^t e^{-\delta(t-s)} \|F_2(s, v(s) + \omega_2(s))\|ds \\ & \leq M\left[\|x_0\| + \Psi(k_0 + \|v\|_\infty)\right] + \frac{ML_1k_0}{\delta} + \frac{M}{\delta} \sup_{s \in \mathbb{R}} \|F_1(s, v(s))\| + M\Phi(k_0 + \|v\|_\infty)\rho_3 \leq k_0, \end{aligned}$$

which implies that

$$(\Gamma^1\omega_1)(t) + (\Gamma^2\omega_2)(t) \in \Omega_{k_0}.$$

Thus  $\Gamma$  maps  $\Omega_{k_0}$  into itself.

Step 3. Show that  $\Gamma^1$  is a contraction on  $\Omega_{k_0}$ .

In fact, for any  $\omega_1(t), \omega_2(t) \in \Omega_{k_0}$  and  $t \in \mathbb{R}^+$ , from (6) it follows that

$$\begin{aligned} & \| [F_1(s, v(s) + \omega_1(s)) - F_1(s, v(s))] - [F_1(s, v(s) + \omega_2(s)) - F_1(s, v(s))] \| \\ & \leq L_1 \|\omega_1(s) - \omega_2(s)\| \text{ for all } s \in \mathbb{R}^+, \omega_1(s), \omega_2(s) \in X. \end{aligned}$$

Thus

$$\begin{aligned} & \|(\Gamma^1\omega_1)(t) - (\Gamma^1\omega_2)(t)\| \\ & \leq \|T(t)[x_0 - g(v|_{\mathbb{R}^+} + \omega_1)] - T(t)[x_0 - g(v|_{\mathbb{R}^+} + \omega_2)]\| \\ & \quad + \left\| \int_0^t T(t-s)[F_1(s, v(s) + \omega_1(s)) - F_1(s, v(s))] - [F_1(s, v(s) + \omega_2(s)) - F_1(s, v(s))]ds \right\| \\ & \leq ML_2\|\omega_1 - \omega_2\|_0 + ML_1 \int_0^t e^{-\delta(t-s)} \|\omega_1(s) - \omega_2(s)\|ds \\ & \leq \left[ ML_2 + \frac{ML_1}{\delta} \right] \|\omega_1 - \omega_2\|_\infty. \end{aligned}$$

As a result, one has

$$\|\Gamma^1\omega_1 - \Gamma^1\omega_2\|_\infty \leq \left[ ML_2 + \frac{ML_1}{\delta} \right] \|\omega_1 - \omega_2\|_\infty.$$



Thus, in view of (12), one obtains the conclusion.

Step 4. Show that  $\Gamma^2$  is completely continuous on  $\Omega_{k_0}$ .

Given  $\varepsilon > 0$ . Let  $\{\omega_k\}_{k=1}^{+\infty} \subset \Omega_{k_0}$  with  $\omega_k \rightarrow \omega_0$  in  $C_0(\mathbb{R}^+, X)$  as  $k \rightarrow +\infty$ . Since  $\beta(t) \in C_0(\mathbb{R}^+)$ , one may choose a  $t_1 > 0$  big enough such that for all  $t \geq t_1$ ,

$$3M\Phi(k_0 + \|v\|_\infty)\beta(t) < \delta\varepsilon.$$

Also, in view of  $(H_1)$ , we have  $F_2(s, v(s) + \omega_k(s)) \rightarrow F_2(s, v(s) + \omega_0(s))$  for all  $s \in [0, t_1]$  as  $k \rightarrow +\infty$ , and

$$\|F_2(\cdot, v(\cdot) + \omega_k(\cdot)) - F_2(\cdot, v(\cdot) + \omega_0(\cdot))\| \leq 2\Phi(k_0 + \|v\|_\infty)\beta(\cdot) \in L^1(0, t_1).$$

Hence, by the Lebesgue dominated convergence theorem we deduce that there exists an  $N > 0$  such that for any  $t \in \mathbb{R}^+$ ,

$$\begin{aligned} \|(\Gamma^2\omega_k)(t) - (\Gamma^2\omega_0)(t)\| &= \left\| \int_0^t T(t-s)F_2(s, v(s) + \omega_k(s))ds - \int_0^t T(t-s)F_2(s, v(s) + \omega_0(s))ds \right\| \\ &\leq M \int_0^{t_1} e^{-\delta(t-s)} \|F_2(s, v(s) + \omega_k(s)) - F_2(s, v(s) + \omega_0(s))\| ds \\ &\quad + 2M\Phi(k_0 + \|v\|_\infty) \int_{t_1}^{\max\{t, t_1\}} e^{-\delta(t-s)} \beta(s) ds \\ &\leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon \end{aligned}$$

whenever  $k \geq N$ . Accordingly,  $\Gamma^2$  is continuous on  $\Omega_{k_0}$ .

In the sequel, we consider the compactness of  $\Gamma^2$ . Since the function

$$t \rightarrow \int_{-\infty}^0 T(t-s)F_2(s, v(s))ds$$

belongs to  $C_0(\mathbb{R}^+, X)$  due to Lemma 2.6 and is independent of  $\omega$ , it suffices to show that the mapping

$$(\Pi\omega)(t) = \int_0^t T(t-s)F_2(s, v(s) + \omega(s))ds, \quad \omega(s) \in C_0(\mathbb{R}^+, X)$$

is compact.

Let  $t \in \mathbb{R}$  be fixed. For given  $\varepsilon_0 > 0$ , from (13) it follows that

$$(\Pi_{\varepsilon_0}\omega)(t) = \int_0^{t-\varepsilon_0} T(t-\varepsilon_0-s)F_2(s, v(s) + \omega(s))ds$$

is uniformly bounded for  $\omega(t) \in \Omega_{k_0}$ . This, together with the compactness of  $T(\varepsilon_0)$ , yields that the set  $\{T(\varepsilon_0)(\Pi_{\varepsilon_0}\omega)(t) : \omega(t) \in \Omega_{k_0}\}$  is relatively compact in  $X$ .

On the other hand

$$\begin{aligned} \|(\Pi\omega)(t) - T(\varepsilon_0)(\Pi_{\varepsilon_0}\omega)(t)\| &\leq \left\| \int_{t-\varepsilon_0}^t T(t-s)F_2(s, v(s) + \omega(s))ds \right\| \\ &\leq M \int_{t-\varepsilon_0}^t e^{-\delta(t-s)} \|F_2(s, v(s) + \omega(s))\| ds \rightarrow 0 \text{ as } \varepsilon_0 \rightarrow 0^+, \end{aligned}$$

this, together with the total boundedness, yields that the set  $\{(\Pi\omega)(t) : \omega(t) \in \Omega_{k_0}\}$  is relatively compact in  $X$  for each  $t \in \mathbb{R}^+$ .

Next, we verify the equicontinuity of the set  $\{(\Pi\omega)(t) : \omega(t) \in \Omega_{k_0}\}$ .

Let  $k > 0$  be small enough and  $t_1, t_2 \in \mathbb{R}^+$ ,  $\omega(t) \in \Omega_{k_0}$ . Then by (6) we have that for the case when  $0 < t_1 < t_2$ ,

$$\begin{aligned} & \|(\Pi\omega)(t_2) - (\Pi\omega)(t_1)\| \\ &= \left\| \int_0^{t_2} T(t_2-s)F_2(s, v(s) + \omega(s))ds - \int_0^{t_1} T(t_1-s)F_2(s, v(s) + \omega(s))ds \right\| \\ &\leq \int_{t_1}^{t_2} \|T(t_2-s)F_2(s, v(s) + \omega(s))\|ds + \int_0^{t_1-k} \| [T(t_2-s) - T(t_1-s)]F_2(s, v(s) + \omega(s)) \|ds \\ &\quad + \int_{t_1-k}^{t_1} \| [T(t_2-s) - T(t_1-s)]F_2(s, v(s) + \omega(s)) \|ds \\ &\leq M\Phi(k_0 + \|v\|_\infty) \int_{t_1}^{t_2} e^{-\delta(t_2-s)}\beta(s)ds \\ &\quad + \Phi(k_0 + \|v\|_\infty) \sup_{t \in [0, t_1-k]} \|T(t_2-s) - T(t_1-s)\| \int_0^{t_1-k} \beta(s)ds \\ &\quad + M\Phi(k_0 + \|v\|_\infty) \int_{t_1-k}^{t_1} (e^{-\delta(t_2-s)} + e^{-\delta(t_1-s)})\beta(s)ds \rightarrow 0 \text{ as } t_2 - t_1 \rightarrow 0, k \rightarrow 0, \end{aligned}$$

and for the case when  $0 = t_1 < t_2$ ,

$$\|(\Pi\omega)(t_2) - (\Pi\omega)(t_1)\| \leq M\Phi(k_0 + \|v\|_\infty) \int_0^{t_2} e^{-\delta(t_2-s)}\beta(s)ds \rightarrow 0 \text{ as } t_2 \rightarrow 0,$$

which verifies that the result follows.

Finally, as

$$\|(\Pi\omega)(t)\| \leq M\Phi(k_0 + \|v\|_\infty) \int_0^t e^{-\delta(t-s)}\beta(s)ds \rightarrow 0 \text{ as } t \rightarrow +\infty$$

uniformly for  $\omega(t) \in \Omega_{k_0}$ , in view of  $\sigma(t) \in C_0(\mathbb{R}^+)$ , we conclude that  $(\Pi\omega)(t)$  vanishes at infinity uniformly for  $\omega(t) \in \Omega_{k_0}$ .

Now an application of Lemma 2.6 justifies the compactness of  $\Pi$ , which together with the representation of  $\Gamma^2$  implies that  $\Gamma^2$  is compact.

Step 5. Show that the nonlocal problem (P) has at least one asymptotically  $\omega$ -periodic mild solution.

Firstly, as shown in Step 3 and Step 4 respectively,  $\Gamma^1$  is a strict contraction and  $\Gamma^2$  is completely continuous. Accordingly, we deduce, thanks to Lemma 2.7, that  $\Gamma$  has at least one fixed point  $\omega(t) \in \Omega_{k_0}$ , furthermore  $\omega(t) \in C_0(\mathbb{R}^+, X)$ .

Then, consider the following coupled system of integral equations

$$\begin{cases} v(t) = \int_{-\infty}^t T(t-s)F_1(s, v(s))ds, & t \in \mathbb{R}^+, \\ \omega(t) = T(t)[x_0 - g(v|_{\mathbb{R}^+} + \omega)] + \int_0^t T(t-s)F(s, v(s) + \omega(s))ds \\ \quad - \int_{-\infty}^t T(t-s)F_1(s, v(s))ds, & t \in \mathbb{R}^+. \end{cases} \quad (14)$$

From the result of step 1, together with the above fixed point  $\omega(t) \in C_0(\mathbb{R}^+, X)$ , it follows that

$$(v(t), \omega(t)) \in C_\omega(\mathbb{R}, X) \times C_0(\mathbb{R}^+, X)$$

is a solution to system (14). Thus

$$x(t) := v(t) + \omega(t) \in AP_\omega(\mathbb{R}^+, X),$$

and it is an asymptotically  $\omega$ -periodic mild solution to the nonlocal problem (P).  $\square$

**Remark 3.7.** Note that the condition (6) in  $(H_1)$  of Theorem 3.6 can be easily extended to the case of  $F_1(t, x)$  being locally Lipschitz continuous:

$$\|F_1(t, x) - F_1(t, y)\| \leq L(r)\|X - Y\|$$

for all  $t \in \mathbb{R}^+$  and  $x, y \in X$  satisfying  $\|x\|, \|y\| \leq r$ .

**Corollary 3.8.** Assume that the hypothesis  $(H_1)$  holds and  $g(x) = 0$ . Then the problem (P) has at least one asymptotically  $\omega$ -periodic mild solution provided that

$$ML_1\delta^{-1} + M\rho_1\rho_3 < 1.$$

In the following, we prove the existence of asymptotically  $\omega$ -periodic mild solutions to the nonlocal problem (P) for the case of  $g$  being completely continuous.

**Theorem 3.9.** Let the hypothesis  $(H_1)$  hold. Assume in addition that  $(H'_2)$  The function  $g : AP_\omega(\mathbb{R}^+, X) \rightarrow X$  is completely continuous, and there exists a nondecreasing function  $\Theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for all  $x \in S_r$ ,

$$\|g(x)\| \leq \Theta(r) \text{ and } \liminf_{r \rightarrow +\infty} \frac{\Theta(r)}{r} = \rho'_2.$$

Then the nonlocal problem (P) has at least one asymptotically  $\omega$ -periodic mild solution provided that

$$M\rho'_2 + ML_1\delta^{-1} + M\rho_1\rho_3 < 1. \quad (15)$$

*Proof.* Let the operator  $\Lambda$  be defined in the same way as in Theorem 3.6 and  $v(t) \in C_\omega(\mathbb{R}, X)$  come from the Step 1 in the proof of Theorem 3.6, is a unique fixed point of  $\Lambda$ .

Consider a mapping  $\Upsilon = \Upsilon_1 + \Upsilon_2$  defined by

$$\begin{aligned} (\Upsilon^1\omega)(t) &= \int_0^t T(t-s)[F_1(s, v(s) + \omega(s)) - F_1(s, v(s))]ds, \quad \omega(s) \in C_0(\mathbb{R}^+, X), \quad t \in \mathbb{R}^+, \\ (\Upsilon^2\omega)(t) &= T(t)[x_0 - g(v|_{\mathbb{R}^+} + \omega)] - \int_{-\infty}^0 T(t-s)F_1(s, v(s))ds \\ &\quad + \int_0^t T(t-s)F_2(s, v(s) + \omega(s))ds, \quad \omega(s) \in C_0(\mathbb{R}^+, X), \quad t \in \mathbb{R}^+. \end{aligned}$$

From our assumptions it follows that  $\Upsilon$  is well defined and maps  $C_0(\mathbb{R}^+, X)$  into itself. Moreover, there exists a constant  $k_0 > 0$  such that

$$\Upsilon_1\omega_1 + \Upsilon_2\omega_2 \in \Omega_{k_0} \text{ for every pair } \omega_1(t), \omega_2(t) \in \Omega_{k_0},$$

(see the Step 2 in the proof of Theorem 3.6 for more details). Thus, to be able to apply Lemma 2.7 to obtain a fixed point of  $\Upsilon$ , we need to prove that  $\Upsilon_1$  is a strict contraction and  $\Upsilon_2$  is completely continuous on  $\Omega_{k_0}$ .

From (15) and the Step 3 in the proof of Theorem 3.6 it follows that  $\mathcal{T}_1$  is a strict contraction. Also, since  $g : AP_\omega(\mathbb{R}^+, X) \rightarrow X$  is completely continuous, it follows from the Step 4 in the proof of Theorem 3.6 that  $\mathcal{T}_2$  is completely continuous. Now, applying Lemma 2.7 we obtain that  $\mathcal{T}$  has a fixed point  $\omega(t) \in C_0(\mathbb{R}^+, X)$ , which gives rise to an asymptotically  $\omega$ -periodic mild solution  $v(t) + \omega(t)$ .  $\square$

**Corollary 3.10.** *The nonlocal problem (P) with  $g(x)$  given by (3) for  $x \in AP_\omega(\mathbb{R}^+, X)$  has at least one asymptotically  $\omega$ -periodic mild solution provided that*

$$M \sum_{i=1}^p |c_i| + \frac{ML_1}{\delta} + M\rho_1\rho_3 < 1.$$

## 4 Applications

In this section, an example is given to illustrate the practical usefulness of the theoretical results established in the preceding section.

Consider the partial differential equation with homogeneous Dirichlet boundary condition and nonlocal initial condition of the form

$$\begin{cases} \frac{\partial u(t, \xi)}{\partial t} = \frac{\partial^2}{\partial \xi^2} u(t, \xi) + \frac{1}{4} \sin t \sin u(t, \xi) \\ \quad + \frac{1}{4} e^{-t} u(t, \xi) \cos u^3(t, \xi), \quad t \in \mathbb{R}^+, \xi \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, \quad t \in \mathbb{R}^+, \\ u(0, \xi) = u_0(\xi) + \sum_{i=1}^4 \frac{1}{16} u(t_i, \xi), \quad \xi \in [0, \pi], \end{cases} \quad (16)$$

where  $0 < t_1 < t_2 < t_3 < t_4 < +\infty$ .

Take  $X = L^2[0, \pi]$  with norm  $\|\cdot\|$  and define  $A : D(A) \subset X \rightarrow X$  given by  $Ax = \frac{\partial^2 x(\xi)}{\partial \xi^2}$  with the domain

$$D(A) = \left\{ x(\cdot) \in X : x'' \in X, x' \in X \text{ is absolutely continuous on } [0, \pi], x(0) = x(\pi) = 0 \right\}.$$

It is well known that  $A$  is self-adjoint, with compact resolvent and is the infinitesimal generator of an analytic as well as compact semigroup  $\{T(t)\}_{t \geq 0}$  satisfying  $\|T(t)\| \leq e^{-t}$  for  $t > 0$ .

Now, let

$$\begin{aligned} x_0 &= u_0(\xi) \in X, \\ F_1(t, x(\xi)) &= \frac{1}{4} \sin t \sin x(\xi) \text{ for all } t \in \mathbb{R} \text{ and } x \in X, \\ F_2(t, x(\xi)) &= \frac{1}{4} e^{-t} x(\xi) \cos x^3(\xi) \text{ for all } t \in \mathbb{R}^+ \text{ and } x \in X, \\ g(x(t)) &= \sum_{i=1}^4 \frac{1}{16} x(t_i) \text{ for } x \in AP_{2\pi}(\mathbb{R}^+, X). \end{aligned}$$

Then it is easy to verify that  $F_1(t, x) \in C_{2\pi}(\mathbb{R} \times X, X)$  and

$$\|F_1(t, x) - F_1(t, y)\| \leq \frac{1}{4} \|x - y\| \text{ for all } t \in \mathbb{R}, x, y \in X,$$

$F_2 : \mathbb{R}^+ \times X \rightarrow X$  is continuous and

$$\|F_2(t, x)\| \leq \frac{1}{4} e^{-t} \|x\| \text{ for all } t \in \mathbb{R}^+, x \in X,$$

which implies  $F_2(t, x) \in C_0(\mathbb{R}^+ \times X, X)$ , furthermore

$$F(t, x) = F_1(t, x) + F_2(t, x) \in AP_{2\pi}(\mathbb{R}^+ \times X, X),$$

$g : AP_{2\pi}(\mathbb{R}^+, X) \rightarrow X$  satisfies

$$\|g(x) - g(y)\| \leq \frac{1}{4} \|u - v\| \text{ for all } x, y \in AP_{2\pi}(\mathbb{R}^+, X).$$

Thus, equation (16) can be reformulated as the abstract nonlocal problem (P) and the assumption  $(H_1)$  as well as the assumption  $(H_2)$  holds with

$$L_1 = \frac{1}{4}, L_2 = \rho_2 = \frac{1}{4}, \Phi(r) = r, \Psi(r) = \|u_0\| + \frac{r}{4}, \beta(t) = \frac{1}{4}e^{-t}, \rho_1 = 1, \rho_3 \leq \frac{1}{4},$$

and

$$M \sum_{i=1}^p |c_i| + \frac{ML_1}{\delta} + M\rho_1\rho_3 \leq \frac{3}{4} < 1.$$

Then from Corollary 3.10 it follows that equation (16) at least has one asymptotically  $2\pi$ -periodic mild solution.

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