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On the boundedness of square function generated by the Bessel differential operator in weighted Lebesque $L_{p,\alpha}$ spaces

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Abstract: In this paper, we consider the square function

$$(Sf)(x) = \left(\int_0^\infty |(f \otimes \Phi_t)(x)|^2 \frac{dt}{t} \right)^{1/2}$$

associated with the Bessel differential operator $B_t = \frac{d^2}{dt^2} + \frac{(2\alpha+1)}{t}\frac{d}{dt}$, $\alpha > -1/2$, $t > 0$ on the half-line $\mathbb{R}_+ = [0, \infty)$. The aim of this paper is to obtain the boundedness of this function in $L_{p,\alpha}$, $p > 1$. Firstly, we proved $L_{2,\alpha}$-boundedness by means of the Bessel-Plancherel theorem. Then, its weak-type $(1, 1)$ and $L_{p,\alpha}$, $p > 1$ boundedness are proved by taking into account vector-valued functions.

Keywords: Generalized translation, Generalized convolution, Bessel translation operator, Bessel transform, Bessel Plancherel formula, Bessel differential operator, Square function

MSC: 42B35, 42A85

1 Introduction

The classical square function is defined by

$$(Sf)(x) = \left(\int_0^\infty |(f * \Phi_t)(x)|^2 \frac{dt}{t} \right)^{1/2}, \Phi \in S(\mathbb{R}^n)$$

where $S(\mathbb{R}^n)$ is the Schwartz space consisting of infinitely differentiable and rapidly decreasing functions, $\int_{\mathbb{R}^n} \Phi(x)dx = 0$ and $\Phi_t(x) = t^{-n}\Phi(\frac{x}{t})$, $t > 0$.

This function plays an important role in Fourier harmonic analysis, theory of functions and their applications. It has direct connection with L_2-estimates and Littlewood-Paley theory. Moreover, there are a lot of diverse variants of square functions and their various applications (see, Daly and Phillips [7], Jones, Ostrovskii and Rosenblatt [18], Kim [20], Aliev and Bayrakci [5], Keles and Bayrakci [19], etc.)

The Bessel differential operator B_t,

$$B_t = \frac{d^2}{dt^2} + \frac{(2\alpha+1)}{t}\frac{d}{dt}, \quad \alpha > -1/2, \ t > 0$$

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and the Laplace-Bessel differential operator Δ_B,

$$\Delta_B = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \left(\frac{\partial^2}{\partial x_n^2} + \frac{(2\alpha+1)}{x_n} \frac{\partial}{\partial x_n} \right), \ \alpha > -1/2, \ x_n > 0$$

are known as important technical tools in analysis and its applications.

The relevant Fourier-Bessel harmonic analysis, associated with the Bessel differential operator B_t (or the Laplace-Bessel differential operator Δ_B) has been a research area for many mathematicians such as Levitan [24, 25], Kipriyanov and Klyuchantsev [21], Trimeche [32], Lyakhov [26], Stempak [30], Gadjiev and Aliev [10, 11], Aliev and Bayrakci [3, 4], Aliev and Saglik [6], Ekincioglu and Serbetci [9], Hasanov [17], Guliyev [14–16], and others.

The Bessel translation operator is one of the most important generalized translation operators on the half-line $\mathbb{R}_+ = [0, \infty)$, [24, 32]. It is used while studying various problems connected with Bessel operators (see, [22], [27] and bibliography therein).

In this paper, the square function associated with the Bessel differential operator B_t is introduced on the half-line $\mathbb{R}_+ = [0, \infty)$ and its $L_{2,\alpha}-$ boundedness by means of the Bessel-Plancherel theorem is proved. Then, $(1, 1)$ weak-type and $L_{p,\alpha}$, $1 < p < \infty$ boundedness of this function are obtained by taking into account vector-valued functions. For this, some necessary definitions and auxiliary facts are given in Section 2. The main results of the paper are formulated and proved in Section 3.

2 Preliminaries

Let $\mathbb{R}_+ = [0, \infty)$, $C(\mathbb{R}_+)$ be the set of continuous functions on $\mathbb{R}_+$, $C^{(k)}(\mathbb{R}_+)$, the set of even k-times differentiable functions on $\mathbb{R}_+$ and $S(\mathbb{R})$ be the Schwartz space consisting of infinitely differentiable and rapidly decreasing functions on $\mathbb{R}$ and $S^+(\mathbb{R}_+)$ be the subspace of even functions on $S(\mathbb{R})$.

For a fixed parameter $\alpha > -1/2$, let $L_{p,\alpha} = L_{p,\alpha}(\mathbb{R}_+)$ be the space of measurable functions f defined on $\mathbb{R}_+$ and the norm

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p \, x^{2\alpha+1} dx \right)^{1/p}, \quad 1 \le p < \infty \tag{1}$$

is finite. In the case $p = \infty$, we identify L_∞ with C_0, the corresponding space of continuous functions vanishing at infinity.

Denoted by T^s, $s \in \mathbb{R}_+$ the Bessel translation operator acts according to the law

$$T^s f(t) = c_\alpha \int_0^\pi f(\sqrt{s^2 - 2st\cos\xi + t^2})(\sin\xi)^{2\alpha} d\xi, \tag{2}$$

where

$$c_\alpha = \left(\int_0^\pi (\sin\xi)^{2\alpha} d\xi \right)^{-1} = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \tag{3}$$

and the following relations are known [25] :

$$T^s f(t) = T^t f(s) \ ; \ T^s T^\tau f(t) = T^\tau T^s f(t);$$
$$T^s f(t) = T^{-s} f(-t) \ ; \ T^0 f(t) = f(t);$$
$$\int_0^\infty \left(T^s f(t)\right) g(t) t^{2\alpha+1} dt = \int_0^\infty f(t)(T^s g(t)) t^{2\alpha+1} dt. \tag{4}$$

It is not difficult to see the following inequality

$$\left\| T^s f \right\|_\infty \le \|f\|_\infty$$

that is, T^s is a continuous operator in C_0. Moreover, for $1 \le p < \infty$ and $f \in S^+(\mathbb{R}_+)$ it is shown that

$$|T^s f(t)|^p \le T^s(|f(t)|^p). \tag{5}$$

For this, we define a measure on the $[0, \pi]$ by $d\mu(\varphi) = c_\alpha (\sin\varphi)^{2\alpha}\, d\varphi$, where c_α is defined by (3). By using (2) and the Hölder inequality, we have

$$\begin{aligned} |T^s f(t)| &= \Big| \int_0^\pi f\left(\sqrt{s^2 - 2st\cos\xi + t^2}\right)\, d\mu(\varphi) \Big| \\ &\le \left(\int_0^\pi |f\left(\sqrt{s^2 - 2st\cos\xi + t^2}\right)|^p d\mu(\varphi) \right)^{1/p} \left(\int_0^\pi d\mu(\varphi) \right)^{1/q} \\ &= (T^s(|f(t)|^p))^{1/p}, \quad \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

Further, by using (5) and (4) we obtain

$$\begin{aligned} \|T^s f\|_{p,\alpha}^p &= \int_0^\infty |T^s f(t)|^p\, t^{2\alpha+1} dt \le \int_0^\infty T^s(|f(t)|^p)\, t^{2\alpha+1} dt \\ &= \int_0^\infty |f(t)|^p (T^s 1)\, t^{2\alpha+1} dt = \int_0^\infty |f(t)|^p\, t^{2\alpha+1} dt \\ &= \|f\|_{p,\alpha}^p. \end{aligned} \tag{6}$$

As $S^+(\mathbb{R}_+)$ is dense $L_{p,\alpha}$ for $p < \infty$, (6) stays valid for every function in $f \in L_{p,\alpha}$.

Note that T^s, $s \in \mathbb{R}_+$ is closely connected with the Bessel differential operator

$$B_t = \frac{d^2}{dt^2} + \frac{(2\alpha+1)}{t}\frac{d}{dt}, \ \alpha > -1/2, \ t > 0.$$

It is known that the function $u(t,s) = T^s f(t), f \in C^2(\mathbb{R}_+)$ is the solution the following Cauchy problem, (see [8, 25]):

$$\begin{cases} B_t u(t,s) = B_s u(t,s) \\ u(t,0) = f(t)\,, \ \frac{\partial u}{\partial s}(t,0) = 0 \end{cases}.$$

The Bessel transform of order $\alpha > -1/2$ of a function $f \in L_{1,\alpha}$ is defined by

$$(\mathcal{B}f)(\lambda) = \int_0^\infty f(t)\, j_\alpha(\lambda t)\, t^{2\alpha+1} dt \quad \lambda \ge 0, \tag{7}$$

and the inverse Bessel transform is given by the formula

$$\mathcal{B}^{-1} = 2^\alpha \Gamma(\alpha+1))^{-2} \mathcal{B}$$

where

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\, z^{-\alpha} J_\alpha(z), \quad (\alpha > -1/2, \ 0 < z < \infty)$$

is the normalized Bessel function and $J_\alpha(z)$ is the Bessel function of the first kind. From the following integral presentation for $j_\alpha(t)$ (see[13], Eq. 8.411(8))

$$j_\alpha(t) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_{-1}^1 \left(1 - u^2\right)^{\alpha-1/2} \cos(tu)\ du \tag{8}$$

we have

$$|j_\alpha(t)| \le 1, \quad t \in \mathbb{R} \tag{9}$$

and the equality takes place only at $t = 0$. We also note that, by using (8) and the Riemann-Lebesgue Lemma, we have

$$\lim_{\lambda\to\infty} (\mathcal{B}f)(\lambda) = 0.$$

Moreover, from (9) we have

$$|(\mathcal{B}f)(\lambda)| \le \int_0^\infty |f(t)|\, |j_\alpha(\lambda t)|\, t^{2\alpha+1} dt \le \|f\|_{1,\alpha}$$

and thus $\|\mathcal{B}f\|_\infty \le \|f\|_{1,\alpha}$ is obtained.

The asymptotic formula for $J_\alpha(r)$ is as follows ([28]):

$$J_\alpha(r) = O(r^{-1/2}), \quad r \to \infty. \tag{10}$$

Then, the following asymptotic formula for $j_\alpha(r)$ is obtained easily:

$$j_\alpha(r) = O(r^{-\alpha-1/2}), \quad r \to \infty. \tag{11}$$

The following Lemmas will be needed in proving the main results containing important properties of Bessel transform.

Lemma 2.1 ([25]). *Let $f \in L_{1,\alpha}$ then*

$$(\mathcal{B}f(at))(x) = a^{-2\alpha-2} (\mathcal{B}f)\left(\frac{x}{a}\right), \quad a > 0.$$

Lemma 2.2 ([25], Bessel-Plancherel formula). *Let $f \in L_{1,\alpha} \cap L_{2,\alpha}$ then*

$$\|\mathcal{B}f\|_{2,\alpha} = \|f\|_{2,\alpha}. \tag{12}$$

The generalized convolution generated by the Bessel translation operator for $f, g \in L_{1,\alpha}$ is defined by

$$(f \otimes g)(s) = \int_0^\infty T^s f(t) g(t)\, t^{2\alpha+1} dt. \tag{13}$$

The convolution operation makes sense if the integral on the right-hand side of (13) is defined; in particular, if $f, g \in S^+(\mathbb{R}_+)$, then the convolution $f \otimes g$ also belongs to $S^+(\mathbb{R}_+)$.

Now, we list some properties of generalized convolution as follows: (see details in [25])

$$\begin{aligned} f \otimes g &= g \otimes f, \\ (f \otimes g) \otimes h &= f \otimes (g \otimes h), \\ \mathcal{B}(f \otimes g)(\lambda) &= (\mathcal{B}f)(\lambda)(\mathcal{B}g)(\lambda). \end{aligned} \tag{14}$$

Further, by using (6) and the Hölder inequality it is not difficult to prove the corresponding Young inequality

$$\|f \otimes g\|_{p,\alpha} \le \|f\|_{1,\alpha} \|g\|_{p,\alpha}, \quad f \in L_{1,\alpha}, \ g \in L_{p,\alpha}, \ 1 \le p \le \infty.$$

3 Main results and proofs

In this part, the $L_{2,\alpha}$ boundedness of the square function generated by the Bessel differential operator is proved by Bessel-Plancherel formula, then its $(1, 1)$ weak-type and $L_{p,\alpha}$, $1 < p < \infty$ boundedness is obtained by using vector-valued functions.

Definition 3.1. *Let* $\Phi \in S^+(\mathbb{R}^+)$ *and* $\int_0^\infty \Phi(x)x^{2\alpha+1}dx = 0$. *The square function associated with the Bessel differential operator is defined by*

$$(\mathcal{S}f)(x) = \left(\int_0^\infty |(f \otimes \Phi_t)(x)|^2 \frac{dt}{t} \right)^{1/2} \tag{15}$$

where $\Phi_t(x) = t^{-2\alpha-2}\Phi\left(\frac{x}{t}\right)$, $t > 0$, $\alpha > -\frac{1}{2}$.

An important trend in mathematical analysis and applications is to investigate convolution-type operators. Convolution type square functions have a very direct connection with L_2-estimates by the Plancherel theorem.

For this reason, we have proved $L_{2,\alpha}$-boundedness of the square function (15), associated with the Bessel differential operator by using Bessel-Plancherel formula (12) in the following.

Theorem 3.2. *Let the square function* $\mathcal{S}f$ *be defined as (15). If* $f \in L_{2,\alpha}$ *then there is* $c > 0$ *such that*

$$\|\mathcal{S}f\|_{2,\alpha} \le c\|f\|_{2,\alpha}.$$

Proof. Firstly, let $f \in S^+(\mathbb{R}_+)$. By making use of the Fubini theorem and Bessel-Plancherel formula, we have

$$\begin{aligned}
\|\mathcal{S}f\|_{2,\alpha}^2 &= \int_0^\infty \left(\int_0^\infty |(f \otimes \Phi_t)(x)|^2 \frac{dt}{t} \right) x^{2\alpha+1}dx \\
&= \int_0^\infty \int_0^\infty |(f \otimes \Phi_t)(x)|^2 x^{2\alpha+1}dx\frac{dt}{t} \\
&= \int_0^\infty \|f \otimes \Phi_t\|_{2,\alpha}^2 \frac{dt}{t} \\
&= \int_0^\infty \|\mathcal{B}(f \otimes \Phi_t)\|_{2,\alpha}^2 \frac{dt}{t} \\
&= \int_0^\infty \int_0^\infty |\mathcal{B}(f \otimes \Phi_t)(x)|^2 x^{2\alpha+1}dx\frac{dt}{t}.
\end{aligned}$$

Taking into account (14) and then using Fubini theorem, we get

$$\begin{aligned}
\|\mathcal{S}f\|_{2,\alpha}^2 &= \int_0^\infty \int_0^\infty |(\mathcal{B}f)(x)|^2 |(\mathcal{B}\Phi_t)(x)|^2 x^{2\alpha+1}dx\frac{dt}{t} \\
&= \int_0^\infty |(\mathcal{B}f)(x)|^2 \left(\int_0^\infty |(\mathcal{B}\Phi_t)(x)|^2 \frac{dt}{t} \right) x^{2\alpha+1}dx.
\end{aligned} \tag{16}$$

Since $\Phi_t(x) = t^{-2\alpha-2}\Phi\left(\frac{x}{t}\right)$, then using Lemma 1, we have

$$\begin{aligned}
(\mathcal{B}\Phi_t)(x) &= t^{-2\alpha-2}(\mathcal{B}\Phi(\frac{x}{t})) \\
&= t^{-2\alpha-2}t^{2\alpha+2}(\mathcal{B}\Phi)(tx) \\
&= (\mathcal{B}\Phi)(tx).
\end{aligned}$$

Thus

$$\int_0^\infty |(\mathcal{B}\Phi_t)(x)|^2 \frac{dt}{t} = \int_0^\infty |(\mathcal{B}\Phi)(tx)|^2 \frac{dt}{t}$$
$$(\text{set } \tau = tx)$$

$$= \int_0^\infty |(\mathcal{B}\Phi)(\tau)|^2 \frac{d\tau}{\tau}.$$

By taking this into account in the formula (16) and using (12) we have

$$\begin{aligned} \|\mathcal{S}f\|_{2,\alpha}^2 &= c \int_0^\infty |(\mathcal{B}f)(x)|^2 x^{2\alpha+1} dx \\ &= c\|f\|_{2,\alpha}^2 \end{aligned} \tag{17}$$

where $c = \int_0^\infty |(\mathcal{B}\Phi)(\tau)|^2 \frac{d\tau}{\tau}$. Let us show that $c < \infty$.

$$\begin{aligned} \int_0^\infty |(\mathcal{B}\Phi)(\tau)|^2 \frac{d\tau}{\tau} &= \int_0^1 |(\mathcal{B}\Phi)(\tau)|^2 \frac{d\tau}{\tau} + \int_1^\infty |(\mathcal{B}\Phi)(\tau)|^2 \frac{d\tau}{\tau} \\ &= I_1 + I_2. \end{aligned}$$

Firstly, let us estimate I_1. Since $\int_0^\infty \Phi(x) x^{2\alpha+1} dx = 0$, we have

$$|(\mathcal{B}\Phi)(\tau)| \le \int_0^\infty |\Phi(t)|\, |j_\alpha(\tau t) - 1|\, t^{2\alpha+1} dt$$

and taking into account (8) for the normalized Bessel function $j_\alpha(t)$ we get

$$\begin{aligned} |j_\alpha(\tau t) - 1| &\le \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_{-1}^1 \left(1-u^2\right)^{\alpha-1/2} |\cos(\tau t u) - 1|\, du \\ &= \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_{-1}^1 \left(1-u^2\right)^{\alpha-1/2} \sin^2\left(\frac{\tau t u}{2}\right) du \\ &\le c_1 t^2 \tau^2. \end{aligned}$$

Therefore,

$$|(\mathcal{B}\Phi)(\tau)| \le c_1 \tau^2 \int_0^\infty |\Phi(t)| t^{2\alpha+3} dt = c_2 \tau^2$$

and

$$I_1 = \int_0^1 |(\mathcal{B}\Phi)(\tau)|^2 \frac{d\tau}{\tau} = c_2^2 \int_0^1 \tau^4 \frac{d\tau}{\tau} = c_3 < \infty.$$

Now we estimate I_2. For this, we need the following asymptotic formula for $j_\alpha(r)$, (cf.(11)):

$$|j_\alpha(u)| \le \left\{ \begin{array}{cc} c_4, & 0 < u \le 1 \\ \dfrac{c_5}{u^{\alpha+\frac{1}{2}}}, & u > 1 \end{array} \right\} \le \frac{c_6}{u^{\alpha+\frac{1}{2}}}, \quad c_6 = \max\{c_4, c_5\}.$$

Hence

$$\begin{aligned} |(\mathcal{B}\Phi)(\tau)| &\le \int_0^\infty |\Phi(t)|\, |j_\alpha(\tau t)|\, t^{2\alpha+1} dt \\ &\le \int_0^\infty |\Phi(t)| \frac{c_6}{\tau^{\alpha+\frac{1}{2}} t^{\alpha+\frac{1}{2}}} t^{2\alpha+1} dt \\ &= \frac{c_7}{\tau^{\alpha+\frac{1}{2}}}, \quad \alpha > -1/2 \end{aligned}$$

and we have

$$I_2 = \int_1^\infty |(\mathcal{B}\Phi)(\tau)|^2 \frac{d\tau}{\tau} \le c_7 \int_1^\infty \frac{1}{\tau^{2\alpha+1}} \frac{d\tau}{\tau} = c_8 < \infty.$$

For arbitrary $f \in L_{2,\alpha}$, we will take into account that the Schwartz space $S^+(\mathbb{R}_+)$ is dense in $L_{2,\alpha}$. Namely, let (f_n) be a sequence of functions in $S^+(\mathbb{R}_+)$, which converges to f in $L_{2,\alpha}$-norm.

From the "triangle inequality" $(||u||_{2,\alpha} - ||v||_{2,\alpha})^2 \le ||u-v||^2_{2,\alpha}$, we have

$$\begin{aligned}((Sf_n)(x) - (Sf_m)(x))^2 &= ((\int_0^\infty |(f_n \otimes \Phi_t)(x)|^2 \frac{dt}{t})^{1/2} - (\int_0^\infty |(f_m \otimes \Phi_t)(x)|^2 \frac{dt}{t})^{1/2})^2 \\ &\le \int_0^\infty |((f_n \otimes \Phi_t) - (f_m \otimes \Phi_t))(x)|^2 \frac{dt}{t} \\ &= \int_0^\infty |((f_n - f_m) \otimes \Phi_t)|^2 \frac{dt}{t}.\end{aligned}$$

and

$$|(Sf_n(x) - (Sf_m)(x)| \le (\int_0^\infty |((f_n - f_m) \otimes \Phi_t)|^2 \frac{dt}{t})^{1/2} = S(f_n - f_m)(x).$$

Hence, by (3.17) we get

$$||Sf_n - Sf_m||_{2,\alpha} \le ||S(f_n - f_m)||_{2,\alpha} \le c||f_n - f_m||_{2,\alpha}.$$

This shows that the sequence (Sf_n) converges to (Sf) in $L_{2,\alpha}$–norm. Thus

$$||Sf||_{2,\alpha} \le c||f||_{2,\alpha}\,, \quad \forall\, f \in L_{2,\alpha}$$

and the proof is complete. □

Now, taking into account vector-valued functions spaces, we will obtain $L_{p,\alpha}(\mathbb{R}_+)$, $1 < p < \infty$ boundedness of the square function associated with the Bessel differential operator.

For this, necessary definitions and theorems are given below. The first theorem is well known as the Marcikiewicz interpolation theorem for the vector-valued functions. The other theorem is the extension of Benedek-Calderon-Panzone principle.

Let H be a seperable Hilbert space. We say that a function f defined on $\mathbb{R}_+ = [0,\infty)$ and with values in H is measurable if the scalar valued function $(f(x), h)$ is measurable for every h in H, where $(,)$ denotes the inner product of H and h denotes an arbitrary vector of H. Throughout the text, the absolute value $|.|_H$ denotes the norm in H. Moreover, let H_1 and H_2 be two seperable Hilbert spaces, and $B(H_1, H_2)$ denote the Banach spaces of bounded linear operators A from H_1 to H_2 endowed with the norm

$$|A|_{B(H_1,H_2)} = |A| = \sup_{h \in H_1} \left(\frac{|Ah|_{H_2}}{|h|_{H_1}} \right).$$

Let $L_{p,\alpha}(\mathbb{R}_+, H)$ be the space of measurable functions $f(x)$ from $\mathbb{R}_+$ to H with the norm

$$\|f\|_{L_{p,\alpha}(\mathbb{R}_+,H)} = ||f||_{p,\alpha} = \left(\int_0^\infty |f(x)|_H^p \, x^{2\alpha+1} dx \right)^{1/p}, \quad 1 \le p < \infty$$

is finite. If $p = \infty$, then the norm

$$\|f\|_{L_\infty(\mathbb{R}_+,H)} = ess\sup_{x \in \mathbb{R}^n} |f(x)|_H$$

is finite, (see for details, [28]; p.27-30, [29]; p.45-46 [31]; p.307-309).

Theorem 3.3 ([31], Theorem 2.1, p.307). *Let be A a sublinear operator defined on $L^0_\infty(\mathbb{R}_+, H_1)$, i.e., compactly supported, bounded H_1-valued functions, with values in $M(\mathbb{R}_+, H_2)$, i,e., the space of measurable, H_2–valued function. Suppose in addition that for $f \in L^0_\infty(\mathbb{R}_+, H_1)$*

$$\lambda |\{|Af|_{H_2} > \lambda\}| \le c_1 \|f\|_{1,\alpha}$$

and

$$\lambda^r |\{|Af|_{H_2} > \lambda\}| \le c_r^r \|f\|_{r,\alpha}^r,$$

where c_1 and c_r are independent of λ and f. Then for each $1 < p < r$, we have that $Af \in L_{p,\alpha}(\mathbb{R}_+, H_2)$ whenever $f \in L_{p,\alpha}(\mathbb{R}_+, H_1)$ and there is a constant $c = c_{1,r,p}$ independent of f such that $\|Af\|_{p,\alpha} \le c\|f\|_{p,\alpha}$.

Theorem 3.4 ([31], Theorem 2.2, p.307). *Suppose a linear operator A defined in $L^0_\infty(\mathbb{R}_+, H_1)$ and with values in $M(\mathbb{R}_+, H_2)$ verifies*

$$\lambda^r |\{|Af|_{H_2} > \lambda\}| \le c_1^r \|f\|_{r,\alpha}^r, \quad \text{some } r > 1$$

and if f has support in $B(x_0, R)$ and integral 0, then there are constants $c_2, c_3 > 1$ independent of f so that

$$\int_{\mathbb{R}^n / B(x_0, c_2 R)} |Af(x)|_{H_2} dx \le c_3 \|f\|_{1,\alpha}. \tag{18}$$

Then

$$\lambda |\{|Af|_{H_2} > \lambda\}| \le c \|f\|_{1,\alpha}.$$

Now let $H_1 = \mathbb{R}_+$ and $H_2 = L_{2,\alpha}(\mathbb{R}_+, \frac{dt}{t})$, $\alpha > -1/2$ be the Hilbert space of square integrable functions on the half-line with respect to the measure $\frac{dt}{t}$ and the norm

$$|\varphi|_{H_2} = \left(\int_0^\infty |\varphi(t)|^2 \frac{dt}{t} \right)^{1/2}.$$

Since $\Phi \in S^+(\mathbb{R}^+)$ and $\int_0^\infty \Phi(x) x^{2\alpha+1} dx = 0$ then we define $K(x)$ to be the H_2-valued function given by

$$K(x) = t^{-2\alpha-2} \Phi\left(\frac{x}{t}\right) = \Phi_t(x).$$

So, the square function associated with the Bessel differential operator $(Sf)(x)$ is the linear operator $(Af)(x) = (f \otimes K)(x)$ and Af takes its values in H_2.

Thus, the condition (18) is equivalent to the following inequality

$$\int_{x \ge 2y} |T^y K(x) - K(x)|_{H_2} x^{2\alpha+1} dx \le c, \quad y \in \mathbb{R}_+. \tag{19}$$

Now let us calculate (19). For this, since $\Phi \in S^+(\mathbb{R}^+)$, we take

$$|\Phi(x)| \le c (1+x)^{-(q+\theta)}, \quad q = 2\alpha + 2, \ \theta > 0 \text{ and } |\Phi_t(x)| \le \frac{c t^\theta}{(1+x)^{(q+\theta)}}$$

and for $0 < \epsilon < \min\{\theta, q\}$ by using Hölder inequality we have

$$\int_{x \ge 2y} |T^y K(x) - K(x)|_{H_2} x^{2\alpha+1} dx = \int_{x \ge 2y} x^{-\frac{\epsilon+q}{2}} |T^y \Phi_t(x) - \Phi_t(x)|_{H_2} x^{\frac{\epsilon+q}{2}} x^{2\alpha+1} dx$$

$$\le \left(\int_{x \ge 2y} x^{-(\epsilon+q)} x^{2\alpha+1} dx \right)^{1/2} \left(\int_{x \ge 2y} |T^y \Phi_t(x) - \Phi_t(x)|_{H_2}^2 x^{\epsilon+q} x^{2\alpha+1} dx \right)^{1/2}$$

$$\leq cy^{-\frac{\epsilon}{2}}\left(\int\limits_{x\geq 2y} x^{\epsilon+q}\left(\int\limits_0^\infty |T^y\Phi_t(x)-\Phi_t(x)|^2\frac{dt}{t}\right)x^{2\alpha+1}dx\right)^{1/2}$$

$$\leq cy^{-\frac{\epsilon}{2}}\left(\int\limits_0^\infty t^{-2q}\left(\int\limits_{x\geq 2y} x^{\epsilon+q}|T_t^y\Phi(\frac{x}{t})-\Phi(\frac{x}{t})|^2x^{2\alpha+1}dx\right)\frac{dt}{t}\right)^{1/2}.$$

Since

$$|T_t^y\Phi(\frac{x}{t})-\Phi(\frac{x}{t})|\leq c\left(\frac{t}{x}\right)^{q+\epsilon}$$

then we get

$$\int\limits_{x\geq 2y}|T^yK(x)-K(x)|_{H_2}x^{2\alpha+1}dx$$

$$\leq cy^{-\frac{\epsilon}{2}}\left(\int\limits_0^\infty t^{-q+\epsilon}\left(\int\limits_0^\infty |T_t^y\Phi(\frac{x}{t})-\Phi(\frac{x}{t})|x^{2\alpha+1}dx\right)\frac{dt}{t}\right)^{1/2}$$

$$\leq cy^{-\frac{\epsilon}{2}}\left(\int\limits_0^\infty t^{-q+\epsilon}(t^q 2||\Phi_t||_{1,\alpha})\frac{dt}{t}\right)^{1/2}\leq cy^{-\frac{\epsilon}{2}}\left(\int\limits_0^y t^{\epsilon-1}dt+\int\limits_y^\infty\frac{dt}{t^{\epsilon+1}}\right)^{1/2}$$

$$\leq cy^{-\frac{\epsilon}{2}}y^{\frac{\epsilon}{2}}=c.$$

Finally, by using Theorem 3.4, we see that the square function associated with the Bessel differential operator $\mathcal{S}f$ is of weak-type $(1,1)$ and since we have already verified the $L_{2,\alpha}(\mathbb{R}_+)$-boundedness then by the Marcinkiewicz interpolation theorem for the vector-valued functions, (Theorem 3.3) Sf is also of type (p,p), $1<p<2$ and consequently, by a simple duality argument $\mathcal{S}f$ is of type (p,p), $1<p<\infty$.

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