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Research Article

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On Banach and Kuratowski Theorem, K-Lusin sets and strong sequences

Abstract: In 2003 Bartoszyński and Halbeisen published the results on various equivalences of Kuratowski and Banach theorem from 1929 concerning some aspect of measure theory. They showed that the existence of the so called BK-matrix related to Banach and Kuratowski theorem is equivalent to the existence of a K-Lusin set of cardinality continuum. On the other hand, in 1965 Efimov introduced the strong sequences method and using this method proved some well-known theorems in dyadic spaces. The goal of this paper is to show that the existence of such a K-Lusin set is equivalent to the existence of strong sequences of the same cardinality. Some applications of this results are also shown.

Keywords: GCH, Consistency results, BK-matrix, Lusin set, Strong sequences

MSC: 03E05, 03E10, 03E20, 03E35

1 Introduction

In the paper [1] Bartoszyński and Halbeisen proved some equivalences of the existence of K-Lusin sets of size $2^{\aleph_0}$ and showed that the existence of such a set is independent of ZFC+¬CH. Some of their results were inspired by a paper of Banach and Kuratowski [2] published in 1929, in which the authors solved a problem in measure theory concerning the existence of a non-vanishing σ-additive finite measure on the real line which is defined for every set of reals by using the following combinatorial result, (see [1]).

Banach and Kuratowski Theorem. *Under the assumption of CH, there is an infinite matrix $A_k^i \subseteq [0, 1]$ (where $i, k \in \omega$) such that*
(i) for each $i \in \omega$, $[0, 1] = \bigcup_{k \in \omega} A_k^i$,
(ii) for each $i \in \omega$, if $k \neq k'$ then $A_k^i \cap A_{k'}^i = \varnothing$,
(iii) for every sequence $k_0, k_1, …, k_i, …$ of ω the set

$$\bigcap_{i \in \omega}(A_0^i \cup A_1^i \cup … \cup A_{k_i}^i)$$

is at most countable.

The matrix from the theorem above is called in the literature as a *BK-matrix*.

Following [1] recall that if $\mathcal{F} \subseteq {}^\omega\omega$ then

$$\lambda(\mathcal{F}) = \min\{\eta: \forall_{g \in {}^\omega\omega} |\{f \in \mathcal{F}: f \leq g\}| < \eta\},$$

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where $f, g \in {}^\omega\omega, f \preceq g$ iff $|\{n \in \omega : f(n) > g(n)\}| < \omega$ and

$$\mathfrak{l} = \min\{\lambda(\mathcal{F}) : \mathcal{F} \subseteq {}^\omega\omega \wedge |\mathcal{F}| = \mathfrak{c}\}.$$

The crucial point in the Banach and Kuratowski proof, ([2], Theorem II), is the following result

Fact 1.1. *The existence of BK-Matrix is equivalent to $\mathfrak{l} = \aleph_1$.*

Obviously CH implies that $\mathfrak{l} = \aleph_1$. Among other theorems, Bartoszyński and Halbeisen showed that the existence of a K-Lusin set of cardinality $\mathfrak{c}$ is equivalent to $\mathfrak{l} = \aleph_1$, ([1], Lemma 2.3), and to the existence of a concentrated set of cardinality $\mathfrak{c}$, ([1], Proposition 3.4).

On the other hand, Efimov in [3] introduced the combinatorial method so called strong sequences method and used it for proving some well-known theorems in dyadic spaces (among others the Marczewski theorem on cellularity, the Shanin theorem on a calibre).

The goal of this paper is to show that the existence of a special kind of a strong sequence is equivalent to the existence of a K-Lusin set of cardinality $\mathfrak{c}$, which is an easy consequence of $\mathfrak{l} = \aleph_1$ and of the existence of a concentrated set of cardinality $\mathfrak{c}$. The results in this paper will be proved in a generalized version which generates further equivalences, especially between the existence of generalized strong sequences and the so called generalized BK-matrix as a kind of generalized independent family, (see [4–6]).

2 On the existence of generalized K-Lusin sets and generalized strong sequences

Let $\kappa \geq \omega$ be a regular cardinal. In the set ${}^\kappa\kappa$ we consider the following relation, if $f, g \in {}^\kappa\kappa$ then $g \preceq f$ iff $|\{\alpha \in \kappa : g(\alpha) > f(\alpha)\}| < \kappa$.

A set $K \subseteq {}^\kappa\kappa$ is said to be *closed* iff for all $f \in K$ and for all $g \in {}^\kappa\kappa$ if $f \preceq g$ then $g \in K$. A closed set $K \subseteq {}^\kappa\kappa$ is κ-*compact* iff there is a function $f \in {}^\kappa\kappa$ such that $K \subseteq \{g \in {}^\kappa\kappa : g \preceq f\}$.

Definition 2.1. *A set X of size 2^κ is a generalized K-Lusin set if $|X \cap K| < 2^\kappa$ for every κ-compact set $K \subseteq {}^\kappa\kappa$.*

Now let $\mathcal{F} \subseteq {}^\kappa\kappa$ then

$$\lambda_\kappa(\mathcal{F}) = \min\{\tau : \forall_{g \in {}^\kappa\kappa} |\{f \in \mathcal{F} : f \preceq g\}| < \tau\},$$

$$\mathfrak{l}_\kappa = \min\{\lambda_\kappa(\mathcal{F}) : \mathcal{F} \subseteq {}^\kappa\kappa \wedge |\mathcal{F}| = 2^\kappa\}.$$

Assuming GCH we obtain $\mathfrak{l}_\kappa = 2^\kappa$. It follows from the next lemma.

Lemma 2.2 (GCH). *There exists a family $\mathcal{F} \subseteq {}^\kappa\kappa$ of cardinality κ^+ such that*

$$\lambda_\kappa(\mathcal{F}) = 2^\kappa.$$

Proof. Let

$$ {}^\kappa\kappa = \{g_\alpha : \alpha < \kappa^+\}.$$

We will construct a sequence $\{f_\alpha \in {}^\kappa\kappa : \alpha < \kappa^+\}$ such that
(i) $f_\beta \preceq f_\alpha$ for $\beta < \alpha < \kappa^+$
(ii) $g_\alpha \preceq f_\alpha$ for $\alpha < \kappa^+$
by transfinite recursion.

Assume that for $\alpha < \kappa^+$ the sequence $\{f_\beta \in {}^\kappa\kappa : \beta < \alpha\}$ fulfilling (i) and (ii) has been defined. Since κ^+ is regular, we have ${}^\kappa\kappa \smallsetminus \{f_\beta \in {}^\kappa\kappa : \beta < \alpha\} \neq \varnothing$.

Let α be a successor. Take $g_\alpha \in {}^\kappa\kappa$ and suppose that for each $f_\alpha \in {}^\kappa\kappa \smallsetminus \{f_\beta \in {}^\kappa\kappa : \beta < \alpha\}$ at least one of the conditions (i) or (ii) does not hold.

If (ii) does not hold then $|\{f \in {}^{\kappa}\kappa : f \leq g_{\beta}\}| = 2^{\kappa}$ for $\beta < \alpha$. A contradiction. The argumentation for proving that (i) holds is similar.

If α is limit, then we take $f_{\alpha} = \bigcup_{\beta < \alpha} f_{\beta}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The following lemma is an easy consequence of the above definitions and Lemma 2.2.

Lemma 2.3 (GCH). *The following are equivalent*
(a) $I_{\kappa} = 2^{\kappa}$
(b) there exists a generalized K-Lusin set of cardinality 2^{κ}.

Now we define a generalized strong sequence, (see also [6]). Let α be a cardinal number. We say that $A \subseteq {}^{\kappa}\kappa$ is *an α-directed set* iff for each $B \subset A$ of cardinality less than α there exists $f \in {}^{\kappa}\kappa$ such that $g \leq f$ for all $g \in B$.

Definition 2.4. *Let α and η be cardinals. A sequence $(H_{\phi})_{\phi < \eta}$, where $H_{\phi} \subset X$, is called an α-strong sequence if:*
1° H_{ϕ} is α-directed for all $\phi < \eta$,
2° $H_{\psi} \cup H_{\phi}$ is not α-directed for all $\phi < \psi < \eta$.

The following result is true.

Theorem 2.5 (GCH). *The following are equivalent*
(a) $I_{\kappa} = 2^{\kappa}$
(b) there exists a κ^{+}-strong sequence $(H_{\xi})_{\xi < \kappa^{+}}$ with $|H_{\xi}| < \kappa^{+}$ for all $\xi < \kappa^{+}$.

Proof. By Lemma 2.2 there exists a family $\mathcal{F} \subseteq {}^{\kappa}\kappa$ of cardinality κ^{+} with $\lambda_{\kappa}(\mathcal{F}) = 2^{\kappa}$. Choose $f_0 \in {}^{\kappa}\kappa$ arbitrarily. Let $H_0 = \{f \in \mathcal{F} : f \leq f_0\}$. Hence $|H_0| < 2^{\kappa}$. Let H_0 be the first element of a κ^{+}-strong sequence.

Assume that the κ^{+}-strong sequence $(H_{\xi})_{\xi < \psi}$ for $\psi < \kappa^{+}$ such that

$$H_{\xi} = \{f \in \mathcal{F} \setminus \bigcup_{\eta < \xi} H_{\eta} : f \leq f_{\xi}\},$$

where $f_{\xi} \in {}^{\kappa}\kappa \setminus \{f_{\eta} : \eta < \xi\}, f_{\xi} \perp f_{\eta}, (f_{\xi} \perp f_{\eta}$ means $f_{\eta} \not\leq f_{\xi}$ and $f_{\xi} \not\leq f_{\eta}$ for $\eta < \xi$) has been defined.

Since $\psi < \kappa^{+}$, $|\mathcal{F}| = \kappa^{+}$ and κ^{+} is regular, we have that there exists $f \in {}^{\kappa}\kappa \setminus \{f_{\xi} : \xi < \kappa^{+}\}$ such that $f \perp f_{\xi}$ for any $\xi < \psi$. Name such f by f_{ψ}. Let $H_{\psi} = \{f \in \mathcal{F} \setminus \bigcup_{\xi < \psi} H_{\xi} : f \leq f_{\psi}\}$. Obviously $|H_{\psi}| < \kappa^{+}$, (because of our assumptions). By the construction $H_{\xi} \cup H_{\psi}, \xi < \psi$ is not κ^{+}-directed, (because $f_{\psi} \perp f_{\xi}$ for any $\xi < \psi$).

Now let $(H_{\xi})_{\xi < \kappa^{+}}$ be a κ^{+}-strong sequence with $H_{\xi} \subseteq {}^{\kappa}\kappa$ and $|H_{\xi}| < \kappa^{+}$ for each $\xi < \kappa^{+}$. According to Definition 2.4 we have that for all $\xi < \psi$ there exist $h_{\xi} \in H_{\xi}, h_{\psi} \in H_{\psi}$ such that $h_{\xi} \perp h_{\psi}$. For any $\psi < \kappa^{+}$ let

$$H'_{\psi} = \{h_{\psi} \in H_{\psi} : \xi < \psi \Rightarrow \exists_{h_{\xi} \in H_{\xi}} h_{\xi} \perp h_{\psi}\}.$$

Since $H_{\psi}, \psi < \kappa^{+}$ is κ^{+}-directed, there exists $\bar{h}_{\psi} \in {}^{\kappa}\kappa$ such that $h \leq \bar{h}_{\psi}$, for all $h_{\psi} \in H'_{\psi}$. Thus we can describe H_{ψ} as follows

$$H_{\psi} = \{f \in {}^{\kappa}\kappa : f \leq \bar{h}_{\psi}\}.$$

Let $\mathcal{F} = \bigcup \{H_{\xi} : \xi < \kappa^{+}\}$. For any $g \in {}^{\kappa}\kappa$ consider $\{f \in \mathcal{F} : f \leq g\}$. Suppose that there exists $g_o \in {}^{\kappa}\kappa$ such that $|\{f \in \mathcal{F} : f \leq g_0\}| = \kappa^{+}$. But by definition of $\mathcal{F}$, there exists ξ_0 such that $\{f \in \mathcal{F} : f \leq g_0\} \subseteq H_{\xi_0}$. Hence $|H_{\xi_0}| = \kappa^{+}$. A contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The next corollary follows from Theorem 2.5 and Lemma 2.3.

Corollary 2.6 (GCH). *The following are equivalent*
(a) there exists a generalized K-Lusin set of cardinality κ^{+}
(b) there exists an κ^{+}-strong sequence $(H_{\xi})_{\xi < \kappa^{+}}$, $H_{\xi} \subseteq {}^{\kappa}\kappa$ with $|H_{\xi}| < \kappa^{+}$ for any $\xi < \kappa^{+}$.

For a cardinal α we introduce the following notation, (see [7])

$$\hat{s}_{\alpha} = sup\{\eta : \text{there exists an } \alpha\text{-strong sequence of cardinality } \eta\}.$$

Then by Theorem 2.5 we have

Corollary 2.7 (GCH). $\mathfrak{l}_\kappa \leq \hat{s}_{\kappa^+}$.

3 Some further results

Definition 3.1. *Let Q be a countable dense subset of the interval $[0, 1]$. Then $X \subseteq [0, 1]$ is called concentrated on Q if every open set of $[0, 1]$ containing Q, contains all but countably many elements of X.*

Assuming CH, Lemma 2.3 and Proposition 3.4 in [1] and Theorem 2.5 we have the following proposition.

Proposition 3.2 (CH). *The following are equivalent*
(a) there exists a $\aleph_1$-strong sequence $(H_\xi)_{\xi < 2^{\aleph_0}}$, with $|H_\xi| < 2^{\aleph_0}$, for any $\xi < 2^{\aleph_0}$,
(b) $\mathfrak{l} = \aleph_1$,
(c) there exists a K-Lusin set of cardinality $\mathfrak{c}$,
(d) there exists a concentrated set of cardinality $\mathfrak{c}$,
(e) there exists a BK-matrix.

The following fact is obvious, (see [1], Lemma 2.2).

Fact 3.3. *Every Lusin set is a K-Lusin set.*

The following corollary is immediate.

Corollary 3.4. *If there exists a Lusin set of cardinality $\mathfrak{c}$ ($\mathfrak{c}$ - regular) then there is an $\aleph_1$-strong sequence $(H_\xi)_{\xi < 2^{\aleph_0}}$ with $|H_\xi| < 2^{\aleph_0}$ for any $\xi < 2^{\aleph_0}$.*

Following Lemma 8.26 in [8] we obtain a Lusin set of size κ by adding κ many Cohen reals. By Corollary 2.6 we have that it is consistent with ZFC that there exists $\aleph_1$-strong sequence from Proposition 3.2 (a). On the other hand we know that $\mathfrak{l} = \aleph_1$ implies $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \mathfrak{c}$, ([1], Lemma 2.4). It is consistent with ZFC that $\mathfrak{b} > \aleph_1$ or $\mathfrak{d} < \mathfrak{c}$, (cf. [8]). Thus there are no such $\aleph_1$-strong sequences. Summing up we have the following result ([1], Theorem 2.6.).

Theorem 3.5. *The existence of $\aleph_1$-strong sequence $(H_\xi)_{\xi < 2^{\aleph_0}}$ with $|H_\xi| < 2^{\aleph_0}$ for any $\xi < 2^{\aleph_0}$ is independent of ZFC+¬CH.*

In the proof of Proposition 3.2 in [1] the authors used the ω_2-iteration of Miller forcing with countable support over V, a model of $ZFC + CH$ showing that the following property holds. "For all $f \in {}^\omega\omega \cap V[G_{\omega_2}]$ the set $\{\iota : m_\iota \leq f\}$ is countable", where $G_{\omega_2} = \langle m_\iota : \iota < \omega_2 \rangle$ is the corresponding sequence of Miller reals. Using Proposition 3.2 and the previous considerations we obtain the following proposition.

Proposition 3.6. *It is consistent with ZFC that there exists $\aleph_1$-strong sequence $(H_\xi)_{\xi < 2^{\aleph_0}}$ with $|H_\xi| < 2^{\aleph_0}$ for any $\xi < 2^{\aleph_0}$.*

Before Theorem 3.5 there was mentioned that $\mathfrak{l} = \aleph_1$ (so the existence of $\aleph_1$-strong sequence $(H_\xi)_{\xi < 2^{\aleph_0}}$ with $|H_\xi| < 2^{\aleph_0}$ for any $\xi < 2^{\aleph_0}$ by Proposition 3.2) implies that $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \mathfrak{c}$. We can construct a model of ZFC in which although $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \mathfrak{c}$ there is no such an $\aleph_1$-strong sequence $(H_\xi)_{\xi < 2^{\aleph_0}}$ with $|H_\xi| < 2^{\aleph_0}$ for any $\xi < 2^{\aleph_0}$, (compare the proof of Proposition 3.3 in [1]).

Proposition 3.7. *It is consistent with ZFC that $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \mathfrak{c}$ but there is no $\aleph_1$-strong sequence $(H_\xi)_{\xi < \mathfrak{c}}$ with $|H_\xi| < 2^{\aleph_0}$ for any $\xi < \mathfrak{c}$.*

Proof. Let us start with a model M in which $\mathfrak{c} = \aleph_2$ and in which MA holds. Let $G = \langle c_\beta : \beta < \omega_1 \rangle$ be a generic sequence of Cohen reals of length ω_1. The first part of the proposition is fulfilled but there is no $\aleph_1$-strong sequences of required property. Indeed. Let $(H_\xi)_{\xi < \aleph_2}, |H_\xi| < \aleph_2$ be an $\aleph_1$-strong sequence of length $\aleph_2$. Consider $X = \bigcup_{\xi < \aleph_2} H_\xi$. Let $G_\alpha = \langle c_\beta : \beta < \alpha \rangle$ for some α-countable. Let $X' \subseteq X$ be of cardinality $\aleph_2$ and $X' \subseteq M[G_\alpha]$. Now $M[G_\alpha] = M[c]$ for some Cohen real c and $M[c] \Vdash MA(\sigma - \text{centered})$ which implies that $\mathfrak{p} = \mathfrak{c}$. We know that $\mathfrak{p} \leq \mathfrak{b}$, (see e.g. [8]), thus $M[c] \Vdash \mathfrak{b} = \aleph_2$. It means that there exists a function $\bar{g}$ such that $|\{f \in X' : f \leq \bar{g}\}| = \aleph_2$. We obtain a contradiction with the definition of an $\aleph_1$-strong sequence. $\qquad\square$

4 On the existence of generalized BK-matrix and generalized strong sequences

A BK-matrix proposed in [2], (see also [1]), is a special kind of a generalized independent family, well-known in the literature (see [4–6]).

Definition 4.1. *Let $\mathcal{I} = \{\{I_\alpha^\beta : \beta < \lambda_\alpha\} : \alpha < \tau\}$ be a family of partitions of infinite set S with each $\lambda_\alpha \geq 2$ and let κ, λ, θ be cardinals. If for any $J \in [\tau]^{<\theta}$ and for any $f \in \Pi_{\alpha \in J} \lambda_\alpha$ the intersection $\bigcap \{I_\alpha^{f(\alpha)} : \alpha \in J\}$ has cardinality at least μ, then $\mathcal{I}$ is called (θ, μ)-generalized independent family on S. Moreover, if $\lambda_\alpha = \lambda$ for all $\alpha < \tau$, then $\mathcal{I}$ is called a (θ, μ, λ)-generalized independent family on S.*

Using Definition 4.1 we conclude that BK-matrix is $(\omega_1, 1, \omega)$- generalized independent family. Using the proof of Proposition 1.1. in [1] we obtain similar result for a generalized BK-matrix (i.e. BK-matrix considered in $^\kappa\kappa$ instead of $[0, 1]$, where κ is an infinite cardinal). Moreover, using Theorem 2.5 we obtain that

Proposition 4.2 (GCH)**.** *Let κ be an infinite cardinal. On each set of cardinality κ^+ the following are equivalent:*
(a) there exists $(\kappa^+, 1, \kappa)$–generalized independent family of cardinalit 2^κ,
(b) there exists κ^+-strong sequence of cardinality κ^+.

5 Further applications

In [7] there are some results concerning the existence of ω-strong sequences and their dependence on a precalibre. We generalize those results as follows. Let α and κ be cardinals.

Definition 5.1. *A cardinal η is a precalibre for $X \subseteq {}^\kappa\kappa$ if η is infinite and every set $A \in [X]^\eta$ has an α-directed set in the sense of relation $\leq$ of cardinality η, with $\alpha \leq 2^\kappa$.*

Proposition 5.2. *Let κ be an infinite cardinal. If a regular cardinal η is not a precalibre for $X \subseteq {}^\kappa\kappa$, then there exists a κ^+–strong sequence of length η.*

Proof. If η is not a precalibre, then there exists $A \in [X]^\eta$ of cardinality η in which each κ^+-directed subset has cardinality less than η.

Let $f_0 \in A$ be an arbitrary function and let $A_0 \subseteq A$ be a maximal κ^+-directed subset such that $f_0 \in A_0$. Let A_0 be the first element of an κ^+-strong sequence.

Assume that there has been defined the sequence of functions $\{f_\xi : \xi < \psi < \eta\}$ with $f_\xi \perp f_\psi$ for any $\xi < \psi$ such that for each f_ξ there exists a maximal, κ^+-directed set $A_\xi \subseteq A$ such that $f_\xi \in A_\xi$ and $(A_\xi)_{\xi < \psi}$ forms the κ^+–strong sequence. Since $|A_\xi| < \eta$ and η is regular we have that $A \smallsetminus \bigcup_{\xi < \psi} A_\xi$ is non-empty. Hence there exists $f_\zeta \in A \smallsetminus \bigcup_{\xi < \psi} A_\xi$ such that $f_\zeta \perp f_\xi$ for all $\xi < \zeta$ and $A_\zeta \subset A \smallsetminus \bigcup_{\xi < \psi} A_\xi$ is maximal, κ^+-directed such that $f_\zeta \in A_\zeta$. Let A_ζ be the next element of the κ^+–strong sequence. $\qquad\square$

Using Proposition 5.2, Theorem 2.5, Proposition 4.2 and Lemma 2.2 we immediately obtain the following corollary

Corollary 5.3. *Let κ be an infinite cardinal. If a regular cardinal κ is not a precalibre then*
(1) $\mathfrak{l}_\kappa = 2^\kappa$,
(2) there exists $(\kappa^+, 1, \kappa)$-generalized independent family of cardinality 2^κ,
(3) there exists a generalized K-Lusin set of cardinality 2^κ.

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