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On generalized P-reducible Finsler manifolds

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Abstract: The class of generalized P-reducible manifolds (briefly GP-reducible manifolds) was first introduced by Tayebi and his collaborates [1]. This class of Finsler manifolds contains the classes of P-reducible manifolds, C-reducible manifolds and Landsberg manifolds. We prove that every compact GP-reducible manifold with positive or negative character is a Randers manifold. The norm of Cartan torsion plays an important role for studying immersion theory in Finsler geometry. We find the relation between the norm of Cartan torsion, mean Cartan torsion, Landsberg and mean Landsberg curvatures of the class of GP-reducible manifolds. Finally, we prove that every GP-reducible manifold admitting a concurrent vector field reduces to a weakly Landsberg manifold.

Keywords: Generalized P-reducible metric, Randers metric, Landsberg metric

MSC: 53B40, 53C60

1 Introduction

In [2], Tayebi and Sadeghi studied a class of Finsler metrics which contains the class of P-reducible metrics. They called Finsler metrics in this class by GP-reducible metrics. A Finsler metric F on a manifold M is called a GP-reducible metric if its Landsberg curvature is given by

$$L_{ijk} = \lambda C_{ijk} + a_i h_{jk} + a_j h_{ki} + a_k h_{ij},$$

where a_i and λ are scalar functions on TM. In this case, (M, F) is called a GP-reducible manifold and λ is called the character of F. An longstanding open problem in Finsler geometry is to find a Landsberg manifold which is not Berwald manifold. Matsumoto and Shimada proposed a generalization of this problem in the context of P-reducible maniolfds. The class of (α, β)-metrics is a rich class of Finsler metrics with a diverse areas of application. In [2], it is proved that there is no P-reducible (α, β)-metric with vanishing S-curvature which is not C-reducible, thus a partial answer to Matsumoto and Shimada's problem.

The class of Randers metrics are natural Finsler metrics which were first introduced by G. Randers and derived from the research on the four-space of general relativity. His metric is in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is gravitational field and $\beta = b_i(x)y^i$ is the electromagnetic field. Randers metrics have been widely applied in many areas of natural science, including seismic ray theory, biology and physics, etc [3].

We study compact GP-reducible manifold and prove the following rigidity theorem.

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Theorem 1.1. *Every compact GP-reducible manifold of dimension $n \geq 3$ with negative or positive character is a Randers manifold.*

In [4], Shen proved that a Finsler manifold with unbounded Cartan torsion can not be isometrically imbedded into any Minkowski space. Thus, the norm of Cartan torsion plays an important role for studying immersion theory in Finsler geometry. We find the following result on the norm of impotrtant non-Riemannian quantities for a GP-reducible manifold.

Theorem 1.2. *Let (M, F) be a GP-reducible Finsler manifold. Then the following holds*

$$||\mathbf{L}||^2 - \frac{3}{(n+1)}||\mathbf{J}||^2 = \lambda^2\Big[||\mathbf{C}||^2 - \frac{3}{(n+1)}||\mathbf{I}||^2\Big]. \tag{1}$$

In particular, if F has relatively isotropic Landsberg curvature $\mathbf{L} = cF\mathbf{C}$, then (1) reduces to

$$||\mathbf{C}|| = \sqrt{\frac{3}{n+1}}||\mathbf{I}||, \tag{2}$$

provided that $\lambda \neq \pm cF$.

It is known that for a P-reducible manifold (1) holds with $\lambda = 0$. Thus, we can consider (1) as a generealization of this well-known fact.

Studying geometric vector fields plays a prominent role in Differential Geometry. In [5], K. Yano introduced the notion of concurrent vector fields on an affine manifold $(M, \triangledown)$. Then, F. Brickell and K. Yano studied this kind of geometric vector fields in Riemannian geometry considering the Levi-Civita connection as $\triangledown$ [6]. Here, we prove that addmiting a concurrect vector field on a GP-reducible manifold reduces it to a relatively isotropic mean Landsberg metric manifold. More precisely, we have the following.

Theorem 1.3. *Let (M, F) be a GL-reducible Finsler manifold admitting a concurrent vector field. Then F is a weakly Landsberg.*

Throughout this paper, we use the Cartan connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2 Preliminaries

A Finsler metric on an n-dimensional C^∞ manifold M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on $TM_0 = TM \setminus \{0\}$, (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, (iii) for each $y \in T_xM$, the quadratic form $\mathbf{g}_y : T_xM \otimes T_xM \to \mathbb{R}$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]\Big|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]\Big|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C}$=0 if and only if F is Riemannian.

Taking a trace of Cartan torsion yields the mean Cartan torsion $\mathbf{I}_y$. Let (M, F) be an n-dimensional Finsler manifold. For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_i),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y \in TM_0}$ is called the mean Cartan torsion.

For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\},$$

and $h_{ij} = g_{ij} - F_{y^i}F_{y^j}$ is the angular metric. F is said to be C-reducible if $\mathbf{M}_y = 0$.

Lemma 2.1. ([7]) A Finsler metric F on a manifold M of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

The horizontal covariant derivative of the Cartan torsion $\mathbf{C}$ along geodesics gives rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ is defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where

$$L_{ijk} := C_{ijk|s}y^s. \tag{3}$$

The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$ (see [8, 9]).

The horizontal covariant derivative of the mean Cartan torsion $\mathbf{I}$ along geodesics gives rise to the mean Landsberg curvature $\mathbf{J}_y : T_xM \to \mathbb{R}$ which is defined by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := I_{i|s}y^s. \tag{4}$$

The family $\mathbf{J} := \{\mathbf{J}_y\}_{y\in TM_0}$ is called the mean Landsberg curvature. A Finsler metric is called a weakly Landsberg metric if $\mathbf{J} = 0$.

A Finsler metric F is called *GP-reducible* if its Landsberg curvature is given by

$$L_{ijk} = \lambda C_{ijk} + a_ih_{jk} + a_jh_{ki} + a_kh_{ij}, \tag{5}$$

where a_i and λ are scalar functions on TM. By definition, if $a_i = 0$ then F reduces to a general relatively isotropic Landsberg metric and if $\lambda = 0$ then F is P-reducible [10]. Thus, the study of this class of Finsler spaces will enhance our understanding of the geometric meaning of C-reducible and P-reducible metrics.

Remark 2.2. *Let F be a GP-reducible metric with character λ. Then for every positive constant c, the Finsler metric cF is also GP-reducible metric with character $c^2\lambda$. Hence, the class of GP-reducible metrics is closed under homothetic transformations.*

Finsler geometry is a natural generalization of Riemannain one. Study of geometric vector fields has been extended from Riemannian geometry to Finsler geometry. Concurrent vector fields in Finslerian setting have been studied extensively (for example see [11, 12]). Let us consider a vector field $X^i(x)$ in (M, F). This field is called *concurrent* if it satisfies the following

$$X^i_{|j} = \delta^i_j, \tag{6}$$

$$X^i_{,j} = 0. \tag{7}$$

One can also use covariant derivative with respect to Berwald connection to define concurrent vector fields. It is well knwon that the two defintions are the same.

Let us mention a standard example of concurrent vector field. Let V be a vector space and (x^i) be a standard chart of V. Suppose that F is a Minkowski norm on V. Then, the radial vector field $X = x^i\frac{\partial}{\partial x^i}$ is a concurrent vector field of (V, F).

3 Proof of Theorem 1.1

In this section, we will prove a generalized version of Theorem 1.1. Indeed we study complete GP-reducible manifold of positive (or negative) character with the assumption that F has bounded Matsumoto torsion. For this aim, we remark the following.

Lemma 3.1. ([2]) Let (M, F) be a GP-reducible Finsler manifold. Then the Matsumoto torsion of F satisfies the following

$$M_{ijk|s}y^s = \lambda(x, y)M_{ijk}. \tag{8}$$

Proof. Let F be a GP-reducible metric

$$L_{ijk} = \lambda C_{ijk} + a_i h_{jk} + a_j h_{ki} + a_k h_{ij}. \tag{9}$$

Contracting (9) with g^{ij} and using relations $g^{ij}h_{ij} = n - 1$ and $g^{ij}(a_i h_{jk}) = g^{ij}(a_j h_{ik}) = a_k$ imply that

$$J_k = \lambda I_k + (n + 1)a_k. \tag{10}$$

Then

$$a_i = \frac{1}{n+1}\Big[J_i - \lambda I_i\Big]. \tag{11}$$

Putting (11) into (9) yields

$$L_{ijk} - \frac{1}{n+1}(J_i h_{jk} + J_j h_{ki} + J_k h_{ij}) = \lambda\Big\{C_{ijk} - \frac{1}{n+1}(I_i h_{jk} + I_j h_{ki} + I_k h_{ij})\Big\}. \tag{12}$$

It is easy to see that (12) is equivalent to (8). □

Now, we are going to prove the main result of this section.

Theorem 3.2. *Let (M, F) be an n-dimensional complete GP-reducible manifold ($n \geq 3$) with negative or positive character . Suppose that F has bounded Matsumoto torsion. Then F is a Randers metric.*

Proof. By definition, the norm of the Matsumoto torsion at $x \in M$ is given by

$$\|\mathbf{M}\|_x := \sup_{y,u,v,w \in T_xM \setminus \{0\}} \frac{F(y)|\mathbf{M}_y(u, v, w)|}{\sqrt{\mathbf{g}_y(u, u)\mathbf{g}_y(v, v)\mathbf{g}_y(w, w)}}.$$

Suppose that the Matsumoto torsion satisfies

$$\mathbf{M}_y(u, u, u) = M_{ijk}(x, y)u^i u^j u^k \neq 0$$

for some $y, u \in T_xM_0$ with $F(x, y) = 1$. Let $\sigma(t)$ be the unit speed geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Let $U(t)$ denote the linear parallel vector field along σ with $U(0) = u$. Let

$$\mathcal{M}(t) := \mathbf{M}_{\dot{\sigma}(t)}\Big(U(t), U(t), U(t)\Big) = M_{ijk}\Big(\sigma(t), \dot{\sigma}(t)\Big)U^i(t)U^j(t)U^k(t).$$

It followos from (8)

$$\mathcal{M}'(t) + \check{}(t)\mathcal{M}(t) = 0. \tag{13}$$

Remark 2.2 permits us to assume that $\lambda(t) \leq -1$ or $\lambda(t) \geq 1$. The general solution of (13) is given by

$$\mathcal{M}(t) = e^{-\int_0^t \lambda(t)dt}\mathcal{M}(0).$$

By assumption, $\mathcal{M}(0) = \mathbf{M}_y(u, u, u) \neq 0$. Letting $t \to -\infty$ or $t \to \infty$ implies that $\mathcal{M}(t)$ is unbounded, contradicting with boundednes of Matsumoto's torsion. Thus the Matsumoto torsion of F vanishes. By Lemma 2.1, F reduces to a Randers metric. □

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

Proof. Let F be a GP-reducible metric

$$L_{ijk} = \lambda C_{ijk} + a_i h_{jk} + a_j h_{ki} + a_k h_{ij}. \tag{14}$$

Multiplying (14) with $g^{mi} g^{pj} g^{qk}$ yields

$$L^{ijk} = \lambda C^{ijk} + a^i h^{jk} + a^j h^{ki} + a^k h^{ij}, \tag{15}$$

where $a^i = g^{im} a_m$. The following hold

$$C_{ijk} h^{ij} = I_k, \quad h_{ij} h^{jk} = h_i^k, \quad h_{ij} h^{ij} = n - 1. \tag{16}$$

Multiplying (14) with (15) implies that

$$L_{ijk} L^{ijk} = (\lambda C_{ijk} + a_i h_{jk} + a_j h_{ki} + a_k h_{ij})(\lambda C^{ijk} + a^i h^{jk} + a^j h^{ki} + a^k h^{ij})$$

equivalently

$$||\mathbf{L}||^2 = \lambda^2 ||\mathbf{C}||^2 + 6\lambda I_k a^k + 3(n+1)||\mathbf{a}||^2, \tag{17}$$

where $||\mathbf{a}||^2 := a_s a^s$. Thus

$$6\lambda I_k a^k = ||\mathbf{L}||^2 - \lambda^2 ||\mathbf{C}||^2 - 3(n+1)||\mathbf{a}||^2. \tag{18}$$

Contracting (14) with g^{jk} implies that

$$J_k = \lambda I_k + (n+1) a_k. \tag{19}$$

Multiplying (19) with g^{ik} yields

$$J^k = \lambda I^k + (n+1) a^k. \tag{20}$$

By (19) and (20), we get

$$J_k J^k = \big(\lambda I_k + (n+1) a_k\big)\big(\lambda I^k + (n+1) a^k\big)$$

equivalently

$$||\mathbf{J}||^2 = \lambda^2 ||\mathbf{I}||^2 + 2(n+1)\lambda I_k a^k + (n+1)^2 ||\mathbf{a}||^2. \tag{21}$$

Thus

$$2(n+1)\lambda I_k a^k = ||\mathbf{J}||^2 - \lambda^2 ||\mathbf{I}||^2 - (n+1)^2 ||\mathbf{a}||^2. \tag{22}$$

By (18) and (22), we get (1). □

5 Proof of Theorem 1.3

Suppose that $X^i(x)$ is a concurrent vector field on a Finsler manifold, then from Ricci identities, we get the following integrability conditions:

$$X^h R_{hijk} = 0, \tag{23}$$

$$X^h P_{hijk} - C_{ijk} = 0, \tag{24}$$

$$X^h S_{hijk} = 0. \tag{25}$$

Since P_{hijk} are skew symmetric in h and i, we have from (24)

$$X^i C_{ijk} = 0. \tag{26}$$

It is well known that

$$P_{hijk} = C_{ijk|h} - C_{hjk|i} + C_{hjr} C^r_{ik|0} - C_{ijr} C^r_{hk|0}. \tag{27}$$

From (7), one can see that

$$X^h C_{hjk|i} = -C_{ijk}. \tag{28}$$

Plugging (27) and (28) into (24), we get

$$X^h C_{ijk|h} = 0. \tag{29}$$

Let $X_i(x)$ denote the covariant components of X^i. Define a 1-form β as follows $\beta := X_i y^i$. Put $m_i := X_i - \beta \frac{y_i}{F^2}$. Then, it is proved that the following hold

$$h_{ij} X^j = m_i \neq 0, \tag{30}$$

$$h_{ij} X^i X^j = m^2 \neq 0, \tag{31}$$

where $m^2 = g_{ij} m^i m^j$ and $m^i = g^{ij} m_j$ [12]. Contracting (5) by $X^i X^j$ and using (7) and (11), we obtain

$$X^i X^j h_{ij} J_k = 0.$$

Using (31), we get $J_k = \lambda I_k$. Hence, we get Theorem 1.3.

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