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Existence of solutions for a shear thickening fluid-particle system with non-Newtonian potential

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Abstract: This paper is concerned with a compressible shear thickening fluid-particle interaction model for the evolution of particles dispersed in a viscous non-Newtonian fluid. Taking the influence of non-Newtonian gravitational potential into consideration, the existence and uniqueness of strong solutions are established.

Keywords: Existence, Strong solutions, Compressible, Non-Newtonian fluid

MSC: 76A05, 76A10

1 Introduction

We consider a compressible non-Newtonian fluid-particle interaction model which reads as follows

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + \rho \Psi_x - \lambda\left[(u_x^2 + \mu_1)^{\frac{p-2}{2}} u_x\right]_x + (P + \eta)_x = -\eta \Phi_x, \quad (x,t) \in \Omega_T \\ (|\Psi_x|^{q-2}\Psi_x)_x = 4\pi g(\rho - \frac{1}{|\Omega|}\int\limits_{\Omega} \rho \mathrm{d}x), \\ \eta_t + [\eta(u - \Phi_x)]_x = \eta_{xx} \end{cases} \tag{1}$$

with the initial and boundary conditions

$$\begin{cases} (\rho, u, \eta)|_{t=0} = (\rho_0, u_0, \eta_0), & x \in \Omega, \\ u|_{\partial\Omega} = \Psi|_{\partial\Omega} = 0, & t \in [0, T], \end{cases} \tag{2}$$

and the no-flux condition for the density of particles

$$(\eta_x + \eta \Phi_x)|_{\partial\Omega} = 0, \quad t \in [0, T]. \tag{3}$$

where $\rho, u, \eta, P(\rho) = a\rho^\gamma$ denote the fluid density, velocity, the density of particle in the mixture and pressure respectively, Ψ denotes the non-Newtonian gravitational potential and the given function $\Phi(x)$ denotes the external potential. $a > 0, \gamma > 1, \mu_1 > 0, p > 2, 1 < q < 2, \lambda > 0$ is the viscosity coefficient and $\beta \neq 0$ is a constant. Ω is a one-dimensional bounded interval, for simplicity we only consider $\Omega = (0,1)$, $\Omega_T = \Omega \times [0, T]$.

In fact, there are extensive studies concerning the theory of strong and weak solutions for the multi-dimensional fluid-particle interaction models for the newtonian case. In [1], Carrillo *et al.* discussed the

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global existence and asymptotic behavior of the weak solutions for a fluid-particle interaction model. Subsequently, Fang *et al*. [2] obtained the global classical solution in dimension one. In dimension three, Ballew and Trivisa [3, 4] established the global existence of weak solutions and the existence of weakly dissipative solutions under reasonable physical assumptions on the initial data. In addition, Constantin and Masmoudi [5] obtained the global existence of weak solutions for a coupled incompressible fluid-particle interaction model in 2D case followed the spirit of reference [6].

The non-Newtonian fluid is an important type of fluid because of its immense applications in many fields of engineering fluid mechanics such as inks, paints, jet fuels etc., and biological fluids such as blood (see [7]). Many researchers turned to the study of this type of fluid under different conditions both theoretically and experimentally. For details, we refer the readers to [8-12] and the references therein. To our knowledge, there seems to be a very few mathematical results for the case of the fluid-interaction model systems with non-Newtonian gravitational potential. There are still no existence results to problem (1)-(3) when $p > 2$, $1 < q < 2$ which describes that the motion of the compressible viscous isentropic gas flow is driven by a non-Newtonian gravitational force.

We are interested in the existence and uniqueness of strong solutions on a one dimensional bounded domain. The strong nonlinearity of (1) bring us new difficulties in getting the upper bound of ρ and the method used in [2] is not suitable for us. Motivated by the work of Cho et al. [13, 14] on Navier-Stokes equations, we establish local existence and uniqueness of strong solutions by the iteration techniques.

Throughout the paper we assume that $a = \lambda = 1$. In the following sections, we will use simplified notations for standard Sobolev spaces and Bochner spaces, such as $L^p = L^p(\Omega)$, $H_0^1 = H_0^1(\Omega)$, $C(0,T;H^1) = C(0,T;H^1(\Omega))$.

We state the definition of strong solution as follows:

Definition 1.1. *The (ρ, u, Φ, η) is called a strong solution to the initial boundary value problem (1)-(3), if the following conditions are satisfied:*
(i)

$$\begin{aligned}
&\rho \in L^\infty(0,T_*;H^1(\Omega)), u \in L^\infty(0,T_*;W_0^{1,p}(\Omega)\cap H^2(\Omega)),\\
&\eta \in L^\infty(0,T_*;H^2(\Omega)), \Psi \in L^\infty(0,T_*;H^2(\Omega)), \rho_t \in L^\infty(0,T_*;L^2(\Omega)),\\
&u_t \in L^2(0,T_*;H_0^1(\Omega)), \Psi_t \in L^\infty(0,T_*H^1(\Omega)), \sqrt{\rho}u_t \in L^\infty(0,T_*;L^2(\Omega)),\\
&\eta_t \in L^\infty(0,T_*;L^2(\Omega)), ((u_x^2+\mu_1)^{\frac{p-2}{2}}u_x)_x \in L^2(0,T_*;L^2(\Omega))
\end{aligned}$$

(ii) *For all $\varphi \in L^\infty(0,T_*;H^1(\Omega))$, $\varphi_t \in L^\infty(0,T_*;L^2(\Omega))$, for a.e. $t \in (0,T)$, we have*

$$\int_\Omega \rho\varphi(x,t)\mathrm{d}x - \int_0^t\int_\Omega (\rho\varphi_t + \rho u\varphi_x)(x,s)\mathrm{d}x\mathrm{d}s = \int_\Omega \rho_0\varphi(x,0)\mathrm{d}x \tag{4}$$

(iii) *For all $\phi \in L^\infty(0,T_*;W_0^{1,p}(\Omega)\cap H^2(\Omega))$, $\phi_t \in L^2(0,T_*;H_0^1(\Omega))$, for a.e. $t \in (0,T)$, we have*

$$\begin{aligned}
\int_\Omega \rho u\phi(x,t)\mathrm{d}x - \int_0^t\int_\Omega \{&\rho u^2\phi_x - \rho\Psi_x\phi - \lambda(u_x^2+\mu_1)^{\frac{p-2}{2}}u_x\phi_x\\
&+ (P+\eta)\phi_x + \eta\Phi_x\phi\}(x,s)\mathrm{d}x\mathrm{d}s = \int_\Omega \rho_0u_0\phi(x,0)\mathrm{d}x
\end{aligned} \tag{5}$$

(iv) *For all $\vartheta \in L^\infty(0,T_*;H^2(\Omega))$, $\vartheta_t \in L^\infty(0,T_*H^1(\Omega))$, for a.e. $t \in (0,T)$, we have*

$$-\int_0^t\int_\Omega |\Psi_x|^{q-2}\Psi_x\vartheta_x(x,s)\mathrm{d}x\mathrm{d}s = \int_0^t\int_\Omega 4\pi g(\rho - \frac{1}{|\Omega|}\int_\Omega \rho\mathrm{d}x)\vartheta(x,0)\mathrm{d}x\mathrm{d}s \tag{6}$$

(v) *For all $\psi \in L^\infty(0,T_*;H^2(\Omega))$, $\psi_t \in L^\infty(0,T_*;L^2(\Omega))$, for a.e. $t \in (0,T)$, we have*

$$\int_\Omega \eta\psi(x,t)\mathrm{d}x - \int_0^t\int_\Omega [\eta(u-\Phi_x)-\eta_x]\psi_x(x,s)\mathrm{d}x\mathrm{d}s = \int_\Omega \eta_0\psi(x,0)\mathrm{d}x \tag{7}$$

1.1 Main results

Theorem 1.2. *Let $\mu_1 > 0$ be a positive constant and $\Phi \in C^2(\Omega)$, and assume that the initial data (ρ_0, u_0, η_0) satisfy the following conditions*

$$0 \le \rho_0 \in H^1(\Omega), u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \eta_0 \in H^2(\Omega)$$

and the compatibility condition

$$-\left[(u_{0x}^2 + \mu_1)^{\frac{p-2}{2}} u_{0x}\right]_x + \left(P(\rho_0) + \eta_0\right)_x + \eta_0 \Phi_x = \rho_0^{\frac{1}{2}}(g + \beta \Phi_x), \tag{8}$$

for some $g \in L^2(\Omega)$. Then there exist a $T_* \in (0, +\infty)$ and a unique strong solution (ρ, u, η) to (1)-(3) such that

$$\begin{cases} \rho \in L^\infty(0, T_*; H^1(\Omega)), \rho_t \in L^\infty(0, T_*; L^2(\Omega)), \\ u \in L^\infty(0, T_*; W_0^{1,p}(\Omega) \cap H^2(\Omega)), u_t \in L^2(0, T_*; H_0^1(\Omega)), \\ \eta \in L^\infty(0, T_*; H^2(\Omega)), \eta_t \in L^\infty(0, T_*; L^2(\Omega)), \\ \Psi \in L^\infty(0, T_*; H^2(\Omega)), \Psi_t \in L^\infty(0, T_* H^1(\Omega)), \\ \sqrt{\rho} u_t \in L^\infty(0, T_*; L^2(\Omega)), ((u_x^2 + \mu_1)^{\frac{p-2}{2}} u_x)_x \in L^2(0, T_*; L^2(\Omega)). \end{cases} \tag{9}$$

2 A priori Estimates for Smooth Solutions

In this section, we will prove the local existence of strong solutions. By virtue of the continuity equation $(1)_1$, we deduce the conservation of mass

$$\int_\Omega \rho(t) \mathrm{d}x = \int_\Omega \rho_0 \mathrm{d}x := m_0, \quad (t > 0, m_0 > 0).$$

Provided that (ρ, u, η) is a smooth solution of (1)-(3) and $\rho_0 \ge \delta$, where $0 < \delta \ll 1$ is a positive number. We denote by $M_0 = 1 + \mu_1 + \mu_1^{-1} + |\rho_0|_{H^1} + |g|_{L^2}$, and introduce an auxiliary function

$$Z(t) = \sup_{0 \le s \le t} (1 + |u(s)|_{W_0^{1,p}} + |\rho(s)|_{H^1} + |\eta_t(s)|_{L^2} + |\eta(s)|_{H^1} + |\sqrt{\rho} u_t(s)|_{L^2}).$$

Then we estimate each term of $Z(t)$ in terms of some integrals of $Z(t)$, apply arguments of Gronwall-type and thus prove that $Z(t)$ is locally bounded.

2.1 Estimate for $|u|_{W_0^{1,p}}$

By using $(1)_1$, we rewrite the $(1)_2$ as

$$\rho u_t + \rho u u_x + \rho \Psi_x - \left[(u_x^2 + \mu_1)^{\frac{p-2}{2}} u_x\right]_x + (P + \eta)_x = -\eta \Phi_x. \tag{10}$$

Multiplying (10) by u_t, integrating (by parts) over Ω_T, we have

$$\iint_{\Omega_T} \rho |u_t|^2 \mathrm{d}x\mathrm{d}s + \iint_{\Omega_T} (u_x^2 + \mu_1)^{\frac{p-2}{2}} u_x u_{xt} \mathrm{d}x\mathrm{d}s = -\iint_{\Omega_T} (\rho u u_x + \rho \Psi_x + P_x + \eta_x + \eta \Phi_x) u_t \mathrm{d}x\mathrm{d}s. \tag{11}$$

We deal with each term as follows:

$$\int_\Omega (u_x^2 + \mu_1)^{\frac{p-2}{2}} u_x u_{xt} \mathrm{d}x = \frac{1}{2} \int_\Omega (u_x^2 + \mu_1)^{\frac{p-2}{2}} (u_x^2)_t \mathrm{d}x = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Big(\int_0^{u_x^2} (s + \mu_1)^{\frac{p-2}{2}} \mathrm{d}s \Big) \mathrm{d}x$$

$$\int_0^{u_x^2} (s + \mu_1)^{\frac{p-2}{2}} \mathrm{d}s = \int_{\mu_1}^{u_x^2 + \mu_1} t^{\frac{p-2}{2}} \mathrm{d}t = \frac{2}{p} [(u_x^2 + \mu_1)^{\frac{p}{2}} - \mu_1^{\frac{p}{2}}] \ge \frac{2}{p} |u_x|^p - \frac{2}{p} \mu_1^{\frac{p}{2}}$$

$$-\iint_{\Omega_T} P_x u_t \mathrm{d}x\mathrm{d}s = \iint_{\Omega_T} P u_{xt} \mathrm{d}x\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega_T} P u_x \mathrm{d}x\mathrm{d}s - \iint_{\Omega_T} P_t u_x \mathrm{d}x\mathrm{d}s.$$

Since from $(1)_1$ we get

$$P_t = -\gamma P u_x - P_x u \tag{12}$$

$$-\iint_{\Omega_T} (\eta_x + \eta\Phi_x) u_t \mathrm{d}x\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega_T} (\eta_x + \eta\Phi_x) u \mathrm{d}x\mathrm{d}s - \iint_{\Omega_T} (\eta_x + \eta\Phi_x)_t u \mathrm{d}x\mathrm{d}s$$

$$\begin{aligned}-\iint_{\Omega_T} (\eta_x + \eta\Phi_x)_t u \mathrm{d}x\mathrm{d}s &= \iint_{\Omega_T} \eta_t (u_x - \Phi_x u) \mathrm{d}x\mathrm{d}s \\ &= -\iint_{\Omega_T} [\eta_x - \eta(u - \Phi_x)](u_x - \Phi_x u)_x \mathrm{d}x\mathrm{d}s.\end{aligned}$$

Substituting the above into (11), we obtain

$$\begin{aligned}&\int_0^t |\sqrt{\rho} u_t(s)|_{L^2}^2 \mathrm{d}s + \frac{1}{p}\int_\Omega |u_x(t)|^p \mathrm{d}x \\ &\le C + \int_\Omega |P u_x| \mathrm{d}x + \iint_{\Omega_T} (|\rho u u_x u_t| + |\rho \Psi_x u_t| + |\gamma P u_x^2| + |P_x u u_x|) \mathrm{d}x\mathrm{d}s \\ &\quad + \iint_{\Omega_T} \big(|\eta_x u_{xx}| + |\eta_x \Phi_x u_x| + |\eta_x \Phi_{xx} u| + |\eta u u_{xx}| + |\eta u^2 \Phi_{xx}| + |\eta u \Phi_x u_x| \\ &\qquad + |\eta \Phi_x u_{xx}| + |\eta \Phi_x \Phi_{xx} u| + |\eta \Phi_x^2 u_x|\big) \mathrm{d}x\mathrm{d}s.\end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned}&\int_0^t |\sqrt{\rho} u_t(s)|_{L^2}^2 \mathrm{d}s + |u_x(t)|_{L^p}^p \\ &\le C + C\int_0^t \big(|\rho|_{L^\infty} |u|_{L^\infty}^2 |u_x|_{L^p}^2 + |\rho|_{L^\infty} |\Psi_{xx}|_{L^2}^2 + |P|_{L^\infty} |u_x|_{L^p}^2 + |P_x|_{L^2} |u|_{L^\infty} |u_x|_{L^p} \\ &\quad + |\eta_x|_{L^2} |u_{xx}|_{L^2} + |\eta_x|_{L^2} |u_x|_{L^p} + |\eta_x|_{L^2} |u|_{L^\infty} + |\eta|_{L^\infty} |u|_{L^\infty} |u_{xx}|_{L^2} + |\eta|_{L^\infty} |u|_{L^\infty}^2 \\ &\quad + |\eta|_{L^\infty} |u|_{L^\infty} |u_x|_{L^p} + |\eta|_{L^\infty} |u_{xx}|_{L^2} + |\eta|_{L^\infty} |u|_{L^\infty} + |\eta|_{L^\infty} |u_x|_{L^p}\big) \mathrm{d}s + C|P(t)|_{L^2}^{\frac{p}{p-1}}.\end{aligned} \tag{13}$$

On the other hand, multiplying $(1)_3$ by Ψ and integrating over Ω, we get

$$\begin{aligned}\int_\Omega |\Psi_x|^q \mathrm{d}x &= -\int_\Omega (|\Psi_x|^{q-2}\Psi_x)_x \Psi \mathrm{d}x = -4\pi g\Big(\int_\Omega \rho\Psi \mathrm{d}x - m_0 \int_\Omega \Psi \mathrm{d}x\Big) \\ &\le 8\pi g m_0 |\Psi|_{L^\infty} \le 8\pi g m_0 |\Psi_x|_{L^q} \le \frac{1}{q}|\Phi_x|_{L^q}^q + \frac{1}{p}(8\pi.g m_0)^p\end{aligned}$$

Then we have

$$\int_\Omega |\Psi_x|^q \mathrm{d}x \le C(m_0), \quad 1 < q < 2.$$

Differentiating $(1)_3$ with respect to x, multiplying it by Ψ_x and integrating over Ω, we have

$$\int_\Omega (|\Psi_x|^{q-2}\Psi_x)_x \Psi_{xx} \mathrm{d}x = -4\pi g \int_\Omega \rho_x \Psi_x \mathrm{d}x.$$

By virtue of

$$\int_\Omega (|\Psi_x|^{q-2}\Psi_x)_x \Psi_{xx} \mathrm{d}x = \int_\Omega (|\Psi_x|^{q-4}[(q-2)\Psi_x^2 + \Psi_x^2]\Psi_{xx}^2 \mathrm{d}x$$

$$\geq C\int_{\Omega}|\Psi_x|^{q-2}\Psi_{xx}^2\mathrm{d}x \geq C|\Psi_x|_{L^\infty}^{q-2}|\Psi_{xx}|_{L^2}^2$$
$$\geq C|\Psi_{xx}|_{L^2}^{q-2}|\Psi_{xx}|_{L^2}^2 \geq C|\Psi_{xx}|_{L^2}^q$$

and

$$-4\pi g\int_{\Omega}\rho_x\Psi_x\mathrm{d}x \leq C\int_{\Omega}|\rho_x||\Psi_x|\mathrm{d}x \leq C|\rho_x|_{L^2}^p + C|\Psi_x|_{L^2}^q \leq C|\rho_x|_{L^2}^p + C(\varepsilon)|\Psi_{xx}|_{L^2}^q.$$

Therefore,

$$|\Psi_{xx}|_{L^2} \leq CZ^{\frac{p}{q}}(t). \tag{14}$$

We deal with the term of $|u_{xx}|_{L^2}$. Notice that

$$|\big[(u_x^2+\mu_1)^{\frac{p-2}{2}}u_x\big]_x| \geq \mu_1^{\frac{p-2}{2}}|u_{xx}|.$$

Then

$$|u_{xx}| \leq C|\rho u_t + \rho u u_x + \rho\Psi_x + (P+\eta)_x + \eta\Phi_x|.$$

Taking the above inequality by L^2 norm, we get

$$\begin{aligned}|u_{xx}|_{L^2} &\leq C|\rho u_t + \rho u u_x + \rho\Psi_x + (P+\eta)_x + \eta\Phi_x|_{L^2}\\ &\leq C(|\rho|_{L^\infty}^{\frac{1}{2}}|\sqrt{\rho}u_t|_{L^2} + |\rho|_{L^\infty}|u|_{L^\infty}|u_x|_{L^2} + |\rho|_{L^2}|\Psi_{xx}|_{L^2} + |P_x|_{L^2} + |\eta_x|_{L^2} + |\eta|_{L^2}|\Phi_x|_{L^2}).\end{aligned}$$

Hence, we deduce that

$$|u_{xx}|_{L^2} \leq CZ^{\max(\frac{p}{q}+1,3)}(t). \tag{15}$$

Moreover, using $(1)_1$, we have

$$\begin{aligned}|P(t)|_{L^2}^{\frac{p}{p-1}} \leq \int_{\Omega}|P(t)|^2\mathrm{d}x &= \int_{\Omega}|P(0)|^2\mathrm{d}x + \int_0^t\frac{\partial}{\partial s}\Big(\int_{\Omega}P(s)^2\mathrm{d}x\Big)\mathrm{d}s\\ &\leq \int_{\Omega}|P(0)|^2\mathrm{d}x + 2\int_0^t\int_{\Omega}P(s)\gamma\rho^{\gamma-1}(-\rho_x u - \rho u_x)\mathrm{d}x\mathrm{d}s\\ &\leq C + C\int_0^t|P|_{L^\infty}|\rho|_{L^\infty}^{\gamma-1}|\rho|_{H^1}|u_x|_{L^p}\mathrm{d}s\\ &\leq C(1+\int_0^t Z^{2\gamma+1}(s)\mathrm{d}s).\end{aligned} \tag{16}$$

Combining (13)-(16), yields

$$\int_0^t|\sqrt{\rho}u_t(s)|_{L^2}^2(s)\mathrm{d}s + |u_x(t)|_{L^p}^p \leq C(1+\int_0^t Z^{\max(\frac{2p}{q},2\gamma+3)}(s)\mathrm{d}s), \tag{17}$$

where C is a positive constant, depending only on M_0.

2.2 Estimate for $|\rho|_{H^1}$

From $(1)_3$, taking it by L^2 norm, we get

$$|\eta_{xx}|_{L^2} \leq |\eta_t + (\eta(u-\Phi_x))_x|_{L^2}$$

$$\begin{aligned}&\le |\eta_t|_{L^2} + |\eta_x|_{L^2}|u|_{L^\infty} + C|\eta_x|_{L^2} + |\eta|_{H^1}|u_x|_{L^p} + C|\eta|_{H^1}\\&\le CZ^2(t).\end{aligned}\tag{18}$$

Multiplying $(1)_1$ by ρ, integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\rho|^2\mathrm{d}s + \int_\Omega (\rho u)_x\rho\mathrm{d}x = 0.$$

Integrating by parts, using Sobolev inequality, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho(t)|_{L^2}^2 \le \int_\Omega |u_x||\rho|^2\mathrm{d}x \le |u_{xx}|_{L^2}|\rho|_{L^2}^2.\tag{19}$$

Differentiating $(1)_1$ with respect to x, and multiplying it by ρ_x, integrating over Ω, using Sobolev inequality, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\rho_x|^2\mathrm{d}x &= -\int_\Omega [\frac{3}{2}u_x(\rho_x)^2 + \rho\rho_x u_{xx}](t)\mathrm{d}x\\&\le C[|u_x|_{L^\infty}|\rho_x|_{L^2}^2 + |\rho|_{L^\infty}|\rho_x|_{L^2}|u_{xx}|_{L^2}]\\&\le C|\rho|_{H^1}^2|u_{xx}|_{L^2}.\end{aligned}\tag{20}$$

From (19)and (20), by Gronwall's inequality, it follows that

$$\sup_{0\le t\le T}|\rho(t)|_{H^1}^2 \le |\rho_0|_{H^1}^2\exp\{C\int_0^t |u_{xx}|_{L^2}\mathrm{d}s\} \le C\exp(C\int_0^t Z^{\max(\frac{p}{q}+1,3)}(s)\mathrm{d}s).\tag{21}$$

Besides, by using $(1)_1$, we can also get the following estimates:

$$|\rho_t(t)|_{L^2} \le |\rho_x(t)|_{L^2}|u(t)|_{L^\infty} + |\rho(t)|_{L^\infty}|u_x(t)|_{L^2} \le CZ^2(t).\tag{22}$$

2.3 Estimate for $|\eta_t|_{L^2}$ and $|\eta|_{H^1}$

Multiplying $(1)_4$ by η, integrating the resulting equation over Ω_T, using the boundary conditions (3), Young's inequality, we have

$$\begin{aligned}\int_0^t |\eta_x(s)|_{L^2}^2\mathrm{d}s + \frac{1}{2}|\eta(t)|_{L^2}^2 &\le \iint_{\Omega_T} (|\eta u\eta_x| + |\eta\Phi_x\eta_x|)\mathrm{d}x\mathrm{d}s\\&\le \frac{1}{4}\int_0^t |\eta_x(s)|_{L^2}^2\mathrm{d}s + C\int_0^t |u_x|_{L^p}^2|\eta|_{H^1}^2\mathrm{d}s + C\int_0^t |\eta|_{H^1}^2 + C\\&\le \frac{1}{4}\int_0^t |\eta_x(s)|_{L^2}^2\mathrm{d}s + C(1+\int_0^t Z^4(t)\mathrm{d}s).\end{aligned}\tag{23}$$

Multiplying $(1)_4$ by η_t, integrating (by parts) over Ω_T, using the boundary conditions (3), Young's inequality, we have

$$\begin{aligned}\int_0^t |\eta_t(s)|_{L^2}^2\mathrm{d}s + \frac{1}{2}|\eta_x(t)|_{L^2}^2 &\le \iint_{\Omega_T} |\eta(u-\Phi_x)\eta_{xt}|\mathrm{d}x\mathrm{d}s\\&\le \frac{1}{4}\int_0^t |\eta_{xt}(s)|_{L^2}^2\mathrm{d}s + C\int_0^t |\eta|_{H^1}^2|u_x|_{L^p}^2\mathrm{d}s + C\int_0^t |\eta|_{H^1}^2\mathrm{d}s + C\end{aligned}$$

$$\leq \frac{1}{4}\int_0^t |\eta_{xt}(s)|_{L^2}^2 \mathrm{d}s + C(1 + \int_0^t Z^4(t)\mathrm{d}s). \tag{24}$$

Differentiating $(1)_4$ with respect to t, multiplying the resulting equation by η_t, integrating (by parts) over Ω_T, we get

$$\begin{aligned}
\int_0^t |\eta_{xt}(s)|_{L^2}^2 \mathrm{d}s + \frac{1}{2}|\eta_t(t)|_{L^2}^2 &= \iint_{\Omega_T} (\eta(u-\Phi_x))_t \eta_{xt} \mathrm{d}x\mathrm{d}s \\
&\leq C + \iint_{\Omega_T} (|\eta_t u \eta_{xt}| + |\eta_t \Phi_x \eta_{xt}| + |\eta_x u_t \eta_t| + |\eta u_{xt} \eta_t|)\mathrm{d}x\mathrm{d}s \\
&\leq C(1 + \int_0^t (|\eta_t|_{L^2}^2 ||u_x|_{L^p}^2 + |\eta_t|_{L^2}^2 + |\eta_x|_{L^2}^2 |\eta_t|_{L^2}^2 + |\eta|_{H^1}^2 |\eta_t|_{L^2}^2)\mathrm{d}x) \\
&\quad + \frac{1}{2}\int_0^t |\eta_{xt}|_{L^2}^2 + \frac{1}{2}\int_0^t |u_{xt}|_{L^2}^2 \\
&\leq C(1 + \int_0^t Z^{2\gamma+6}(s)\mathrm{d}s).
\end{aligned} \tag{25}$$

Combining (23)-(25), we get

$$|\eta|_{H^1}^2 + |\eta_t|_{L^2}^2 + \int_0^t (|\eta_x|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta_{xt}|_{L^2}^2)(s)\mathrm{d}s \leq C(1 + \int_0^t Z^{2\gamma+6}(s)\mathrm{d}s). \tag{26}$$

2.4 Estimate for $|\sqrt{\rho}u_t|_{L^2}$

Differentiating equation (10) with respect to t, multiplying the result equation by u_t, and integrating it over Ω with respect to x, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u_t|^2 \mathrm{d}x + \int_\Omega \left[(u_x^2+\mu_1)^{\frac{p-2}{2}} u_x\right]_t u_{xt}\mathrm{d}x \\
&= \int_\Omega [(\rho u)_x(u_t^2 + u u_x u_t + \Psi_x u_t) - \rho u_x u_t^2 - \rho \Psi_{xt} u_t - (P+\eta)_t u_{xt} - \eta_t \Phi_x u_t]\mathrm{d}x.
\end{aligned} \tag{27}$$

Note that

$$\left[(u_x^2+\mu_1)^{\frac{p-2}{2}} u_x\right]_t u_{xt} = (u_x^2+\mu_1)^{\frac{p-4}{2}}(p-1)(u_x^2+\mu_1)u_{xt}^2 \geq \mu_1^{\frac{p-2}{2}} u_{xt}^2.$$

Combining (12), (27) can be rewritten into

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u_t|^2\mathrm{d}x + \int_\Omega |u_{xt}|^2 \mathrm{d}x \\
&\leq 2\int_\Omega \rho|u||u_t||u_{xt}|\mathrm{d}x + \int_\Omega \rho|u||u_x|^2|u_t|\mathrm{d}x + \int_\Omega \rho|u|^2|u_{xx}||u_t|\mathrm{d}x \\
&\quad + \int_\Omega \rho|u|^2|u_x||u_{xt}|\mathrm{d}x + \int_\Omega \rho|u||\Psi_{xx}||u_t|\mathrm{d}x + \int_\Omega \rho|u||\Psi_x||u_{xt}|\mathrm{d}x \\
&\quad + \int_\Omega \rho|u_x||u_t|^2\mathrm{d}x + \int_\Omega \gamma|P||u_x||u_{xt}|\mathrm{d}x + \int_\Omega |P_x||u||u_{xt}|\mathrm{d}x \\
&\quad + \int_\Omega |\eta_t||u_{xt}|\mathrm{d}x + \int_\Omega |\eta_t||\Phi_x||u_t|\mathrm{d}x + \int_\Omega \rho|\Psi_{xt}||u_t|\mathrm{d}x = \sum_{j=1}^{12} I_j.
\end{aligned} \tag{28}$$

By using Sobolev inequality, Hölder inequality and Young's inequality, (14),(15), we estimate each term of I_j as follows

$$
\begin{aligned}
I_1 &= 2\int_\Omega \rho|u||u_t||u_{xt}|\mathrm{d}x \le 2|\rho|_{L^\infty}^{\frac{1}{2}}|u|_{L^\infty}|\sqrt{\rho}u_t|_{L^2}|u_{xt}|_{L^2} \le CZ^5(t)+\frac{1}{7}|u_{xt}|_{L^2}^2\\
I_2 &= \int_\Omega \rho|u||u_x|^2|u_t|\mathrm{d}x \le |\rho|_{L^\infty}^{\frac{1}{2}}|u|_{L^2}|u_x|_{L^p}^2|\sqrt{\rho}u_t|_{L^2} \le CZ^5(t)\\
I_3 &= \int_\Omega \rho|u|^2|u_{xx}||u_t|\mathrm{d}x \le |\rho|_{L^\infty}^{\frac{1}{2}}|u|_{L^\infty}^2|u_{xx}|_{L^2}|\sqrt{\rho}u_t|_{L^2} \le CZ^{\max(\frac{p}{q}+5,7)}(t)\\
I_4 &= \int_\Omega \rho|u|^2|u_x||u_{xt}|\mathrm{d}x \le |\rho|_{L^\infty}|u|_{L^\infty}^2|u_x|_{L^2}|u_{xt}|_{L^2} \le CZ^8(t)+\frac{1}{7}|u_{xt}|_{L^2}^2\\
I_5 &= \int_\Omega \rho|u||\Psi_{xx}||u_t|\mathrm{d}x \le |\rho|_{L^\infty}^{\frac{1}{2}}|u|_{L^\infty}|\Psi_{xx}|_{L^2}|\sqrt{\rho}u_t|_{L^2} \le CZ^{\frac{q}{p}+3}(t)\\
I_6 &= \int_\Omega \rho|u||\Psi_x||u_{xt}|\mathrm{d}x \le |\rho|_{L^\infty}|u|_{L^\infty}|\Psi_{xx}|_{L^2}|u_{xt}|_{L^2} \le CZ^{\frac{2p}{q}+4}(t)+\frac{1}{7}|u_{xt}|_{L^2}^2\\
I_7 &= \int_\Omega \rho|u_x||u_t|^2\mathrm{d}x \le |u_x|_{L^\infty}|\sqrt{\rho}u_t|_{L^2}^2 \le CZ^{\max(\frac{p}{q}+3,5)}(t)\\
I_8 &= \int_\Omega \gamma|P||u_x||u_{xt}|\mathrm{d}x \le C|P|_{L^\infty}|u_x|_{L^p}|u_{xt}|_{L^2} \le CZ^{2\gamma+2}(t)+\frac{1}{7}|u_{xt}|_{L^2}^2\\
I_9 &= \int_\Omega |P_x||u||u_{xt}|\mathrm{d}x \le |P_x|_{L^2}|u|_{L^\infty}|u_{xt}|_{L^2} \le CZ^{2\gamma+2}(t)+\frac{1}{7}|u_{xt}|_{L^2}^2\\
I_{10} &= \int_\Omega |\eta_t||u_{xt}|\mathrm{d}x \le |\eta_t|_{L^2}|u_{xt}|_{L^2} \le CZ^2(t)+\frac{1}{7}|u_{xt}|_{L^2}^2\\
I_{11} &= \int_\Omega |\eta_t||\Phi_x||u_t|\mathrm{d}x \le |\eta_t|_{L^2}|\Phi_x|_{L^2}|u_t|_{L^\infty} \le CZ^2(t)+\frac{1}{7}|u_{xt}|_{L^2}^2\\
I_{12} &= \int_\Omega \rho|\Psi_{xt}||u_t|\mathrm{d}x \le C|\rho|_{L^\infty}^{\frac{1}{2}}|\Psi_{xt}|_{L^2}|\sqrt{\rho}u_t|_{L^2},
\end{aligned}
$$

where C is a positive constant, depending only on M_0.

Next, we deal with the term $|\Psi_{xt}|_{L^2}$ of I_{12}. Differentiating $(1)_3$ with respect to t, multiplying it by Ψ_t, integrating over Ω and using Young's inequality, we obtain

$$\int_\Omega (|\Psi_x|^{q-2}\Psi_x)_t\Psi_{xt}\mathrm{d}x = -4\pi g\int_\Omega \rho_t\Psi_t\mathrm{d}x.$$

By virtue of

$$
\begin{aligned}
\int_\Omega (|\Psi_x|^{q-2}\Psi_x)_t\Psi_{xt}\mathrm{d}x &= \int_\Omega (|\Psi_x|^{q-4}[(q-2)\Psi_x^2+\Psi_x^2]\Psi_{xt}^2\mathrm{d}x\\
&\ge C\int_\Omega |\Psi_x|^{q-2}\Psi_{xt}^2\mathrm{d}x \ge C|\Psi_x|_{L^\infty}^{q-2}|\Psi_{xt}|_{L^2}^2\\
&\ge C|\Psi_{xx}|_{L^2}^{q-2}|\Psi_{xt}|_{L^2}^2
\end{aligned}
$$

and

$$-4\pi g\int_\Omega \rho_t\Psi_t\mathrm{d}x \le C\int_\Omega |\rho_t||\Psi_t|\mathrm{d}x \le C|\rho_t|_{L^2}^2+C|\Psi_t|_{L^2}^2 \le C|\rho_t|_{L^2}^2+C(\varepsilon)|\Psi_{xt}|_{L^2}^2,$$

then $|\Psi_{xt}|_{L^2} \le CZ^{2-\frac{p(2-q)}{q}}(t)$. Therefore,

$$I_{12} = \int_\Omega \rho|\Psi_{xt}||u_t|\mathrm{d}x \le C|\rho|_{L^\infty}^{\frac{1}{2}}|\Psi_{xt}|_{L^2}|\sqrt{\rho}u_t|_{L^2} \le Z^{4+\frac{p(2-q)}{q}}(t).$$

Substituting $I_j (j = 1, 2, \dots, 12)$ into (28), and integrating over $(\tau, t) \subset (0, T)$ on the time variable, we have

$$|\sqrt{\rho}u_t(t)|_{L^2}^2 + \int_\tau^t |u_{xt}|_{L^2}^2(s)\mathrm{d}s \le |\sqrt{\rho}u_t(\tau)|_{L^2}^2 + C\int_\tau^t Z^{\max(\frac{2p}{q}+4,8)}(s)\mathrm{d}s. \tag{29}$$

To obtain the estimate of $|\sqrt{\rho}u_t(t)|_{L^2}^2$, we need to estimate $\lim_{\tau\to 0}|\sqrt{\rho}u_t(\tau)|_{L^2}^2$. Multiplying (10) by u_t and integrating over Ω, we have

$$\int_\Omega \rho|u_t|^2\mathrm{d}x \le 2\int_\Omega (\rho|u|^2|u_x|^2 + \rho|\Psi_x|^2 + \rho^{-1}\big| - \big[(u_x^2+\mu_1)^{\frac{p-2}{2}}u_x\big]_x + (P+\eta)_x + \eta\Phi_x\big|^2)\mathrm{d}x.$$

According to the smoothness of (ρ, u, η), we obtain

$$\begin{aligned}
&\lim_{\tau\to 0}\int_\Omega \big(\rho|u|^2|u_x|^2 + \rho|\Psi_x|^2 + \rho^{-1}\big| - \big[(u_x^2+\mu_1)^{\frac{p-2}{2}}u_x\big]_x + (P+\eta)_x + \eta\Phi_x\big|^2\big)\mathrm{d}x \\
&= \int_\Omega \big(\rho|u_0|^2|u_{0x}|^2 + \rho_0|\Psi_x|^2 + \rho_0^{-1}\big| - \big[(u_{0x}^2+\mu_1)^{\frac{p-2}{2}}u_{0x}\big]_x + (P_0+\eta_0)_x + \eta_0\Phi_x\big|^2\big)\mathrm{d}x \\
&\le |\rho_0|_{L^\infty}|u_0|_{L^\infty}^2|u_{0x}|_{L^2}^2 + |\rho_0|_{L^\infty}|\Psi_x|_{L^2}^2 + |g|_{L^2}^2 + \beta|\Phi_x|_{L^2}^2 \le C.
\end{aligned}$$

Therefore, taking a limit on τ in (29), as $\tau \to 0$, we conclude that

$$|\sqrt{\rho}u_t(t)|_{L^2}^2 + \int_0^t |u_{xt}|_{L^2}^2(s)\mathrm{d}s \le C(1 + \int_0^t Z^{\max(\frac{2p}{q}+4,8)}(s)\mathrm{d}s), \tag{30}$$

where C is a positive constant, depending only on M_0.

Combining the estimates of (15), (18), (21), (22), (17), (26), (30) and the definition of $Z(t)$, we conclude that

$$Z(t) \le \tilde{C}\exp(\tilde{\tilde{C}}\int_0^t Z^{\max(\frac{2p}{q}+4,8)}(s)\mathrm{d}s), \tag{31}$$

where $\tilde{C}, \tilde{\tilde{C}}$ are positive constants, depending only on M_0. This means that there exist a time $T_1 > 0$ and a constant $C > 0$, such that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0\le t\le T_1}(|\rho|_{H^1} + |u|_{W_0^{1,p}\cap H^2} + |\eta|_{H^2} + |\eta_t|_{L^2} + |\sqrt{\rho}u_t|_{L^2} + |\rho_t|_{L^2}) \\
&\quad + \int_0^{T_1}(|\sqrt{\rho}u_t|_{L^2}^2 + |u_{xt}|_{L^2}^2 + |\eta_x|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta_{xt}|_{L^2}^2)\mathrm{d}s \le C.
\end{aligned} \tag{32}$$

3 Proof of the Main Theorem

In this section, our proof will be based on the usual iteration argument and some ideas developed in [13, 14]. Precisely, we construct the approximate solutions, by using the iterative scheme, inductively, as follows: first define $u^0 = 0$ and assuming that u^{k-1} was defined for $k \ge 1$, let ρ^k, u^k, η^k be the unique smooth solution to the following problems:

$$\begin{aligned}
&\rho_t^k + \rho_x^k u^{k-1} + \rho^k u_x^{k-1} = 0 \\
&\rho^k u_t^k + \rho^k u^{k-1}u_x^k + \rho^k\Psi_x^k - \big[((u_x^k)^2+\mu_1)^{\frac{p-2}{2}}u_x^k\big]_x + P_x^k + \eta_x^k = -\eta^k\Phi_x \\
&(|\Psi_x^k|^{q-2}\Psi_x^k)_x = 4\pi g(\rho^k - m_0) \\
&\eta_t^k + (\eta^k(u^{k-1} - \Phi_x))_x = \eta_{xx}^k
\end{aligned}$$

with the initial and boundary conditions

$$(\rho^k, u^k, \eta^k)|_{t=0} = (\rho_0, u_0, \eta_0)$$
$$u^k|_{\partial\Omega} = (\eta_x^k + \eta^k \Phi_x)|_{\partial\Omega} = 0$$

with the process, the nonlinear coupled system has been deduced into a sequence of decoupled problems and each problem admits a smooth solution. And the following estimates hold

$$\begin{aligned}\operatorname*{ess\,sup}_{0\le t\le T_1} &(|\rho^k|_{H^1} + |u^k|_{W_0^{1,p}\cap H^2} + |\eta^k|_{H^2} + |\eta_t^k|_{L^2} + |\sqrt{\rho^k}u_t^k|_{L^2} + |\rho_t^k|_{L^2})\\ &+ \int_0^{T_1} (|\sqrt{\rho^k}u_t^k|_{L^2}^2 + |u_{xt}^k|_{L^2}^2 + |\eta_x^k|_{L^2}^2 + |\eta_t^k|_{L^2}^2 + |\eta_{xt}^k|_{L^2}^2)\mathrm{d}s \le C,\end{aligned} \tag{33}$$

where C is a generic constant depending only on M_0, but independent of k.

In addition, we first find ρ^k from the initial problem

$$\rho_t^k + u^{k-1}\rho_x^k + u_x^{k-1}\rho^k = 0, \quad \text{and} \quad \rho^k|_{t=0} = \rho_0$$

with smooth function u^{k-1}, obviously, there is a unique solution ρ^k to the above problem and also by a standard argument, we could obtain that

$$\rho^k(x,t) \ge \delta \exp\Big[-\int_0^{T_1} |u_x^{k-1}(.,s)|_{L^\infty}\mathrm{d}s\Big] > 0, \text{for all } t \in (0, T_1).$$

Next, we have to prove that the approximate solution (ρ^k, u^k, η^k) converges to a solution to the original problem (1) in a strong sense. To this end, let us define

$$\bar\rho^{k+1} = \rho^{k+1} - \rho^k, \quad \bar u^{k+1} = u^{k+1} - u^k, \quad \bar\Psi^{k+1} = \Psi^{k+1} - \Psi^k, \quad \bar\eta^{k+1} = \eta^{k+1} - \eta^k,$$

then we can verify that the functions $\bar\rho^{k+1}, \bar u^{k+1}, \bar\eta^{k+1}$ satisfy the system of equations

$$\bar\rho_t^{k+1} + (\bar\rho^{k+1}u^k)_x + (\rho^k \bar u^k)_x = 0 \tag{34}$$

$$\begin{aligned}&\rho^{k+1}\bar u_t^{k+1} + \rho^{k+1}u^k\bar u_x^{k+1} - \big([((u_x^{k+1})^2+\mu_1)^{\frac{p-2}{2}}u_x^{k+1}]_x - [((u_x^k)^2+\mu_1)^{\frac{p-2}{2}}u_x^k]_x\big)\\ &= -\bar\rho^{k+1}(u_t^k + u^k u_x^k + \Psi_x^{k+1}) - (P^{k+1} - P^k)_x + \rho^k(\bar u^k u_x^k - \bar\Psi_x^{k+1}) - \bar\eta_x^{k+1} - \bar\eta^{k+1}\Phi_x\end{aligned} \tag{35}$$

$$(|\Psi_x^{k+1}|^{q-2}\Psi_x^{k+1})_x - (|\Psi_x^k|^{q-2}\Psi_x^k)_x = 4\pi g\bar\rho^{k+1} \tag{36}$$

$$\bar\eta_t^{k+1} + (\eta^k \bar u^k)_x + (\bar\eta^{k+1}(u^k - \Phi_x))_x = \bar\eta_{xx}^{k+1}. \tag{37}$$

Multiplying (34) by $\bar\rho^{k+1}$, integrating over Ω and using Young's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}|\bar\rho^{k+1}|_{L^2}^2 &\le C|\bar\rho^{k+1}|_{L^2}^2|u_x^k|_{L^\infty} + |\rho^k|_{H^1}|\bar u_x^k|_{L^2}|\bar\rho^{k+1}|_{L^2}\\ &\le C|u_{xx}^k|_{L^2}|\bar\rho^{k+1}|_{L^2}^2 + C_\zeta|\rho^k|_{H^1}^2|\bar\rho^{k+1}|_{L^2}^2 + \zeta|\bar u_x^k|_{L^2}^2\\ &\le C_\zeta|\bar\rho^{k+1}|_{L^2}^2 + \zeta|\bar u_x^k|_{L^2}^2,\end{aligned} \tag{38}$$

where C_ζ is a positive constant, depending on M_0 and ζ for all $t < T_1$ and $k \ge 1$.

Multiplying (35) by $\bar u^{k+1}$, integrating over Ω and using Young's inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^{k+1}|\bar u^{k+1}|^2\mathrm{d}x + \int_\Omega \big([((u_x^{k+1})^2+\mu_1)^{\frac{p-2}{2}}u_x^{k+1}]_x - [((u_x^k)^2+\mu_1)^{\frac{p-2}{2}}u_x^k]_x\big)\bar u^{k+1}\mathrm{d}x\\ &\le C\int_\Omega \big(|\bar\rho^{k+1}|(|u_t^k| + |u^k u_x^k| + |\Psi_x^{k+1}|)|\bar u^{k+1}| + |P_x^{k+1} - P_x^k||\bar u^{k+1}| + |\rho^k|\bar u^k||u_x^k||\bar u^{k+1}|\\ &\quad + |\rho^k||\bar\Psi_x^{k+1}||\bar u^{k+1}| + |\bar\eta_x^{k+1}||\bar u^{k+1}| + |\bar\eta^{k+1}\Phi_x||\bar u^{k+1}|\big)\mathrm{d}x.\end{aligned} \tag{39}$$

Let

$$\sigma(s)=(s^2+\mu_1)^{\frac{p-2}{2}}s,$$

then

$$\sigma'(s)=\left((s^2+\mu_1)^{\frac{p-2}{2}}s\right)'=(s^2+\mu_1)^{\frac{p-4}{2}}\left((p-1)s^2+\mu_1\right)\geq\mu_1^{\frac{p-2}{2}}.$$

We estimate the second term of (39) as follows

$$\begin{aligned}&\int_\Omega\left([((u_x^{k+1})^2+\mu_1)^{\frac{p-2}{2}}u_x^{k+1}]_x-[((u_x^k)^2+\mu_1)^{\frac{p-2}{2}}u_x^k]_x\right)\bar{u}^{k+1}\mathrm{d}x\\&=\int_\Omega\int_0^1\sigma'(\theta u_x^{k+1}+(1-\theta)u_x^k)\mathrm{d}\theta|\bar{u}_x^{k+1}|^2\mathrm{d}x\geq\mu_1^{\frac{p-2}{2}}\int_\Omega|\bar{u}_x^{k+1}|^2\mathrm{d}x.\end{aligned}\tag{40}$$

Similarly, multiplying (36) by $\bar{\Psi}^{k+1}$, integrating over Ω, we get

$$\int_\Omega\left[(|\Psi_x^{k+1}|^{q-2}\Psi_x^{k+1})_x-(|\Psi_x^k|^{q-2}\Psi_x^k)_x\right]\bar{\Psi}^{k+1}\mathrm{d}x=4\pi g\int_\Omega\bar{\rho}^{k+1}\bar{\Psi}^{k+1}\mathrm{d}x,\tag{41}$$

since

$$\begin{aligned}&\int_\Omega\left[|\Psi_x^{k+1}|^{q-2}\Psi_x^{k+1}-|\Psi_x^k|^{q-2}\Psi_x^k\right]\bar{\Psi}_x^{k+1}\mathrm{d}x\\&=(q-1)\int_\Omega(\int_0^1|\theta\Psi_x^{k+1}+(1-\theta)\Psi_x^k|^{q-2}\mathrm{d}\theta)(\bar{\Psi}_x^{k+1})^2\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int_0^1|\theta\Psi_x^{k+1}+(1-\theta)\Psi_x^k|^{q-2}\mathrm{d}\theta&=\int_0^1\frac{1}{|\theta\Psi_x^{k+1}+(1-\theta)\Psi_x^k|^{2-q}}\mathrm{d}\theta\\&\geq\int_0^1\frac{1}{(|\Psi_x^{k+1}|+|\Psi_x^k|^{2-q})}\mathrm{d}\theta=\frac{1}{(|\Psi_x^{k+1}|+|\Psi_x^k|)^{2-q}}.\end{aligned}$$

Then

$$\int_\Omega\left[|\Psi_x^{k+1}|^{q-2}\Psi_x^{k+1}-|\Psi_x^k|^{q-2}\Psi_x^k\right]\bar{\Psi}_x^{k+1}\mathrm{d}x\geq\frac{1}{(|\Psi_x^{k+1}(t)|_{L^\infty}+|\Psi_x^k(t)|_{L^\infty})^{2-q}}\int_\Omega(\bar{\Psi}_x^{k+1})^2\mathrm{d}x.$$

That means (41) turns into

$$\int_\Omega(\bar{\Psi}_x^{k+1})^2\mathrm{d}x\leq C|\bar{\rho}^{k+1}|_{L^2}^2.\tag{42}$$

Substituting (40) and (42) into (39), using Young's inequality, yields

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\rho^{k+1}|\bar{u}^{k+1}|^2\mathrm{d}x+\int_\Omega|\bar{u}_x^{k+1}|^2\mathrm{d}x\\&\leq C(|\bar{\rho}^{k+1}|_{L^2}|u_{xt}^k|_{L^2}|\bar{u}_x^{k+1}|_{L^2}+|\bar{\rho}^{k+1}|_{L^2}|u_x^k|_{L^p}|u_{xx}^k|_{L^2}|\bar{u}_x^{k+1}|_{L^2}+|\bar{\rho}^{k+1}|_{L^2}|\Psi_x^{k+1}|_{L^2}|\bar{u}_x^{k+1}|_{L^2}\\&\quad+|P^{k+1}-P^k|_{L^2}|\bar{u}_x^{k+1}|_{L^2}+|\rho^k|_{L^2}^{\frac12}|\sqrt{\rho^k}\bar{u}^k|_{L^2}|u_{xx}^k|_{L^2}|\bar{u}_x^{k+1}|_{L^2}+|\rho^k|_{L^\infty}|\bar{\Psi}_x^{k+1}|_{L^2}|\bar{u}_x^{k+1}|_{L^2}\\&\quad+|\bar{\eta}^{k+1}|_{L^2}|\bar{u}_x^{k+1}|_{L^2}+|\bar{\eta}^{k+1}|_{L^2}|\bar{u}_x^{k+1}|_{L^2})\\&\leq B_\zeta(t)|\bar{\rho}^{k+1}|_{L^2}^2+C(|\sqrt{\rho^k}\bar{u}^k|_{L^2}^2+|\bar{\eta}^{k+1}|_{L^2}^2)+\zeta|\bar{u}_x^{k+1}|_{L^2}^2,\end{aligned}\tag{43}$$

where $B_\zeta(t)=C(1+|u_{xt}^k(t)|_{L^2}^2)$, for all $t\leq T_1$ and $k\geq1$. Using (33) we derive

$$\int_0^t B_\zeta(s)\mathrm{d}s\leq C+Ct.$$

Multiplying (37) by $\bar{\eta}^{k+1}$, integrating over Ω, using (33) and Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\bar{\eta}^{k+1}|^2 dx + \int_\Omega |\bar{\eta}_x^{k+1}|^2 dx \\
&\le \int_\Omega |\bar{\eta}^{k+1}||u^k - \Phi_x||\bar{\eta}_x^{k+1}| dx + \int_\Omega (|\eta^k||\bar{u}^k|)_x|\bar{\eta}^{k+1}| dx \\
&\le |\bar{\eta}^{k+1}|_{L^2}|u^k - \Phi_x|_{L^\infty}|\bar{\eta}_x^{k+1}|_{L^2} + |\eta_x^k|_{L^2}|\bar{u}^k|_{L^\infty}|\bar{\eta}^{k+1}|_{L^2} + |\eta^k|_{L^\infty}|\bar{u}_x^k|_{L^2}|\bar{\eta}^{k+1}|_{L^2} \\
&\le C_\zeta|\bar{\eta}^{k+1}|_{L^2}^2 + \zeta|\bar{\eta}_x^{k+1}|_{L^2}^2 + \zeta|\bar{u}_x^k|_{L^2}^2. \qquad (44)
\end{aligned}$$

Collecting (38), (43) and (44), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(|\bar{\rho}^{k+1}(t)|_{L^2}^2 + |\sqrt{\rho^{k+1}}\bar{u}^{k+1}(t)|_{L^2}^2 + |\bar{\eta}^{k+1}(t)|_{L^2}^2\Big) + |\bar{u}_x^{k+1}(t)|_{L^2}^2 + |\bar{\eta}_x^{k+1}|_{L^2}^2 \\
&\le E_\zeta(t)|\bar{\rho}^{k+1}(t)|_{L^2}^2 + C|\sqrt{\rho^k}\bar{u}^k|_{L^2}^2 + C_\zeta|\bar{\eta}^{k+1}|_{L^2}^2 + \zeta|\bar{u}_x^k|_{L^2}^2, \qquad (45)
\end{aligned}$$

where $E_\zeta(t)$ depends only on $B_\zeta(t)$ and C_ζ, for all $t \le T_1$ and $k \ge 1$. Using (33), we have

$$\int_0^t E_\zeta(s) ds \le C + C_\zeta t.$$

Integrating (45) over $(0, t) \subset (0, T_1)$ with respect to t, using Gronwall's inequality, we have

$$\begin{aligned}
&|\bar{\rho}^{k+1}(t)|_{L^2}^2 + |\sqrt{\rho^{k+1}}\bar{u}^{k+1}(t)|_{L^2}^2 + |\bar{\eta}^{k+1}(t)|_{L^2}^2 + \int_0^t |\bar{u}_x^{k+1}(t)|_{L^2}^2 ds + \int_0^t |\bar{\eta}_x^{k+1}|_{L^2}^2 ds \\
&\le C\exp(C_\zeta t)\int_0^t (|\sqrt{\rho^k}\bar{u}^k(s)|_{L^2}^2 + |\bar{u}_x^k(s)|_{L^2}^2) ds. \qquad (46)
\end{aligned}$$

From the above recursive relation, choose $\zeta > 0$ and $0 < T_* < T_1$ such that $C\exp(C_\zeta T_*) < \frac{1}{2}$, using Gronwall's inequality, we deduce that

$$\begin{aligned}
&\sum_{k=1}^K\Big[\sup_{0\le t\le T_*}(|\bar{\rho}^{k+1}(t)|_{L^2}^2 + |\sqrt{\rho^{k+1}}\bar{u}^{k+1}(t)|_{L^2}^2 + |\bar{\eta}^{k+1}(t)|_{L^2}^2 dt \\
&\qquad + \int_0^{T_*} |\bar{u}_x^{k+1}(t)|_{L^2}^2 + \int_0^{T_*} |\bar{\eta}_x^{k+1}(t)|_{L^2}^2 dt\Big] < C. \qquad (47)
\end{aligned}$$

Since all of the constants do not depend on δ, as $k \to \infty$, we conclude that sequence (ρ^k, u^k, η^k) converges to a limit $(\rho^\delta, u^\delta, \eta^\delta)$ in the following convergence

$$\rho \to \rho^\delta \quad \text{in } L^\infty(0, T_*; L^2(\Omega)), \qquad (48)$$

$$u \to u^\delta \quad \text{in } L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; H_0^1(\Omega)), \qquad (49)$$

$$\eta \to \eta^\delta \quad \text{in } L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; H^1(\Omega)), \qquad (50)$$

and there also holds

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0\le t\le T_1}(|\rho^\delta|_{H^1} + |u^\delta|_{W_0^{1,p}\cap H^2} + |\eta^\delta|_{H^2} + |\eta_t^\delta|_{L^2} + |\sqrt{\rho^\delta}u_t^\delta|_{L^2} + |\rho_t^\delta|_{L^2}) \\
&\qquad + \int_0^{T_*}(|\sqrt{\rho}^\delta u_t^\delta|_{L^2}^2 + |u_{xt}^\delta|_{L^2}^2 + |\eta_x^\delta|_{L^2}^2 + |\eta_t^\delta|_{L^2}^2 + |\eta_{xt}^\delta|_{L^2}^2) ds \le C. \qquad (51)
\end{aligned}$$

For each small $\delta > 0$, let $\rho_0^\delta = J_\delta * \rho_0 + \delta$, J_δ is a mollifier on Ω, and $u_0^\delta \in H_0^1(\Omega) \cap H^2(\Omega)$ is a smooth solution of the boundary value problem

$$\begin{cases} -\big[((u_{0x}^\delta)^2 + \mu_1)^{\frac{p-2}{2}} u_{0x}^\delta\big]_x + \big(P(\rho_0^\delta) + \eta_0^\delta\big)_x + \eta_0^\delta\Phi_x = (\rho_0^\delta)^{\frac{1}{2}}(g^\delta + \beta\Phi_x), \\ u_0^\delta(0) = u_0^\delta(1) = 0, \end{cases} \qquad (52)$$

where $g^\delta \in C_0^\infty$ and satisfies $|g^\delta|_{L^2} \le |g|_{L^2}$, $\lim\limits_{\delta\to 0^+} |g^\delta - g|_{L^2} = 0$.

We deduce that $(\rho^\delta, u^\delta, \eta^\delta)$ is a solution of the following initial boundary value problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + \rho\Psi_x - \lambda\big[(u_x^2+\mu_1)^{\frac{p-2}{2}} u_x\big]_x + (P+\eta)_x = -\eta\Phi_x, \\ \big(|\Psi_x|^{q-2}\Psi_x\big)_x = 4\pi g(\rho - \frac{1}{|\Omega|}\int\limits_\Omega \rho \mathrm{d}x), \\ \eta_t + (\eta(u-\Phi_x))_x = \eta_{xx}, \\ (\rho, u, \eta)|_{t=0} = (\rho_0^\delta, u_0^\delta, \eta_0^\delta), \\ u|_{\partial\Omega} = \Psi|_{\partial\Omega} = (\eta_x + \eta\Phi_x)|_{\partial\Omega} = 0, \end{cases}$$

where $\rho_0^\delta \ge \delta, p > 2, 1 < q < 2$.

By the proof of Lemma 2.3 in [11], there exists a subsequence $\{u_0^{\delta_j}\}$ of $\{u_0^\delta\}$, as $\delta_j \to 0^+$,$u_0^\delta \to u_0$ in $H_0^1(\Omega)\cap H^2(\Omega)$, $-(|u_{0x}^{\delta_j}|^{p-2}u_{0x}^{\delta_j})_x \to -(|u_{0x}|^{p-2}u_{0x})_x$ in $L^2(\Omega)$, Hence, u_0 satisfies the compatibility condition (8) of Theorem 1.2. By virtue of the lower semi-continuity of various norms, we deduce that (ρ, u, η) satisfies the following uniform estimate

$$\begin{aligned} &\operatorname*{ess\,sup}_{0\le t\le T_1} (|\rho|_{H^1} + |u|_{W_0^{1,p}\cap H^2} + |\eta|_{H^2} + |\eta_t|_{L^2} + |\sqrt{\rho}u_t|_{L^2} + |\rho_t|_{L^2}) \\ &\qquad + \int\limits_0^{T_*} (|\sqrt{\rho}u_t|_{L^2}^2 + |u_{xt}|_{L^2}^2 + |\eta_x|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta_{xt}|_{L^2}^2)\mathrm{d}s \le C, \end{aligned} \tag{53}$$

where C is a positive constant, depending only on M_0.

The uniqueness of solution can be obtained by the same method as the above proof of convergence, we omit the details here. This completes the proof.

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