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Research Article

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On the power sum problem of Lucas polynomials and its divisible property

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Abstract: The main purpose of this paper is to use the mathematical induction and the properties of Lucas polynomials to study the power sum problem of Lucas polynomials. In the end, we obtain an interesting divisible property.

Keywords: Lucas polynomials, Power sum problem, Mathematical induction, Divisible property

MSC: 11B39

1 Introduction

For any integer $n \geq 0$, the famous Fibonacci polynomials $\{F_n(x)\}$ and Lucas polynomials $\{L_n(x)\}$ are defined by $F_0(x) = 0$, $F_1(x) = 1$, $L_0(x) = 2$, $L_1(x) = x$ and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, $L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$. The general terms of $F_n(x)$ and $L_n(x)$ are given by

$$F_{n+1}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} x^{n-2k}$$

and

$$L_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}, \tag{1}$$

where $\binom{m}{n} = \frac{m!}{n!(m-n)!}$, and $[x]$ denotes the greatest integer $\leq x$.

It is easy to prove the identities

$$F_n(x) = \frac{1}{\sqrt{x^2+4}} \left[\left(\frac{x+\sqrt{x^2+4}}{2} \right)^n - \left(\frac{x-\sqrt{x^2+4}}{2} \right)^n \right]$$

and

$$L_n(x) = \left(\frac{x+\sqrt{x^2+4}}{2} \right)^n + \left(\frac{x-\sqrt{x^2+4}}{2} \right)^n. \tag{2}$$

If we take $x = 1$, then $\{F_n(x)\}$ becomes Fibonacci sequences $\{F_n(1)\}$, and $\{L_n(x)\}$ becomes Lucas sequences $\{L_n(1)\}$.

Since these sequences and polynomials have very important positions in the theory and application of mathematics, many scholars have studied their various properties, and obtained a series of important results,

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some of which can be found in references [1-5], and some other related papers can also be found in [6-16]. For example, in a private communication with Curtis Cooper, R. S. Melham suggested that it would be interesting to discover an explicit expansion for

$$L_1L_2\cdots L_{2m+1}\sum_{k=1}^{n}F_{2k}^{2m+1}$$

as a polynomial in F_{2n+1}. Wiemann and Cooper [1] reported some conjectures of Melham related to the sum $\sum_{k=1}^{n}F_{2k}^{2m+1}$. Kiyota Ozeki [2] proved that

$$\sum_{k=1}^{n}F_{2k}^{2m+1}=\frac{1}{5^m}\sum_{j=0}^{m}\frac{(-1)^j}{L_{2m+1-2j}}\binom{2m+1}{j}\left(F_{(2m+1-2j)(2n+1)}-F_{2m+1-2j}\right).$$

Helmut Prodinger [3] studied the more general summation $\sum_{k=0}^{n}F_{2k+\delta}^{2m+1+\epsilon}$, where $\delta,\ \epsilon\in\{0,\ 1\}$, and obtained many interesting identities.

For the sum of powers of Lucas numbers, Helmut Prodinger [3] also obtained some similar conclusions.

R. S. Melham [4] also proposed the following two conjectures:

Conjecture A. *Let m be a positive integer. Then the sum*

$$L_1L_3L_5\cdots L_{2m+1}\sum_{k=1}^{n}F_{2k}^{2m+1}$$

can be expressed as $\left(F_{2n+1}-1\right)^2P_{2m-1}\left(F_{2n+1}\right)$, where $P_{2m-1}(x)$ is a polynomial of degree $2m-1$ with integer coefficients.

Conjecture B. *Let $m\geq 0$ be an integer. Then the sum*

$$L_1L_3L_5\cdots L_{2m+1}\sum_{k=1}^{n}L_{2k}^{2m+1}$$

can be expressed as $\left(L_{2n+1}-1\right)Q_{2m}\left(L_{2n+1}\right)$, where $Q_{2m}(x)$ is a polynomial of degree $2m$ with integer coefficients.

Wang Tingting and Zhang Wenpeng [5] studied these problems, and proved several conclusions for $\sum_{k=1}^{h}F_k^n(x)$ and $\sum_{k=1}^{h}L_k^n(x)$, where h and n are any positive integers.

As some applications of Theorem 2 in reference [5], they deduced the following:

Corollary A. *Let $h\geq 1$ and $n\geq 0$ be two integers. Then the sum*

$$L_1(x)L_3(x)L_5(x)\cdots L_{2n+1}(x)\sum_{m=1}^{h}L_{2m}^{2n+1}(x)$$

can be expressed as $\left(L_{2h+1}(x)-x\right)Q_{2n}\left(x,\ L_{2h+1}(x)\right)$, where $Q_{2n}(x,y)$ is a polynomial in two variables x and y with integer coefficients and degree $2n$ of y.

Corollary B. *Let $h\geq 1$ and $n\geq 0$ be two integers. Then the sum*

$$L_1(x)L_3(x)L_5(x)\cdots L_{2n+1}(x)\sum_{m=1}^{h}F_{2m}^{2n+1}(x)$$

can be expressed as $\left(F_{2h+1}(x)-1\right)H_{2n}\left(x,\ F_{2h+1}(x)\right)$, where $H_{2n}(x,y)$ is a polynomial in two variables x and y with integer coefficients and degree $2n$ of y.

Therefore, Wang Tingting and Zhang Wenpeng [5] solved Conjecture *B* completely. They also obtained some substantial progress for Conjecture *A*.

It is easy to see that in the above Conjecture A and Conjecture B, the subscripts of F_n and L_n in sums $\sum_{k=1}^{n}F_{2k}^{2m+1}$ and $\sum_{k=1}^{n}L_{2k}^{2m+1}$ are even numbers. Inspired by the above researches, we naturally ask that for

any positive integers n and h with $h \geq 2$, whether there exists a polynomial $U(x)$ with degree > 1 such that the congruence

$$L_1(x)L_3(x)\cdots L_{2n+1}(x)\sum_{m=0}^{h-1} L_{2m+1}^{2n+1}(x) \equiv 0 \bmod U(x). \tag{3}$$

If so, what would $U(x)$ look like? How is it different from the form in Conjecture A and Conjecture B ? Obviously, it is hard to guess the exact form of $U(x)$ from (d) of Theorem 1 in reference [5]. Maybe that is why there is no such formal conjecture in [4].

In this paper, we shall use the mathematical induction and the properties of Lucas polynomials to study this problem, and give an exact polynomial $U(x)$ in (3). The result is detailed in the following theorems.

Theorem A. *For any positive integers n and h with $h \geq 2$, we have*

$$L_1(x)L_3(x)\cdots L_{2n+1}(x)\sum_{m=0}^{h-1} L_{2m+1}^{2n+1}(x) \equiv 0 \bmod x\cdot\left(L_{2h}(x)-2\right).$$

Theorem B. *For any positive integers n and h with $h \geq 2$, we have*

$$L_1(x)L_3(x)\cdots L_{2n+1}(x)\sum_{m=0}^{h-1} F_{2m+1}^{2n+1}(x) \equiv 0 \bmod x\cdot F_{2h}(x).$$

Taking $x = 1$, from Theorem 1 and Theorem 2 we may immediately deduce the following two corollaries:

Corollary A. *For any positive integers n and h with $h \geq 2$, we have*

$$L_1L_3\cdots L_{2n+1}\sum_{m=0}^{h-1} L_{2m+1}^{2n+1} \equiv 0 \bmod \left(L_{2h}-2\right).$$

Corollary B. *For any positive integers n and h with $h \geq 2$, we have*

$$L_1L_3\cdots L_{2n+1}\sum_{m=0}^{h-1} F_{2m+1}^{2n+1} \equiv 0 \bmod F_{2h}.$$

2 Two Lemmas

Lemma A. *For any non-negative integer n and k, we have the identity*

$$L_n\left(L_{2k+1}(x)\right) = L_{n(2k+1)}(x).$$

Proof. Let $\alpha = \frac{x+\sqrt{x^2+4}}{2}$, $\beta = \frac{x-\sqrt{x^2+4}}{2}$, and replace x by $L_{2k+1}(x)$ in (2). Since we have $\alpha^{2k+1}\beta^{2k+1} = -1$,

$$\begin{aligned} &L_{2k+1}(x)+\sqrt{L_{2k+1}^2(x)+4} = \alpha^{2k+1}+\beta^{2k+1}+\sqrt{\left(\alpha^{2k+1}+\beta^{2k+1}\right)^2+4} \\ &= \alpha^{2k+1}+\beta^{2k+1}+\sqrt{\alpha^{2(2k+1)}+\beta^{2(2k+1)}-2+4} \\ &= \alpha^{2k+1}+\beta^{2k+1}+\sqrt{\left(\alpha^{2k+1}-\beta^{2k+1}\right)^2} = 2\alpha^{2k+1} \end{aligned}$$

and

$$\begin{aligned} &L_{2k+1}(x)-\sqrt{L_{2k+1}^2(x)+4} = \alpha^{2k+1}+\beta^{2k+1}-\sqrt{\left(\alpha^{2k+1}+\beta^{2k+1}\right)^2+4} \\ &= \alpha^{2k+1}+\beta^{2k+1}-\sqrt{\left(\alpha^{2k+1}-\beta^{2k+1}\right)^2} = 2\beta^{2k+1}, \end{aligned}$$

from (2) we have the identity

$$L_n\left(L_{2k+1}(x)\right) = \left(\frac{L_{2k+1}+\sqrt{L_{2k+1}^2+4}}{2}\right)^n + \left(\frac{L_{2k+1}-\sqrt{L_{2k+1}^2+4}}{2}\right)^n$$

$$= \alpha^{n(2k+1)} + \beta^{n(2k+1)} = L_{n(2k+1)}(x).$$

This completes the proof of Lemma 1. □

Lemma B. *For any positive integers n and h, we have the congruence*

$$L_1(x)\left(L_{2h(2n+1)}(x) - 2\right) - (2n+1)L_{2n+1}(x)\left(L_{2h}(x) - 2\right) \equiv 0 \bmod x\left(L_{2h}(x) - 2\right).$$

Proof. From (1) we know that $x \mid L_{2n+1}(x)$. Because that $L_1(x) = x$, so in order to prove Lemma 2, we only need to prove the congruence

$$L_{2h(2n+1)}(x) - 2 \equiv 0 \bmod \left(L_{2h}(x) - 2\right). \tag{4}$$

Next we prove (4) by complete induction. It is clear that (4) is true for $n = 0$. If $n = 1$, then apply the identity

$$L_{2h}^3(x) = \left(\alpha^{2h} + \beta^{2h}\right)^3 = L_{6h}(x) + 3L_{2h}(x),$$

we have

$$\begin{aligned} L_{6h}(x) - 2 &= L_{2h}^3(x) - 3L_{2h}(x) - 2 = \left(L_{2h}(x) - 2\right)\left(L_{2h}(x) + 1\right)^2 \\ &\equiv 0 \bmod \left(L_{2h}(x) - 2\right). \end{aligned}$$

That is to say, the congruence (4) is true for $n = 1$.

Suppose that the congruence (4) is true for all integers $0 \le n \le s$. That is,

$$L_{2h(2n+1)}(x) - 2 \equiv 0 \bmod \left(L_{2h}(x) - 2\right) \tag{5}$$

holds for all integers $0 \le n \le s$.

Then for positive integer $n = s + 1$, we have the identities

$$L_{4h}(x) = L_{2h}^2(x) - 2 \equiv 2 \bmod \left(L_{2h}(x) - 2\right)$$

and

$$L_{4h}(x)L_{2h(2s+1)}(x) = L_{2h(2s+3)}(x) + L_{2h(2s-1)}(x),$$

from (5) we can deduce the congruence equations as follows

$$\begin{aligned} L_{2h(2s+3)}(x) - 2 &= L_{4h}(x)L_{2h(2s+1)}(x) - L_{2h(2s-1)}(x) - 2 \\ &\equiv 2L_{2h(2s+1)}(x) - L_{2h(2s-1)}(x) - 2 \\ &\equiv 2\left(L_{2h(2s+1)}(x) - 2\right) - \left(L_{2h(2s-1)}(x) - 2\right) \\ &\equiv 0 \bmod \left(L_{2h}(x) - 2\right). \end{aligned}$$

That is to say, the congruence (4) is true for $n = s + 1$.

Now Lemma 2 follows from (4) and completes the induction. □

3 Proofs of the theorems

In this section, we shall use mathematical induction to complete the proofs of our theorems. Here we only prove Theorem 1. Similarly, we can also deduce Theorem 2 and thus we omit its proving process here. After replacing x by $L_{2m+1}(x)$ in (1), we obtain the following expression with Lemma 1

$$L_{(2n+1)(2m+1)}(x) = L_{2n+1}\left(L_{2m+1}(x)\right) = \sum_{k=0}^{n} \frac{2n+1}{2n+1-k}\binom{2n+1-k}{k} L_{2m+1}^{2n+1-2k}(x)$$

or

$$L_{(2n+1)(2m+1)}(x) - (2n+1)L_{2m+1}(x) = \sum_{k=0}^{n-1} \frac{2n+1}{2n+1-k}\binom{2n+1-k}{k} L_{2m+1}^{2n+1-2k}(x). \tag{6}$$

For any positive integer $h \geq 2$, we first introduce that the identities

$$\begin{aligned}\sum_{m=0}^{h-1} L_{(2n+1)(2m+1)}(x) &= \sum_{m=0}^{h-1} \left(\alpha^{(2n+1)(2m+1)} + \beta^{(2n+1)(2m+1)}\right) \\ &= \alpha^{2n+1} \cdot \frac{\alpha^{2h(2n+1)} - 1}{\alpha^{2(2n+1)} - 1} + \beta^{2n+1} \cdot \frac{\beta^{2h(2n+1)} - 1}{\beta^{2(2n+1)} - 1} \\ &= \frac{\alpha^{2h(2n+1)} - 1}{\alpha^{2n+1} + \beta^{2n+1}} + \frac{\beta^{2h(2n+1)} - 1}{\beta^{2n+1} + \alpha^{2n+1}} = \frac{L_{2h(2n+1)}(x) - 2}{L_{2n+1}(x)},\end{aligned} \tag{7}$$

$$\sum_{m=0}^{h-1} L_{2m+1}(x) = \sum_{m=0}^{h-1} \left(\alpha^{2m+1} + \beta^{2m+1}\right) = \frac{L_{2h}(x) - 2}{L_1(x)}. \tag{8}$$

Then, combining (6), (7) and (8) we have

$$\begin{aligned}\sum_{m=0}^{h-1} \left[L_{(2n+1)(2m+1)}(x) - (2n+1)L_{2m+1}(x)\right] &= \frac{L_{2h(2n+1)}(x) - 2}{L_{2n+1}(x)} - (2n+1) \cdot \frac{L_{2h}(x) - 2}{L_1(x)} \\ &= \sum_{k=0}^{n-1} \frac{2n+1}{2n+1-k}\binom{2n+1-k}{k} \sum_{m=0}^{h-1} L_{2m+1}^{2n+1-2k}(x).\end{aligned} \tag{9}$$

Now we apply (9) and mathematical induction to prove the congruence

$$L_1(x)L_3(x)\cdots L_{2n+1}(x) \sum_{m=0}^{h-1} L_{2m+1}^{2n+1}(x) \equiv 0 \bmod x(L_{2h}(x) - 2). \tag{10}$$

If $n = 1$, then from (9) we have

$$L_1(x)L_3(x)\left[\frac{L_{6h}(x) - 2}{L_3(x)} - 3 \cdot \frac{L_{2h}(x) - 2}{L_1(x)}\right] = L_1(x)L_3(x) \sum_{m=0}^{h-1} L_{2m+1}^3(x). \tag{11}$$

From Lemma 2 we know that

$$L_1(x)L_3(x)\left[\frac{L_{6h}(x) - 2}{L_3(x)} - 3 \cdot \frac{L_{2h}(x) - 2}{L_1(x)}\right] \equiv 0 \bmod x\left(L_{2h}(x) - 2\right). \tag{12}$$

Combining (11) and (12) we know that the congruence (10) is true for $n = 1$.

Suppose that (10) is true for all $1 \leq n \leq s$. That is,

$$L_1(x)L_3(x)\cdots L_{2n+1}(x) \sum_{m=0}^{h-1} L_{2m+1}^{2n+1}(x) \equiv 0 \bmod x(L_{2h}(x) - 2) \tag{13}$$

holds for all $1 \leq n \leq s$.

Then for $n = s + 1$, from (9) we have

$$\begin{aligned}&\frac{L_{2h(2s+3)}(x) - 2}{L_{2s+3}(x)} - (2s+3) \cdot \frac{L_{2h}(x) - 2}{L_1(x)} \\ &= \sum_{k=0}^{s} \frac{2s+3}{2s+3-k}\binom{2s+3-k}{k} \sum_{m=0}^{h-1} L_{2m+1}^{2s+3-2k}(x) \\ &= \sum_{m=0}^{h-1} L_{2m+1}^{2s+3}(x) + \sum_{k=1}^{s} \frac{2s+3}{2s+3-k}\binom{2s+3-k}{k} \sum_{m=0}^{h-1} L_{2m+1}^{2s+3-2k}(x).\end{aligned} \tag{14}$$

Applying Lemma 2 we have the congruence

$$L_1(x)L_3(x)\cdots L_{2s+3}(x)\left[\frac{L_{2h(2s+3)}(x)-2}{L_{2s+3}(x)}-(2s+3)\cdot\frac{L_{2h}(x)-2}{L_1(x)}\right]$$
$$\equiv 0 \bmod x\left(L_{2h}(x)-2\right). \tag{15}$$

If $1 \le k \le s$, then $3 \le 2s+3-2k \le 2s+1$. From the inductive assumption (13) we have

$$L_1(x)L_3(x)\cdots L_{2s+1}(x)L_{2s+3}(x)\sum_{k=1}^{s}\frac{2s+3}{2s+3-k}\binom{2s+3-k}{k}\sum_{m=0}^{h-1}L_{2m+1}^{2s+3-2k}(x)$$
$$\equiv 0 \bmod x\left(L_{2h}(x)-2\right). \tag{16}$$

Combining (14), (15) and (16) we may immediately deduce the congruence

$$L_1(x)L_3(x)\cdots L_{2s+1}(x)L_{2s+3}(x)\sum_{m=0}^{h-1}L_{2m+1}^{2s+3}(x)\equiv 0 \bmod x\left(L_{2h}(x)-2\right).$$

This completes the proof of our theorem by mathematical induction.

Competing interests

The author declare that they have no competing interests.

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