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Super (a, d)-H-antimagic labeling of subdivided graphs

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Abstract: A simple graph $G = (V, E)$ admits an H-covering, if every edge in $E(G)$ belongs to a subgraph of G isomorphic to H. A graph G admitting an H-covering is called an (a, d)-H-antimagic if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \dots, |V(G)| + |E(G)|\}$ such that for all subgraphs H' isomorphic to H the sums $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ form an arithmetic sequence $\{a, a + d, \dots, a + (t - 1)d\}$, where $a > 0$ and $d \geq 0$ are integers and t is the number of all subgraphs of G isomorphic to H. Moreover, if the vertices are labeled with numbers $1, 2, \dots, |V(G)|$ the graph is called super. In this paper we deal with super cycle-antimagicness of subdivided graphs. We also prove that the subdivided wheel admits an (a, d)-cycle-antimagic labeling for some d.

Keywords: H-covering, (Super) (a, d)-H-antimagic labeling, Subdivided graph, Subdivided wheel

MSC: 05C78

1 Introduction

Let $G = (V, E)$ be a finite simple graph with the vertex set $V(G)$ and the edge set $E(G)$. An *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of E belongs to at least one of the subgraphs H_i, $i = 1, 2, \dots, t$. Then it is said that G admits an $(H_1, H_2, \dots, H_t)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an H-*covering*. A bijective function $f : V(G) \cup E(G) \to \{1, 2, \dots, |V(G)| + |E(G)|\}$ is an (a, d)-H-*antimagic labeling* of a graph G admitting an H-covering whenever, for all subgraphs H' isomorphic to H, the H'-weights

$$wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$$

form an arithmetic progression $a, a + d, \dots, a + (t - 1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H. Such a labeling is called *super* if the smallest possible labels appear on the vertices. A graph that admits a (super) (a, d)-H-antimagic labeling is called *(super)* (a, d)-H-*antimagic*. For $d = 0$ it is called H-*magic* and H-*supermagic*, respectively.

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The H-(super)magic labelings were first studied by Gutiérrez and Lladó [1] as an extension of the edge-magic and super edge-magic labelings introduced by Kotzig and Rosa [2] and Enomoto, Lladó, Nakamigawa and Ringel [3], respectively. In [1] are considered star-(super)magic and path-(super)magic labelings of some connected graphs and it is proved that the path P_n and the cycle C_n are P_h-supermagic for some h. Lladó and Moragas [4] studied the cycle-(super)magic behavior of several classes of connected graphs. They proved that wheels, windmills, books and prisms are C_h-magic for some h. Maryati, Salman, Baskoro, Ryan and Miller [5] and also Salman, Ngurah and Izzati [6] proved that certain families of trees are path-supermagic. Ngurah, Salman and Susilowati [7] proved that chains, wheels, triangles, ladders and grids are cycle-supermagic. Maryati, Salman and Baskoro [8] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that the disjoint union of any paths is cP_h-supermagic for some c and h.

The (a, d)-H-antimagic labeling was introduced by Inayah, Salman and Simanjuntak [9]. In [10] there are investigated the super (a, d)-H-antimagic labelings for some shackles of a connected graph H. In [11] was proved that wheels are cycle-antimagic. In [12] it was shoved that if a graph G admits a (super) (a, d)-H-antimagic labeling, where $d = |E(H)| - |V(H)|$, then the disjoint union of m copies of the graph G, denoted by mG, admits a (super) (b, d)-H-antimagic labeling as well. Rizvi, et al. [13] proved the disjoint union of isomorphic copies of fans, triangular ladders, ladders, wheels, and graphs obtained by joining a star $K_{1,n}$ with K_1, and also disjoint union of non-isomorphic copies of ladders and fans are cycle-supermagic.

In this paper we will discuss a super cycle-atimagicness of subdivided graphs. We show that the property to be super (a, d)-H-antimagic is hereditary according to the operation of subdivision of edges. We prove that if a graph G is super cycle-antimagic then the subdivided graph $S(G)$ also admits a super cycle-antimagic labeling. Moreover, we show that the subdivided wheel is super (a, d)-cycle-antimagic for wide range of differences.

2 Subdivided graphs

Let us consider the graph $S(G)$ obtained by subdividing some edges of a graph G, thus by inserting some new vertices to the original graph G. Equivalently, the graph $S(G)$ can by obtained from G by replacing some edges of G by paths. The topic of subdivided graphs has been widely studied in recent years, for example see [14].

Let G be a graph admitting H-covering given by t subgraphs $H_1, H_2, \dots, H_t$ isomorphic to H. Let us consider the subgraphs $S_G(H_i)$, $i = 1, 2, \dots, t$, corresponding to H_i in $S(G)$. If these subgraphs are all isomorphic to a graph, let us denote it by the symbol $S_G(H)$, then the graph $S(G)$ admits $S_G(H)$-covering.

The next theorem shows that the property of being super (a, d)-H-antimagic is hereditary according to the operation of subdivision of edges.

Theorem 2.1. *Let G be a super (a, d)-H-antimagic graph and let H_i, $i = 1, 2, \dots, t$, be all subgraphs of G isomorphic to H. If $S_G(H_i)$, $i = 1, 2, \dots, t$, are all subgraphs of $S(G)$ isomorphic to $S_G(H)$ then the graph $S(G)$ is a super (b, d)-$S(H)$-antimagic graph.*

Proof. Let G be a super (a, d)-H-antimagic graph and let H_i, $i = 1, 2, \dots, t$, be all subgraphs of G isomorphic to H. Let f be a super (a, d)-H-antimagic labeling of G, thus $f : V(G) \cup E(G) \to \{1, 2, \dots, |V(G)| + |E(G)|\}$ such that the vertices of G are labeled with numbers $1, 2, \dots, |V(G)|$ and the weights of subgraphs H_i, $i = 1, 2, \dots, t$,

$$wt_f(H_i) = \sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e)$$

form an arithmetic progression $a, a + d, \dots, a + (t - 1)d$, where $a > 0$ and $d \geq 0$ are two integers, i.e.,

$$\{wt_f(H_i) : i = 1, 2, \dots, t\} = \{a, a + d, \dots, a + (t - 1)d\}. \tag{1}$$

Let us consider the graph $S(G)$ obtained from G by inserting p new vertices, say $v_1, v_2, \dots, v_p$, to the edges of G. Let $S_G(H_i)$, $i = 1, 2, \dots, t$, be all subgraphs of $S(G)$ isomorphic to $S_G(H)$. Then $S(G)$ admits the $S_G(H)$-covering. Let r denote the number of new vertices inserted to every subgraph $S_G(H_i)$, $i = 1, 2, \dots, t$.

We define a labeling g of $S(G)$ in the following way

$$g(v) = \begin{cases} f(v), & \text{if } v \in V(G), \\ |V(G)| + j, & \text{if } v = v_j, j = 1, 2, \dots, p. \end{cases}$$

Evidently, the vertices of $S(G)$ are labeled with distinct numbers $1, 2, \dots, |V(G)| + p$.

Let us choose an orientation of edges in G. According to this orientation we orient the edges in $S(G)$. To an arc uv in G there will correspond the oriented path P_{uv} with initial vertex u and terminal vertex v in $S(G)$. The arcs of $S(G)$ we label such that

$$g(uw) = \begin{cases} f(uv) + p, & \text{if } u \in V(G) \text{ and } uw \text{ is an arc on } P_{uv}, \\ |V(G)| + |E(G)| + 2p + 1 - j, & \text{if } u = v_j, j = 1, 2, \dots, p. \end{cases}$$

The edges are labeled with distinct numbers from the set $|V(G)|+p+1, |V(G)|+p+2, \dots, |V(G)|+|E(G)|+2p$. Now we evaluate the weights of subgraphs $S_G(H_i)$, $i = 1, 2, \dots, t$, under the labeling g. Immediately using the structure of the subgraph $S_G(H_i)$ and the definition of the labeling g we get

$$\begin{aligned} wt_g(S_G(H_i)) &= \sum_{v \in V(G(H_i))} g(v) + \sum_{e \in E(G(H_i))} g(e) \\ &= \sum_{v \in V(H_i)} g(v) + \sum_{v_j \in V(G(H_i))} g(v_j) + \sum_{\substack{e \in E(P(uv)), \\ uv \in E(H_i)}} g(e) \\ &= \sum_{v \in V(H_i)} f(v) + \sum_{v_j \in V(G(H_i))} (|V(G)| + j) + \sum_{e \in E(H_i)} (f(e) + p) \\ &\quad + \sum_{v_j \in V(G(H_i))} (|V(G)| + |E(G)| + 2p + 1 - j) \\ &= \sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e) + |E(H_i)|p + (2|V(G)| + |E(G)| + 2p + 1)r \\ &= wt_f(H_i) + |E(H_i)|p + (2|V(G)| + |E(G)| + 2p + 1)r. \end{aligned}$$

As $|E(H_i)| = |E(H)|$ for $i = 1, 2, \dots, t$ we obtain that the weights of $S_G(H_i)$ depend on the weights of H_i which form an arithmetic sequence with a difference d, see (1). This implies that the set of weights $S_G(H_i)$ also forms an arithmetic sequences with the difference d and the initial term $a + |E(H)|p + (2|V(G)| + |E(G)| + 2p + 1)r$. This concludes the proof. □

Combining Theorem 2.1 with some results on (a, d)-cycle-antimagic graphs we immediately obtain new classes of graphs that are (b, d)-cycle-antimagic. Note, that it is not needed to consider only regular subdivisions of graphs.

3 Subdivided wheels

A wheel W_n is a graph obtained by joining a single vertex to all vertices of a cycle on n vertices. The vertex of degree n is called the central vertex, or the hub vertex, and the remaining vertices are called the rim vertices. The edges adjacent to the central vertex are called spokes and the remaining edges are called rim edges. Let us denote by the symbol $W_n(r, s)$ the graph obtained by inserting r, $r \geq 0$, new vertices to every rim edge and s, $s \geq 0$, new vertices to every spoke in the wheel W_n. Note, that the graph isomorphic to subdivided wheel $W_n(r, 0)$ is also known as the Jahangir graph $J_{n,r+1}$.

In [11] it was proved that wheels are cycle-antimagic.

Theorem 3.1 ([11]). *Let k and $n \geq 3$ be positive integers. The wheel W_n is super $(a,1)$-C_k-antimagic for every $k = 3, 4, \dots, n-1, n+1$.*

Immediately using Theorem 2.1 we obtain that subdivided wheels admit cycle-antimagic labeling with difference 1.

Corollary 3.2. *Let $k, n \geq 3, r \geq 0, s \geq 0$ be integers. The subdivided wheel $W_n(r,s)$ is super $(a,1)$-$C_{k+(k-2)r+2s}$-antimagic for every $k = 3, 4, \dots, n-1, n+1$.*

In the next theorem we will deal with the cycle-antimagicness of the subdivided wheel $W_n(1,1)$. We prove that this graph admits a super (a,d)-C_6-antimagic labeling for $d \in \{0, 1, \dots, 5\}$.

Theorem 3.3. *The subdivided wheel $W_n(1,1)$, $n \geq 3$, is super (a,d)-C_6-antimagic for $d \in \{0, 1, \dots, 5\}$.*

Proof. Let us denote the vertices and edges of $W_n(1,1)$ such that

$$\begin{aligned} V(W_n(1,1)) =& \{c, v_i, u_i, w_i : i = 1, 2, \dots, n\}, \\ E(W_n(1,1)) =& \{cw_i, w_iv_i, v_iu_i, u_iv_{i+1} : i = 1, 2, \dots, n\}, \end{aligned}$$

where the indices are taken modulo n.

For $d = 1$ the result follows from Corollary 3.2. For $d \in \{0, 2, 3, 4, 5\}$ we define a total labeling $g_d : V(W_n(1,1)) \cup E(W_n(1,1)) \to \{1, 2, \dots, 7n+1\}$ in the following way.

$$g_d(c) = 1, \quad \text{for } d = 0, 2, 3, 4, 5,$$

$$g_0(w_i) = \begin{cases} 2, & \text{for } i = 1, \\ n+3-i, & \text{for } 2 \leq i \leq n, \end{cases}$$

$$g_0(u_i) = 3n+2-2i, \quad \text{for } 1 \leq i \leq n,$$

$$g_0(v_i) = \begin{cases} n-1+2i, & \text{for } 2 \leq i \leq n, \\ 3n+1, & \text{for } i = 1, \end{cases}$$

$$g_0(cw_i) = 3n+1+i, \quad \text{for } 1 \leq i \leq n,$$

$$g_0(w_iv_i) = \begin{cases} 5n+1+i, & \text{for } 1 \leq i \leq n-1, \\ 5n+1, & \text{for } i = n, \end{cases}$$

$$g_0(v_iu_i) = \begin{cases} 6n+3-i, & \text{for } 2 \leq i \leq n, \\ 5n+2, & \text{for } i = 1, \end{cases}$$

$$g_0(u_iv_{i+1}) = \begin{cases} 6n+3+i, & \text{for } 1 \leq i \leq n-2, \\ 5n+3+i, & \text{for } n-1 \leq i \leq n, \end{cases}$$

$$g_2(w_i) = 3n+2-i, \quad \text{for } 1 \leq i \leq n,$$

$$g_2(u_i) = 1+2i, \quad \text{for } 1 \leq i \leq n,$$

$$g_2(v_i) = 2i, \quad \text{for } 1 \leq i \leq n,$$

$$g_2(cw_i) = 4n+2-i, \quad \text{for } 1 \leq i \leq n,$$

$$g_2(w_iv_i) = 5n+2-i, \quad \text{for } 1 \leq i \leq n,$$

$$g_2(v_iu_i) = \begin{cases} 5n+2+i, & \text{for } 1 \leq i \leq n-1, \\ 5n+2, & \text{for } i = n, \end{cases}$$

$$g_2(u_iv_{i+1}) = 6n+1+i, \quad \text{for } 1 \leq i \leq n,$$

$$g_3(w_i) = \begin{cases} 2, & \text{for } i = 1, \\ n+3-i, & \text{for } 2 \leq i \leq n, \end{cases}$$

$$g_3(u_i) = 3n + 2 - i, \quad \text{for } 1 \le i \le n,$$

$$g_3(v_i) = \begin{cases} 2n+1, & \text{for } i = 1, \\ n+i, & \text{for } 2 \le i \le n, \end{cases}$$

$$g_3(cw_i) = \begin{cases} 4n+1, & \text{for } i = 1, \\ 3n+i, & \text{for } 2 \le i \le n, \end{cases}$$

$$g_3(w_iv_i) = \begin{cases} 4n+2, & \text{for } i = 1, \\ 5n+3-i, & \text{for } 2 \le i \le n, \end{cases}$$

$$g_3(v_iu_i) = 5n + 2i, \quad \text{for } 1 \le i \le n,$$

$$g_3(u_iv_{i+1}) = 5n + 2i + 1, \quad \text{for } 1 \le i \le n,$$

$$g_4(w_i) = \begin{cases} 2, & \text{for } i = 1, \\ n+3-i, & \text{for } 2 \le i \le n, \end{cases}$$

$$g_4(u_i) = n + 2i, \quad \text{for } 1 \le i \le n,$$

$$g_4(v_i) = \begin{cases} 3n+1, & \text{for } i = 1, \\ n-1+2i, & \text{for } 2 \le i \le n, \end{cases}$$

$$g_4(cw_i) = 3n + 1 + i, \quad \text{for } 1 \le i \le n,$$

$$g_4(w_iv_i) = \begin{cases} 5n+1-i, & \text{for } 1 \le i \le n-1, \\ 5n+1, & \text{for } i = n, \end{cases}$$

$$g_4(v_iu_i) = \begin{cases} 5n+2, & \text{for } i = 1, \\ 6n+3-i, & \text{for } 2 \le i \le n, \end{cases}$$

$$g_4(u_iv_{i+1}) = \begin{cases} 6n+3+i, & \text{for } 1 \le i \le n-2, \\ 5n+3+i, & \text{for } n-1 \le i \le n, \end{cases}$$

$$g_5(w_i) = \begin{cases} 2, & \text{for } i = 1, \\ n+3-i, & \text{for } 2 \le i \le n, \end{cases}$$

$$g_5(v_i) = \begin{cases} 2n+1, & \text{for } i = 1, \\ n+i, & \text{for } 2 \le i \le n, \end{cases}$$

$$g_5(u_i) = 2n + 1 + i, \quad \text{for } 1 \le i \le n,$$

$$g_5(cw_i) = \begin{cases} 3n+i, & \text{for } 2 \le i \le n, \\ 4n+1, & \text{for } i = 1, \end{cases}$$

$$g_5(w_iv_i) = \begin{cases} 4n+2, & \text{for } i = 1, \\ 5n+3-i, & \text{for } 2 \le i \le n, \end{cases}$$

$$g_5(v_iu_i) = 5n + 2i, \quad \text{for } 1 \le i \le n,$$

$$g_5(u_iv_{i+1}) = 5n + 2i + 1, \quad \text{for } 1 \le i \le n.$$

We denote by the symbol C_6^i, $1 \le i \le n$, the 6-cycle such that $C_6^i = cw_iu_iv_iu_{i+1}\, w_{i+1}$, where the index i is taken modulo n. Under the labeling g_d, the weights of C_6^i are as follows.

$$\begin{aligned} wt_g(C_6^i) =& g(c) + g(w_i) + g(u^i) + g(v_i) + g(u_{i+1}) + g(w_{i+1}) + g(cw_i) + g(w_iu_i) \\ &+ g(u_iv_i) + g(v_iu_{i+1}) + g(u_{i+1}w_{i+1}) + g(w_{i+1}c). \end{aligned}$$

It is a simple mathematical exercise to prove that for every i, $1 \le i \le n$, the 6-cycle-weights are:

$$wt_{g_0}(C_6^i) = 35n + 18, \quad \text{for } 1 \le i \le n,$$

$$
\begin{aligned}
wt_{g_2}(C_6^i) &= \begin{cases} 36n+17, & \text{for } i=1,\\ 34n+15+2i, & \text{for } 2\le i\le n,\end{cases}\\
wt_{g_3}(C_6^i) &= 33n+16+3i, \quad \text{for } 1\le i\le n,\\
wt_{g_4}(C_6^i) &= 33n+16+4i, \quad \text{for } 1\le i\le n,\\
wt_{g_5}(C_6^i) &= 32n+15+5i, \quad \text{for } 1\le i\le n.
\end{aligned}
$$

Hence the weights of cycles C_6 form an arithmetic sequence with differences $d = 0, 2, 3, 4, 5$, respectively. This concludes the proof. □

Combining Theorem 2.1 and Theorem 3.3 we immediately obtain the following result.

Theorem 3.4. *The subdivided wheel $W_n(r,s)$, $n \ge 3$, $r \ge 1$ and $s \ge 1$ is super (a,d)-C_{r+2s+3}-antimagic for $d \in \{0,1,2,3,4,5\}$.*

In the next section we will deal with the subdivided wheel $W_n(r,0)$, $n \ge 3$, $r \ge 1$. Let us denote the vertices and the edges of $W_n(r,0)$ such that

$$
\begin{aligned}
V(W_n(r,0)) =& \{c, v_i, u_j^i : 1\le i\le n, 1\le j\le r\}\\
E(W_n(r,0)) =& \{cv_i, v_iu_1^i : 1\le i\le n\} \cup \{u_r^i v_{i+1}\ 1\le i\le n-1\} \cup \{u_r^n v_1\}\\
& \cup \{u_j^i u_{j+1}^i : 1\le i\le n, 1\le j\le r-1\}.
\end{aligned}
$$

The subdivided wheel $W_n(r,0)$, $n \ge 3$, $r \ge 1$, has n vertices of degree 3, nr vertices of degree 2 and one vertex of degree n. The size of $W_n(r,0)$ is $n(r+2)$.

The subdivided wheel $W_n(r,0)$ admits the C_{r+3}-covering consisting of n cycles C_{r+3}. Let us denote these cycles by the symbols C_{r+3}^i, $i = 1,2,\dots,n$, such that $C_{r+3}^i = cv_iu_1^iu_2^i\dots u_r^i\, v_{i+1}$.

The following theorem shows the existence of a super (a,d)-C_{r+3}-antimagic labeling for $W_n(r,0)$ for every odd difference form 1 up to $2r-3$.

Theorem 3.5. *The subdivided wheel $W_n(r,0)$, $n \ge 3$, $r \ge 1$, is super (a,d)-C_{r+3}-antimagic for $d = 1$ when $r = 1$ and for $d \equiv 1 \pmod 2$, $1 \le d \le 2r-3$ when $r \ge 1$.*

Proof. For $r = 1$ the result follows from Corollary 3.2. Let $r \ge 2$ and let d be an odd positive integer, $1 \le d \le 2r-3$. Let $f_d : V(W_n(r,0)) \cup E(W_n(r,0)) \to \{1,2,\dots,n(2r+3)+1\}$ be a labeling of $W_n(r,0)$, $n \ge 3$, $r \ge 2$, defined in the following way.

$$
\begin{aligned}
f_d(c) =& 1,\\
f_d(v_i) =& 1+i, && \text{for } 1\le i\le n,\\
f_d(u_j^i) =& jn+1+i, && \text{for } 1\le i\le n, 1\le j\le r,\\
f_d(u_r^n v_1) =& 2nr+3n+1,\\
f_d(u_r^i v_{i+1}) =& 2nr+2n+1+i, && \text{for } 1\le i\le n-1,\\
f_d(u_j^i u_{j+1}^i) =& nr+3n+jn+2-i, && \text{for } 1\le i\le n, 1\le j\le r-(d+1)/2,\\
f_d(u_j^i u_{j+1}^i) =& nr+2n+1+i+jn, && \text{for } 1\le i\le n, r-(d-1)/2\le j\le r-1,\\
f_d(v_iu_1^i) =& nr+3n+2-i, && \text{for } 1\le i\le n,\\
f_d(cv_i) =& nr+2n+2-i, && \text{for } 1\le i\le n.
\end{aligned}
$$

It is easy to see that f_d is a bijection as

$$
\begin{aligned}
&f_d(c) = 1,\\
&\{f_d(v_i) : 1\le i\le n\} = \{2,3,\dots,n+1\},
\end{aligned}
$$

$$
\begin{aligned}
\{f_d(u_j^i) : 1 \le i \le n,\ 1 \le j \le r\} &= \{n+2, n+3, \dots, nr+n+1\},\\
f_d(u_r^n v_1) &= 2nr+3n+1,\\
\{f_d(u_r^i v_{i+1}) : 1 \le i \le n-1\} &= \{2nr+2n+2, 2nr+2n+3, \dots, 2nr+3n\},\\
\{f_d(u_j^i u_{j+1}^i) : 1 \le i \le n,\ 1 \le j \le r-(d+1)/2\}7 &= \{nr+3n+2, nr+3n+3,\\
&\quad \dots, 2nr+3n-n(d-1)/2+1\},\\
\{f_d(u_j^i u_{j+1}^i) : 1 \le i \le n,\ r-(d-1)/2 \le j \le r-1\} &= \{2nr+3n-n(d-1)/2\\
&\quad +2, 2nr+3n-n(d-1)/2+3, \dots, 2nr+2n+1\},\\
\{f_d(v_i u_1^i) : 1 \le i \le n\} &= \{nr+2n+2, nr+2n+3, \dots, nr+3n+1\},\\
\{f_d(cv_i) : 1 \le i \le n\} &= \{nr+n+2, nr+n+3, \dots, nr+2n+1\}.
\end{aligned}
$$

Under the labeling f_d the weights of cycles C_{r+3}^i, $i = 1, 2, \dots, n-1$, are as follows.

$$
\begin{aligned}
wt_{f_d}(C_{r+3}^i) = \sum_{v \in V(C_{r+3}^i)} f_d(v) + \sum_{e \in E(C_{r+3}^i)} f_d(e) &= f_d(c) + f_d(v_i) + f_d(v_{i+1})\\
&+ \sum_{j=1}^{r} f_d(u_j^i) + f_d(cv_i) + f_d(cv_{i+1}) + f_d(v_i u_1^i) + \sum_{j=1}^{r-1} f_d(u_j^i u_{j+1}^i)\\
&+ f_d(u_r^i v_{i+1}) = 1 + (1+i) + \big(1+(i+1)\big) + \sum_{j=1}^{r}\big(jn+1+i\big)\\
&+ \big(nr+2n+2-i\big) + \big(nr+2n+2-(i+1)\big) + \big(nr+3n+2-i\big)\\
&+ \sum_{j=1}^{r-(d+1)/2} \big(nr+3n+jn+2-i\big)\\
&+ \sum_{j=r-(d-1)/2}^{r-1} \big(nr+2n+1+i+jn\big) + \big(2nr+2n+1+i\big)\\
&= 2nr^2 + 7nr + 7n + 3r + 9 - \tfrac{(d+1)(n+1)}{2} + di.
\end{aligned}
$$

Moreover, the weight of the cycle C_{r+3}^n we get

$$
\begin{aligned}
wt_{f_d}(C_{r+3}^n) = \sum_{v \in V(C_{r+3}^n)} f_d(v) + \sum_{e \in E(C_{r+3}^n)} f_d(e) &= f_d(c) + f_d(v_n) + f_d(v_1)\\
&+ \sum_{j=1}^{r} f_d(u_j^n) + f_d(cv_n) + f_d(cv_1) + f_d(v_n u_1^n) + \sum_{j=1}^{r-1} f_d(u_j^n u_{j+1}^n)\\
&+ f_d(u_r^n v_1) = 1 + (1+n) + 2 + \sum_{j=1}^{r}\big(jn+1+n\big) + \big(nr+n+2\big)\\
&+ \big(nr+2n+1\big) + \big(nr+2n+2\big) + \sum_{j=1}^{r-(d+1)/2} \big(nr+2n+jn+2\big)\\
&+ \sum_{j=r-(d-1)/2}^{r-1} \big(nr+3n+1+jn\big) + \big(2nr+3n+1\big)\\
&= 2nr^2 + 7nr + 6n + 3r + 9 - \tfrac{(d+1)(n+1)}{2}.
\end{aligned}
$$

This proves that f_d is a super (a, d)-C_{r+3}-antimagic labeling of $W_n(r, 0)$ for $d \equiv 1 \pmod 2$, $1 \le d \le 2r-3$ and $a = 2nr^2 + 7nr + 6n + 3r + 9 - (d+1)(n+1)/2$. □

In the next theorem we prove that the graph $W_n(r, 0)$ admits super (a, d)-C_{r+3}-antimagic labelings also for even differences.

Theorem 3.6. *The subdivided wheel $W_n(r, 0)$, $r \ge 1$, is super (a, d)-C_{r+3}-antimagic for $d = 0$ when $r = 1$, $n \ge 5$ and for $d \equiv 0 \pmod 2$, $0 \le d \le 2r-4$ when $r \ge 2$, $n \ge 3$.*

Proof. Lladó and Moragas [4] proved that the wheel W_n, $n \geq 5$ odd, is $(a,0)$-C_3-antimagic. From Corollary 3.2 we obtain that $W_n(r,0)$, $n \geq 5$, $r \geq 1$, is super $(b,0)$-C_{r+3}-antimagic.

Let $r \geq 2$, $n \geq 3$ be positive integers. Let d be an even integer, $0 \leq d \leq 2r-4$. Let $f_d : V(W_n(r,0)) \cup E(W_n(r,0)) \to \{1,2,\dots,n(2r+3)+1\}$ be a labeling of $W_n(r,0)$, $n \geq 3$, $r \geq 1$, defined in the following way.

$$\begin{aligned}
g_d(c) &= 1, && \\
g_d(v_i) &= 2i, && \text{for } 1 \leq i \leq n,\\
g_d(u_1^i) &= 2n-2i+3, && \text{for } 1 \leq i \leq n,\\
g_d(u_j^i) &= jn+1+i, && \text{for } 1 \leq i \leq n,\ 2 \leq j \leq r,\\
g_d(u_r^n v_1) &= nr+n+2, && \\
g_d(u_r^i v_{i+1}) &= nr+2n+2-i, && \text{for } 1 \leq i \leq n-1,\\
g_d(u_j^i u_{j+1}^i) &= 2nr+n+1+i-jn, && \text{for } 1 \leq i \leq n,\ 1 \leq j \leq (1+d/2),\\
g_d(u_j^i u_{j+1}^i) &= 2nr+2n-jn+2-i, && \text{for } 1 \leq i \leq n,\ (2+d/2) \leq j \leq r-1,\\
g_d(v_i u_1^i) &= 2nr+2n+1-i, && \text{for } 1 \leq i \leq n-1,\\
g_d(v_n u_1^n) &= 2nr+2n+1, && \\
g_d(cv_i) &= 2nr+3n+2-i, && \text{for } 1 \leq i \leq n.
\end{aligned}$$

The labeling g_d is a bijection. Under the labeling g_d the weights of cycles C_{r+3}^i, $i = 1,2,\dots,n-1$, are the following.

$$\begin{aligned}
wt_{g_d}(C_{r+3}^i) = &\sum_{v \in V(C_{r+3}^i)} g_d(v) + \sum_{e \in E(C_{r+3}^i)} g_d(e) = g_d(c) + g_d(v_i) + g_d(v_{i+1})\\
&+ \sum_{j=1}^{r} g_d(u_j^i) + g_d(cv_i) + g_d(cv_{i+1}) + g_d(v_i u_1^i) + \sum_{j=1}^{r-1} g_d(u_j^i u_{j+1}^i)\\
&+ g_d(u_r^i v_{i+1}) = 1 + 2i + 2(i+1) + (2n-2i+3)\\
&+ \sum_{j=2}^{r} \big(jn+1+i\big) + \big(2nr+3n+2-i\big) + \big(2nr+3n+2-(i+1)\big)\\
&+ \big(2nr+2n+1-i\big) + \sum_{j=1}^{1+d/2} \big(2nr+n+1+i-jn\big)\\
&+ \sum_{j=2+d/2}^{r-1} \big(2nr+2n-jn+2-i\big) + \big(nr+2n+2-i\big)\\
= &2nr^2 + 8nr + 8n + 3r + 8 - \frac{d(n+1)}{2} + di.
\end{aligned}$$

For the weight of the cycle C_{r+3}^n we obtain:

$$\begin{aligned}
wt_{g_d}(C_{r+3}^n) = &\sum_{v \in V(C_{r+3}^n)} g_d(v) + \sum_{e \in E(C_{r+3}^n)} g_d(e) = g_d(c) + g_d(v_n) + g_d(v_1)\\
&+ \sum_{j=1}^{r} g_d(u_j^n) + g_d(cv_n) + g_d(cv_1) + g_d(v_n u_1^n) + \sum_{j=1}^{r-1} g_d(u_j^n u_{j+1}^n)\\
&+ g_d(u_r^n v_1) = 1 + 2n + 2 + 3 + \sum_{j=2}^{r} \big(jn+1+n\big) + \big(2nr+2n+2\big)\\
&+ \big(2nr+3n+1\big) + \big(2nr+2n+1\big) + \sum_{j=1}^{1+d/2} \big(2nr+2n+1-jn\big)\\
&+ \sum_{j=2+d/2}^{r-1} \big(2nr+n+2-jn\big) + \big(nr+n+2\big)\\
= &2nr^2 + 8nr + 8n + 3r + 8 + \frac{nd}{2} - \frac{d}{2}.
\end{aligned}$$

We showed that g_d is a super (a, d)-C_{r+3}-antimagic labeling of $W_n(r, 0)$ for $d \equiv 0 \pmod 2$, $0 \leq d \leq 2r - 4$ and $a = 2nr^2 + 8nr + 8n + 3r + 8 - dn/2 + d/2$. □

Combining Theorem 3.5 and Theorem 3.6 we immediately obtain that the subdivided wheel $W_n(r, 0)$, $n \geq 5$, is cycle-antimagic for wide range of differences.

Theorem 3.7. *The subdivided wheel $W_n(r, 0)$, $n \geq 5$, is super (a, d)-C_{r+3}-antimagic for $0 \leq d \leq 1$ when $r = 1$ and for $0 \leq d \leq 2r - 3$ when $r \geq 2$.*

Moreover, using Theorem 2.1, we can extend this result also for subdivided wheels in which not only rim edges but also spokes are subdivided.

Theorem 3.8. *The subdivided wheel $W_n(r, s)$, $n \geq 5$, $r \geq 1$, $s \geq 0$, is super (a, d)-C_{r+2s+3}-antimagic for $0 \leq d \leq 1$ when $r = 1$ and for $0 \leq d \leq 2r - 3$ when $r \geq 2$.*

4 Conclusion

In the present paper we showed that the property to be super (a, d)-H-antimagic is hereditary according to the operation of subdivision of edges. We proved that if a graph G is super cycle-antimagic then the subdivided graph $S(G)$ also admits a super cycle-antimagic labeling.

This indicates that it is important to study the antimagic properties of graphs with simple structures which allows us to get result for large graphs. Recently, large graphs have attracted a lot of attention, see [15]. However, the interesting question is whether, for a given graph, it is possible to extend the set of differences also for cases not covered by the general result. It means to find a difference d such that the subdivided graph $S(G)$ is super cycle-antimagic with the difference d but the corresponding graph G is not.

Another interesting directions for further investigation is to deal with the non-uniform subdivision and to find another graph operations that are hereditary according to being cycle-antimagic, or in general H-antimagic.

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