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Directed colimits of some flatness properties and purity of epimorphisms in S-posets

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Abstract: Let S be a pomonoid. In this paper, we introduce some new types of epimorphisms with certain purity conditions, and obtain equivalent descriptions of various flatness properties of S-posets, such as strong flatness, Conditions (E), (E$'$), (P), (P$_w$), (WP), (WP)$_w$, (PWP) and (PWP)$_w$. Thereby, we present other equivalent conditions in the Stenström-Govorov-Lazard theorem for S-posets. Furthermore, we prove that these new epimorphisms are closed under directed colimits. Meantime, this implies that by a new approach we can show that most of flatness properties of S-posets can be transferred to their directed colimit. Finally, we prove that every class of S-posets having a flatness property is closed under directed colimits.

Keywords: S-poset, Flatness property, Pure epimorphism, Directed colimit

MSC: 06F05, 20M50

1 Introduction and preliminaries

Motivated by the work of Lazard and Govorov for modules over a ring, Stenström in 1971 introduced the concept of pure epimorphisms and established the Stenström-Govorov-Lazard theorem in the context of S-acts (see [1]). Based on the method of this theorem, Normak then in [2], used 1-pure epimorphisms to obtain some equivalent descriptions of Condition (E). Recently, Bailey and Renshaw in [3] continued the investigation of purity of epimorphisms. They found that there are some connections between some flatness properties of S-acts and purity of epimorphisms (for example, [3, Propositions 3.11, 3.12]). They also proved that every surjective S-act morphism is pure if and only if it is a directed colimit of split epimorphisms. Moreover, in [4] the authors proved tha every class of S-acts having a flatness property is closed under directed colimits.

In 2005, Bulman-Fleming and Laan [5] introduced the ordered analogues of pure epimorphisms and directed colimits, and extended the Stenström-Govorov-Lazard theorem to S-posets. From the Stenström-Govorov-Lazard theorem for S-posets, we see that a right S-poset A_S is strongly flat, which means that A_S has both Condition (P) and Condition (E), if and only if every surjective S-poset morphism $B_S \to A_S$ is pure. During recent years, a number of papers on flatness properties of S-posets have appeared, but vast majority of them have focused on the homological classification of pomonoids (see, for example, [6-9]). Up to now, the research on purity conditions of epimorphisms which are used to characterize flatness properties of S-posets (with the exception of strong flatness), has not been completed. The present paper addresses some versions of this matter. As applications of the results in this paper, some results on S-acts can be also obtained.

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Preliminary work on flatness properties of S-posets was done by Fakhruddin in the 1980s (see [10, 11]). For background material on S-posets, the reader may consult [6, 12, 13] and the references cited therein.

Let S be a pomonoid. A nonempty poset A is called a *right S-poset* if there exists a right action $A \times S \to A$, $(a, s) \mapsto as$, which satisfies that (1) the action is monotonic in each of the variables, and (2) $a(st) = (as)t$ and $a1 = a$ for all $a \in A$ and $s, t \in S$. Left S-posets are defined analogously. The notations A_S and ${}_SB$ will respectively denote a right and left S-poset, and $\Theta_S = \{\theta\}$ is the one-element right S-poset. By an *S-poset morphism*, we mean a monotone map between S-posets which preserves the S-action. We denote the category of all right (resp., left) S-posets, with S-poset morphisms between them, by Pos-S (resp., S-Pos).

Let A_S be a right S-poset. An *S-poset congruence* θ *on* A_S is an S-act congruence that has the further property that the factor act A/θ can be equipped with a compatible order so that the natural map $A \to A/\theta$ is an S-poset morphism. Because for a given congruence θ the factor S-act A/θ may support several different compatible orders, Bulman-Fleming and Laan in [5] provided a detailed treatment to the factor S-posets A/θ, the essential part was repeated as follows:

Suppose that α is any binary relation on A_S that is reflexive, transitive, and compatible with the S-action. For $a, a' \in A$, we write $a \underset{\alpha}{\leq} a'$ if a so-called *α-chain*

$$a \leq a_1 \,\alpha\, a'_1 \leq a_2 \,\alpha\, a'_2 \leq \cdots \leq a_n \,\alpha\, a'_n \leq a'$$

from a to a' exists in A_S, where each $a_i, a'_i \in A$ for $1 \leq i \leq n$. It can be shown that an S-act congruence θ on an S-poset A_S is an S-poset congruence if and only if $a \,\theta\, a'$ whenever $a \underset{\theta}{\leq} a' \underset{\theta}{\leq} a$.

The next concept will be used frequently in this paper. Let A_S be a right S-poset and let $H \subseteq A \times A$. The *S-poset congruence on A_S induced by H* is described as follows:

$$a \,\nu(H)\, a' \text{ if and only if } a \underset{\alpha(H)}{\leq} a' \underset{\alpha(H)}{\leq} a,$$

where $a \,\alpha(H)\, a'$ if and only if $a = a'$ or

$$a = x_1 s_1, \;\; y_1 s_1 = x_2 s_2, \;\; \cdots, \;\; y_{n-1} s_{n-1} = x_n s_n, \;\; y_n s_n = a'$$

for some $(x_i, y_i) \in H$ and $s_i \in S$, $i = 1, \cdots, n$. The order relation on $A/\nu(H)$ is given by

$$[a]_{\nu(H)} \leq [a']_{\nu(H)} \text{ if and only if } a \underset{\alpha(H)}{\leq} a'.$$

One can show that $\nu(H)$ is the smallest congruence on A_S such that $[x]_{\nu(H)} \leq [x']_{\nu(H)}$ whenever $(x, x') \in H$. The *S-poset congruence on A_S generated by H* is $\theta(H) = \nu(H \cup H^{op})$, and as usual is the smallest congruence on A_S that contains H.

A right S-poset A_S is called *cyclic* if $A = aS$ for some $a \in A$. It is clear that A_S is cyclic if and only if A_S is isomorphic to S/ρ for some right S-poset congruence ρ on S_S. A congruence ρ on an S-poset A_S is called *finitely generated* if $\rho = \nu(H)$ for some finite subset H of $A \times A$. An S-poset A_S is called *finitely presented* if it is isomorphic to $F/\nu(H)$ for some finitely generated free S-poset F_S and some finite subset $H \subseteq F \times F$.

Conditions (E), (P) and (P_w) are formulated below (see for example [5] and [9]).

$$\begin{aligned}
&(\mathrm{E}) : (\forall a \in A)(\forall s, s' \in S)\\
&\qquad (as \leq as' \Longrightarrow (\exists a' \in A)(\exists u \in S)(a = a'u \wedge us \leq us')).\\
&(\mathrm{P}) : (\forall a, a' \in A)(\forall s, s' \in S)\\
&\qquad (as \leq a's' \Longrightarrow (\exists a'' \in A)(\exists u, v \in S)(a = a''u \wedge a' = a''v \wedge us \leq vs')).\\
&(\mathrm{P}_w) : (\forall a, a' \in A)(\forall s, s' \in S)\\
&\qquad (as \leq a's' \Longrightarrow (\exists a'' \in A)(\exists u, v \in S)(a \leq a''u \wedge a''v \leq a' \wedge us \leq vs')).
\end{aligned}$$

It is clear that Condition (P) implies Condition (P_w). According to [9], an S-poset A_S is said to be *strongly flat* if it satisfies Conditions (E) and (P). Equivalently, an S-poset A_S is strongly flat if and only if it is subpullback flat and subequalizer flat [that is, the functor $A_S \otimes -$ (from the category of left S-posets to the category of posets) preserves subpullbacks and subequalizers].

Weak and principal weak versions of Conditions (P) and (P_w) are defined in [7] as follows. An S-poset A_S is said to *satisfy Condition* (WP) if the corresponding ϕ is surjective for every subpullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S. An S-poset A_S is said to *satisfy Condition* $(\mathrm{WP})_w$ if, for all elements $s, t \in S$, all homomorphisms $f \colon {}_S(Ss \cup St) \to {}_SS$, and all $a, a' \in A$, $af(s) \le a'f(t)$ implies $a \otimes s \le a'' \otimes us'$ and $a'' \otimes vt' \le a' \otimes t$ in $A \otimes_S (Ss \cup St)$ for some $a'' \in A$, $u, v \in S$ and $s', t' \in \{s, t\}$ with $f(us') \le f(vt')$. An S-poset A_S is said to *satisfy Condition* (PWP) if the corresponding ϕ is surjective for every subpullback diagram $P(Ss, Ss, f, f, S)$, $s \in S$. An S-poset A_S is said to *satisfy Condition* $(\mathrm{PWP})_w$ if, for all $a, a' \in A$, $s \in S$, $as \le a's$ implies $a \le a''u$, $a''v \le a'$ for some $a'' \in A$, $u, v \in S$ with $us \le vs$.

The definitions of free and projective S-posets can be found, for example, in [9, 14].

By [7] and [9], the relations of the above-mentioned properties are as follows.

$$\begin{array}{ccccccccccc}
\text{free} & \Rightarrow & \text{projective} & \Rightarrow & \text{strongly flat} & \Rightarrow & (\mathrm{P}) & \Rightarrow & (\mathrm{WP}) & \Rightarrow & (\mathrm{PWP}) \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & & & & & (\mathrm{P_w}) & \Rightarrow & (\mathrm{WP})_{\mathrm{w}} & \Rightarrow & (\mathrm{PWP})_{\mathrm{w}}
\end{array}$$

The rest of this paper is organized as follows. In Section 2, we first introduce the concept of 1-pure epimorphisms for S-posets, and obtain an alternative description of Condition (E) which is the ordered version of [2, Proposition 2.2]. In particular, we can give a more brief characterization for Condition (E) by using a new purity epimorphism. Further, characterizations for Condition (E) may serve as a template for Condition (E'), as we show that Condition (E') is equivalent to certain purity conditions of epimorphisms. We also consider similar questions to Conditions (P), (P_w), (WP), $(\mathrm{WP})_w$, (PWP) and $(\mathrm{PWP})_w$. Thereby, we obtain other equivalent conditions in the Stenström-Govorov-Lazard theorem for S-posets. In Section 3, we initiate a study of directed colimits of purity epimorphisms for S-posets, and prove that these new epimorphisms introduced in Section 2 are closed under directed colimits. Then, we deduce that an S-poset epimorphism is pure if and only if it is a directed colimit of split epimorphisms. In Section 4, we consider the behavior of subequlizers and subpullback diagrams under directed colimits. Finally, we show that flatness properties transferred under directed colimits.

2 Purity of epimorphisms for S-posets

In this section, we give equivalent descriptions of some flatness properties of S-posets in accordance with certain purity conditions of epimorphisms.

We first recall the concept of pure epimorphisms for S-posets.

Let $\psi : B_S \to A_S$ be a surjective S-poset morphism. We say that ψ is *a pure (m-pure) epimorphism* [5] if, for every $m \in \mathbb{N}$, $a_1, \cdots, a_m \in A$ and relations

$$a_{\alpha_i} s_i \le a_{\beta_i} t_i \quad (i = 1, 2, \cdots, l),$$

where $\alpha_i, \beta_i \in \{1, \cdots, m\}$ and $s_i, t_i \in S$, there exist $b_1, \cdots, b_m \in B$ such that $\psi(b_r) = a_r$ for all $1 \le r \le m$ and $b_{\alpha_i} s_i \le b_{\beta_i} t_i$ for all i.

We are interested here in the cases $m = 1$ and $m = 2$. In both cases, we shall call ψ *1-pure* and *2-pure*, respectively. It is easy to check that every 2-pure epimorphism is 1-pure, but the following example shows that the converse is not true in general.

Example 2.1. *Let* $S = \{0, 1\}$ *with the usual order. Let*

$$A = \{x, y, z \mid x \cdot 0 = y \cdot 0 = z \cdot 0 = z,\ x \cdot 1 = x,\ y \cdot 1 = y,\ z \cdot 1 = z\}$$

with the only nontrivial order relation is $z < x$ *and* $z < y$, *and*

$$B = \{a, b, c \mid a \cdot 0 = a \cdot 1 = a,\ b \cdot 0 = b \cdot 1 = b,\ c \cdot 0 = c \cdot 1 = c\}$$

with the discrete order. Define an S-poset epimorphism $\psi : B_S \to A_S$ by

$$\psi(a) = x,\ \ \psi(b) = y,\ \ \psi(c) = z.$$

It is not hard to verify that ψ is 1-pure. However, it is not 2-pure, since $x \cdot 0 \le y \cdot 1$ but $a \cdot 0 \nleq b \cdot 1$.

We begin our investigation with Condition (E). The following proposition is an ordered analogue of [2, Proposition 2.2]. The technique of the proof is taken from the unordered case.

Proposition 2.2. *Let A_S be a right S-poset. Then the following assertions are equivalent.*
(1) *A_S has Condition* (E).
(2) *Every surjective S-poset morphism $B_S \to A_S$ is 1-pure.*
(3) *There exists a 1-pure epimorphism $B_S \to A_S$ where B_S is subequalizer flat.*
(4) *Every S-poset morphism $B_S \to A_S$ where B_S is a finitely presented cyclic S-poset may be factorized through a free S-poset.*
(5) *Every S-poset morphism $B_S \to A_S$ where B_S is a finitely presented cyclic S-poset may be factorized through a subequalizer flat S-poset.*

The next proposition gives another description of Condition (E), whose formulation is quite brief. To do this, we require the following.

Definition 2.3. *Let $\psi : B_S \to A_S$ be a surjective S-poset morphism. We say that φ is quasi-1-pure if, for every $a \in A$ and $as \le at$, there exists $b \in B$ such that $\psi(b) = a$ and $bs \le bt$.*

Proposition 2.4. *Let A_S be a right S-poset. Then the following assertions are equivalent.*
(1) *A_S has Condition* (E).
(2) *Every surjective S-poset morphism $B_S \to A_S$ is quasi-1-pure.*
(3) *There exists a quasi-1-pure epimorphism $B_S \to A_S$ where B_S has Condition* (E).

Proof. $(1) \Rightarrow (2)$. Suppose that $\psi : B_S \to A_S$ is an S-poset epimorphism. Let $as \le at$ in A_S for $a \in A$ and $s, t \in S$. Since A_S satisfies Condition (E), there exist $u \in S$ and $a' \in A$ such that $a = a'u$ and $us \le ut$. Applying surjectivity of ψ, obtains $b \in B$ with $\psi(b) = a'$. Further, we have $bus \le but$ in B_S, and $a = a'u = \psi(bu)$, as required.

$(2) \Rightarrow (3)$. From [5, Proposition 2.4] we know that every S-poset is isomorphic to the quotient of a free S-poset. So, without loss of generality, we can always choose an S-poset B_S such that $\psi : B_S \to A_S$ is an S-poset epimorphism and B_S satisfies Condition (E). Then by (2), ψ is quasi-1-pure.

$(3) \Rightarrow (1)$. Let A_S be a right S-poset. Then by (3) there exists a quasi-1-pure epimorphism $\psi : B_S \to A_S$ with B_S satisfies Condition (E). Now assume $as \le at$ in A_S for $a \in A$, $s, t \in S$. Then there exists $b' \in B$ such that $b's \le b't$ in B_S, and $\psi(b') = a$. Further, because B_S satisfies Condition (E), from the inequality $b's \le b't$, yields $b \in B$ and $u \in S$ such that $b' = bu$ and $us \le ut$. Consequently, $a = \psi(b') = \psi(bu) = \psi(b)u$ in A_S, and the proof is complete. □

Similar considerations apply to a generalization of Condition (E) for S-posets.

Definition 2.5. *A right S-poset A_S* satisfies Condition (E′) *if, for all $a \in A$, $s, s', z \in S$ with $as \le as'$ and $sz = s'z$, there exist $a' \in A$, $u \in S$ such that $a = a'u$ and $us \le us'$.*

It is immediate from the above definition that Condition (E) implies Condition (E′), but we will see in the sequel that the converse is not true in general.

Definition 2.6. *Let $\psi : B_S \to A_S$ be a surjective S-poset morphism. We say ψ is*
1. 1′*-pure if, for every $a \in A$ and relations*

$$as_i \le at_i \quad \text{and} \quad s_i z_i = t_i z_i \quad (i = 1, \cdots, n),$$

there exists $b \in B$ such that $\psi(b) = a$ and $bs_i \leq bt_i$ for all i;
2. *quasi-$1'$-pure if, for every $a \in A$ and relations*

$$as \leq at \quad \text{and} \quad sz = tz,$$

there exists $b \in B$ such that $\psi(b) = a$ and $bs \leq bt$.

The following lemma is useful to characterize Condition (E').

Lemma 2.7. *A surjective S-poset morphism $\psi : B_S \to A_S$ is $1'$-pure if and only if for every cyclic right S-poset S/ρ and every S-poset morphism $\alpha : S/\rho \to A_S$, where $\rho = \nu(H)$ for some set H of the form*

$$H = \{(s_i, t_i) \in S \times S \mid s_i z_i = t_i z_i \text{ for some } z_i \in S,\ i = 1, 2, \cdots, n\},$$

there exists an S-poset morphism $\beta : S/\rho \to B_S$ such that the diagram

$$\begin{tikzcd} B \arrow[rr, "\psi"] & & A \\ & S/\rho \arrow[ul, "\beta"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutes.

Proof. **Necessity**. Suppose that $\psi : B_S \to A_S$ is a $1'$-pure epimorphism and suppose that $\alpha : S/\rho \to A_S$ is an S-poset morphism, where ρ is an S-poset congruence on S_S induced by a set of the form $\{(s_i, t_i) \in S \times S \mid s_i z_i = t_i z_i$ for some $z_i \in S,\ i = 1, 2, \cdots, n\}$. Then we have $\alpha([1])s_i \leq \alpha([1])t_i$ in A_S and $s_i z_i = t_i z_i$ for all i. By the assumption, there is an element $b \in B$ such that $\alpha([1]) = \psi(b)$ and $bs_i \leq bt_i$ for all i. Now we define a mapping $\beta : S/\rho \to B_S$ by $\beta([s]) = bs$. It is easy to see that β is a well-defined S-poset morphism such that $\alpha = \psi\beta$.

Sufficiency. Let $\psi : B_S \to A_S$ be an S-poset epimorphism and let $as_i \leq at_i$ and $s_i z_i = t_i z_i$ for $a \in A$, $s_i, t_i, z_i \in S$, $i = 1, 2, \cdots, n$. Consider the S-poset congruence ρ on S_S induced by the set $\{(s_i, t_i) | i = 1, 2, \cdots, n\}$. Then $\alpha : S/\rho \to A_S$, given by $\alpha([s]) = as$ for $s \in S$, is a well-defined S-poset morphism. By the assumption, there exists an S-poset morphism $\beta : S/\rho \to B_S$ such that $\alpha = \psi\beta$. Thus, we can deduce $a = \alpha([1]) = \psi(\beta([1]))$, and $\beta([1])s_i = \beta([s_i]) \leq \beta([t_i]) = \beta([1])t_i$ in B_S for all i, exactly as needed. □

For the sake of completeness we now prove the following

Proposition 2.8. *For any right S-poset A_S, the following statements are equivalent.*
(1) *A_S has Condition (E').*
(2) *Every surjective S-poset morphism $B_S \to A_S$ is $1'$-pure.*
(3) *There exists a $1'$-pure epimorphism $B_S \to A_S$, where B_S has Condition (E').*
(4) *Every S-poset morphism $S/\rho \to A_S$ may be factorized through a free right S-poset, where ρ is as in Lemma 2.7.*
(5) *Every surjective S-poset morphism $B_S \to A_S$ is quasi-$1'$-pure.*
(6) *There exists a quasi-$1'$-pure epimorphism $B_S \to A_S$, where B_S has Condition (E').*

Proof. (1) ⇒ (2). Let, first, $\psi : B_S \to A_S$ be a surjective S-poset morphism, and let $as_i \leq at_i$ and $s_i z_i = t_i z_i$ for some $a \in A$ and $s_i, t_i, z_i \in S$, $i = 1, 2, \cdots, n$. We start by applying Condition (E') for $i = 1$, and obtain $a_1 \in A$ and $u_1 \in S$ with $a = a_1 u_1$ and $u_1 s_1 \leq u_1 t_1$. We substitute the expression $a = a_1 u_1$ into the relation $as_2 \leq at_2$ (for $i = 2$), and get $a_1 u_1 s_2 \leq a_1 u_1 t_2$ and $u_1 s_2 z_2 = u_1 t_2 z_2$. Again using Condition (E'), yields $a_2 \in A$ and $u_2 \in S$ such that $a_1 = a_2 u_2$ and $u_2 u_1 s_2 \leq u_2 u_1 t_2$, and so we have $a = a_2 u_2 u_1$. Continuing in this way, we end up with expressions $a = a_n u_n \cdots u_1$ and $(u_n \cdots u_1)s_n \leq (u_n \cdots u_1)t_n$ (for $i = n$). Denoting $u = u_n \cdots u_1$, we have $a = a_n u$ and $us_i \leq ut_i$ for all i. Applying surjectivity of ψ, there exists $b \in B$ with $\psi(b) = a_n$. Therefore, $a = a_n u = \psi(bu)$, and $bus_i \leq but_i$ in B_S for all i, as desired.

The implications $(2) \Rightarrow (3)$ and $(5) \Rightarrow (6)$ follow from [5, Proposition 2.4].

The implications $(2) \Rightarrow (5)$ and $(3) \Rightarrow (6)$ are all clear.

$(3) \Rightarrow (4)$. Suppose that $\alpha : S/\rho \to A_S$ is an S-poset morphism, where $\rho = \nu(H)$, for some set $H = \{(s_i, t_i) \in S \times S \mid s_i z_i = t_i z_i$ for some $z_i \in S,\ i = 1, 2, \cdots, n\}$. By (3) there exists a $1'$-pure epimorphism $\psi : B_S \to A_S$, where B_S has Condition (E$'$). In view of Lemma 2.7, there exists an S-poset morphism $\beta : S/\rho \to B_S$ such that $\alpha = \psi\beta$. Then we see $\beta([1])s_i \leq \beta([1])t_i$ in B_S and $s_i z_i = t_i z_i$ for all i. Also, as B_S satisfies Condition (E$'$), there exist $b \in B$ and $u \in S$ with $\beta([1]) = bu$ and $us_i \leq ut_i$ for all i. So the mappings $\phi : S/\rho \to S_S$ and $\varphi : S_S \to A_S$, defined by $\phi([s]) = us$ and $\varphi(s) = \psi(bs)$, respectively, are S-poset morphisms such that $\alpha = \varphi\phi$, as required.

$(4) \Rightarrow (1)$. Let $as \leq at$ and $sz = tz$ for $a \in A$ and $s, t, z \in S$. We consider the S-poset congruence ρ on S_S induced by the pair (s, t). Then the mapping $\alpha : S/\rho \to A_S$, defined by $\alpha([u]) = au$, is a well-defined S-poset morphism. By assumption there are a free S-poset B_S and two S-poset morphisms $\beta : S/\rho \to B_S$ and $\psi : B_S \to A_S$ such that $\alpha = \psi\beta$. Then $[s] \leq [t]$ implies $\beta([1])s \leq \beta([1])t$ in B_S. Since B_S is free, but it must also be Condition (E$'$), from the last inequality and $sz = tz$ we obtain $b \in B$ and $v \in S$ with $\beta([1]) = bv$ and $vs \leq vt$. Therefore, we can calculate $\psi(b)v = \psi(bv) = \psi\beta([1]) = \alpha([1]) = a$, as required.

$(6) \Rightarrow (1)$. It is similar to the implication $(3) \Rightarrow (1)$ of Proposition 2.4. □

As mentioned before, Condition (E$'$) does not imply Condition (E). The example below is illustrative of the fact.

Example 2.9. *Let $S = \{1, x \mid x^2 = 1\}$ with the order of S to be discrete. Clearly, S is a pomonoid. Now consider an S-poset epimorphism $\psi : S_S \to \Theta_S$. It is not hard to verify that ψ is quasi-$1'$-pure, and so Θ_S satisfies Condition* (E$'$) *by Proposition 2.8. However, ψ is not quasi-1-pure, because $\theta \cdot 1 \leq \theta \cdot x$, but there cannot exist $s \in S$ such that $\psi(s) = \theta$ and $s \cdot 1 \leq s \cdot x$. It follows from Proposition 2.4 that Θ_S does not satisfy Condition* (E).

As we known that the situations of Conditions (P) and (P$_w$) are unknown since now, here we wish to consider them.

Definition 2.10. Let $\psi : B_S \to A_S$ be a surjective S-poset morphism. We say ψ is *(weakly) quasi-2-pure,* if for any $a, a' \in A$ and $as \leq a't$, there exist $b, b' \in B$ such that $bs \leq b't$, $a = (\leq)\ \psi(b)$ and $\psi(b') = (\leq)\ a'$.

Note that, quasi-2-purity and quasi-1-purity are two unrelated notions. Indeed, on the one hand, it follows easily from Example 2.1 that quasi-1-purity does not imply quasi-2-purity. On the other hand, if $S = x^* \cup \{0\}$, where x^* is the monogenic free monoid generated by x, equipped with the order in which

$$1 < x < x^2 < \cdots$$

and 0 is isolated. Then S is a pomonoid. Consider the natural epimorphism $f : S_S \to S/\rho(1, x)$, where $\rho(1, x)$ is an S-poset congruence on S_S generated by the pair $(x, 1)$. Since $x\ \rho(x, 1)\ 1$, this implies $[1]x \leq [1]1$, but there cannot exist $u \in S\backslash\{0\}$ with $ux \leq u$, and so f is not quasi-1-pure. But, it is easy to check that f is quasi-2-pure.

Now, we provide equivalent descriptions of Conditions (P) and (P$_w$).

Proposition 2.11. *Let A_S be a right S-poset. Then the following assertions are equivalent.*
(1) *A_S has Condition* (P) (*Condition* (P$_w$)).
(2) *Every surjective S-poset morphism $B_S \to A_S$ is* (*weakly*) *quasi-2-pure.*
(3) *There exists a* (*weakly*) *quasi-2-pure epimorphism $B_S \to A_S$ where B_S has Condition* (P).

Proof. $(1) \Rightarrow (2)$. We deal only with Condition (P), the proof for Condition (P$_w$) being similar. Now suppose that $\psi : B_S \to A_S$ is an S-poset epimorphism and suppose that $a, a' \in A$, $s, t \in S$ are such that $as \leq a't$ in A_S. Since A_S satisfies Condition (P), there exist $u, v \in S$ and $a'' \in A$ such that $a = a''u$, $a' = a''v$ and $us \leq vt$. Applying surjectivity of ψ, we obtain $b \in B$ with $\psi(b) = a''$. Thus, we have $a = a''u = \psi(bu)$, $a' = a''v = \psi(bv)$, and $bus \leq bvt$ in B_S, as required.

$(2) \Rightarrow (3)$. This is given by [5, Proposition 2.4].

$(3) \Rightarrow (1)$. Let A_S be a right S-poset. By (3) there exists a quasi-2-pure epimorphism $\psi : B_S \to A_S$, where B_S satisfies Condition (P). Now suppose that $as \leq a't$ in A_S for $a, a' \in A$, $s, t \in S$. Then we see that $bs \leq b't$ in B_S, $\psi(b') = a'$ and $\psi(b) = a$ for some $b, b' \in B$. Also, because B_S satisfies Condition (P), from the inequality $bs \leq b't$, we obtain $b'' \in B$ and $u, v \in S$ with $b = b''u$, $b' = b''v$ and $us \leq vt$. Consequently, $a = \psi(b) = \psi(b''u) = \psi(b'')u$ and $a' = \psi(b') = \psi(b''v) = \psi(b'')v$ in A_S, and the proof is complete. □

Since 2-pure epimorphisms are quasi-2-pure, we have the following corollary, which is the S-poset version of [3, Proposition 3.12]

Corollary 2.12. *Let S be a pomonoid and let $\psi : B_S \to A_S$ be an S-poset epimorphism in which B_S has Condition* (P). *If ψ is 2-pure then A_S has Condition* (P).

From [9] we remark that Condition (P) implies Condition (P_w). But we can show that this implication is strict by using an example of a purity epimorphism.

Example 2.13. *Let $S = \{1, x \mid x^2 = x\}$ with the order of S to be discrete. Let $A = \{a, b \mid a \cdot 1 = a,\ a \cdot x = b \cdot x = b \cdot 1 = b\}$ with the order $a < b$. Define an S-poset epimorphism $f : S_S \to A_S$ by $f(1) = a$ and $f(x) = b$. Since $a \cdot 1 \leq b \cdot x$, and we want to reach $1 \cdot 1 \leq x \cdot x$, but this is impossible. Hence f is not quasi-2-pure, and so by Proposition 2.11, A_S fails to satisfy Condition* (P). *However, it is easy to see that f is weakly quasi-2-pure, and so A_S satisfies Condition (P_w) by Proposition 2.11.*

Bulman-Fleming and Laan in [5] established the Stenström-Govorov-Lazard Theorem in the context of S-posets. Further, synthesizing Propositions 2.2, 2.4 and 2.11, we can deduce the following

Corollary 2.14. *Let A_S be a right S-poset. Then the following assertions are equivalent.*

(1) *A_S is strongly flat.*
(2) *A_S has Conditions* (P) *and* (E).
(3) *Every surjective S-poset morphism $B_S \to A_S$ is pure.*
(4) *Every S-poset morphism $B_S \to A_S$ where B_S is finitely presented factors through a finitely generated, free S-poset.*
(5) *A_S is isomorphic to a directed colimit of a family of finitely generated, free S-posets.*
(6) *A_S is subpullback flat and subequalizer flat.*
(7) *Every surjective S-poset morphism $B_S \to A_S$ is 2-pure.*
(8) *There exists a 2-pure epimorphism $B_S \to A_S$ where B_S is strongly flat.*
(9) *Every surjective S-poset morphism $B_S \to A_S$ is quasi-2-pure and (quasi-)1-pure.*
(10) *There exists a quasi-2-pure and (quasi-)1-pure epimorphism $B_S \to A_S$ where B_S is strongly flat.*

We make immediate use of the above ideas to the next propositions, which describe Conditions (WP), $(\mathrm{WP})_w$, (PWP) and $(\mathrm{PWP})_w$ by certain purity epimorphisms.

Definition 2.15. *Let $\psi : B_S \to A_S$ be a surjective S-poset morphism. We say ψ is*

- *(weakly) quasi-w-2-pure if, for all elements $a, a' \in A$, $s, t \in S$, all S-poset morphisms $f : {}_S(Ss \cup St) \to {}_SS$, and $af(s) \leq a'f(t)$, there exist $b, b' \in B$ and $s', t' \in \{s, t\}$ such that $bf(s') \leq b'f(t')$, $a \otimes s = (\leq)\ \psi(b) \otimes s'$ and $\psi(b') \otimes t' = (\leq)\ a' \otimes t$ in $A \otimes_S (Ss \cup St)$.*
- *(weakly) quasi-pw-2-pure if, for any $a, a' \in A$ and $as \leq a's$, there exist $b, b' \in B$ such that $bs \leq b's$, $a = (\leq)\ \psi(b)$ and $\psi(b') = (\leq)\ a'$.*

Clearly, quasi-w-2-purity implies weakly quasi-w-2-purity, and quasi-pw-2-purity implies weakly quasi-pw-2-purity. But, it follows easily from Example 2.13 that these two implications are strict.

Notice from the proof of Proposition 2.11 that we can deduce the following results.

Proposition 2.16. *Let A_S be a right S-poset. Then the following assertions are equivalent.*
(1) *A_S has Condition* (WP) (*Condition* $(\mathrm{WP})_w$).
(2) *Every surjective S-poset morphism $B_S \to A_S$ is* (*weakly*) *quasi-w-2-pure.*
(3) *There exists a* (*weakly*) *quasi-w-2-pure epimorphism $B_S \to A_S$ where B_S has Condition* (WP).

Proposition 2.17. *Let A_S be a right S-poset. Then the following assertions are equivalent.*
(1) *A_S has Condition* (PWP) (*Condition* $(\mathrm{PWP})_w$).
(2) *Every surjective S-poset morphism $B_S \to A_S$ is* (*weakly*) *quasi-pw-2-pure.*
(3) *There exists a* (*weakly*) *quasi-pw-2-pure epimorphism $B_S \to A_S$ where B_S has Condition* (PWP).

In [15], the authors introduced a generalization of Condition (P) called Condition (P') for S-acts. In this way, we introduce Conditions (P') and (P'_w) for S-posets.

Definition 2.18. *Let A_S be a right S-poset. We say that*
1. *A_S satisfies Condition (P') if, for all $a \in A$, $s, s', z \in S$ with $as \leq a's'$ and $sz = s'z$, there exist $a'' \in A$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us \leq vs'$.*
2. *A_S satisfies Condition (P'_w) if, for all $a \in A$, $s, s', z \in S$ with $as \leq a's'$ and $sz = s'z$, there exist $a'' \in A$, $u, v \in S$ such that $a \leq a''u$, $a''v \leq a'$ and $us \leq vs'$.*

Now, we define two kinds of purity conditions of epimorphisms in order to consider Conditions (P') and (P'_w).

Definition 2.19. *Let $\psi : B_S \to A_S$ be a surjective S-poset morphism. We say ψ is*
1. *quasi-$2'$-pure if, for every $a, a' \in A$ and relations*
$$as \leq a't \quad \text{and} \quad sz = tz,$$
there exist $b, b' \in B$ such that $a = \psi(b)$, $\psi(b') = a'$, and $bs \leq bt$.
2. *weakly quasi-$2'$-pure if, for every $a, a' \in A$ and relations*
$$as \leq a't \quad \text{and} \quad sz = tz,$$
there exist $b, b' \in B$ such that $a \leq \psi(b)$, $\psi(b') \leq a'$, and $bs \leq bt$.

Similarly to the argument of Proposition 2.11 one could prove the following result.

Proposition 2.20. *Let A_S be a right S-poset. Then the following assertions are equivalent.*
(1) *A_S has Condition* (P') (*Condition* (P'_w)).
(2) *Every surjective S-poset morphism $B_S \to A_S$ is* (*weakly*) *quasi-$2'$-pure.*
(3) *There exists a* (*weakly*) *quasi-$2'$-pure epimorphism $B_S \to A_S$ where B_S has Condition* (P').

Concluding this section, we summarize in the following Table 1 the results concerning equivalent descriptions of some flatness properties for S-posets (starting with strong flatness and ending with Condition $(\mathrm{PWP})_w$) by certain purity conditions of epimorphisms.

3 Directed colimits for *S*-posets

In this section, we show that these purity epimorphisms of S-posets introduced in Section 2 are preserved under directed colimits.

Directed colimits (also called direct limits) of families of right S-posets are introduced in [5], but note that the definition of directed colimits is omitted in that paper. For completeness, we will provide a full definition of directed colimits for S-posets.

Table 1. Equivalent descriptions of some flatness properties

$\Longleftrightarrow$	**Purity conditions of epimorphisms $\psi : B_S \to A_S$**
A_S **is strongly flat**	ψ **is pure or 2-pure**
A_S **has Condition (E)**	ψ **is 1-pure or quasi-1-pure**
A_S **has Condition (E′)**	ψ **is 1′-pure or quasi-1′-pure**
A_S **has Condition (P)**	ψ **is quasi-2-pure**
A_S **has Condition (P′)**	ψ **is quasi-2′-pure**
A_S **has Condition (P_w)**	ψ **is weakly quasi-2-pure**
A_S **has Condition ($\mathrm{P'}_w$)**	ψ **is weakly quasi-2′-pure**
A_S **has Condition (WP)**	ψ **is quasi-w-2-pure**
A_S **has Condition $(\mathrm{WP})_w$**	ψ **is weakly quasi-w-2-pure**
A_S **has Condition (PWP)**	ψ **is quasi-pw-2-pure**
A_S **has Condition $(\mathrm{PWP})_w$**	ψ **is weakly quasi-pw-2-pure**

Let I be a quasi-ordered (that is, a reflexive and transitive relation) set. *A direct system* in Pos-S is a collection of right S-posets $(A_i)_{i\in I}$ and a collection of right S-poset morphisms $\phi_{i,j} : A_i \to A_j$ $(i \leq j)$ with the following properties:

(i) $\phi_{i,i} = 1_{A_i}$ for all $i \in I$;
(ii) $\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k}$, whenever $i \leq j \leq k$.

The colimit of the system $(A_i, \phi_{i,j})$ is a right S-poset A_S together with right S-poset morphisms $\phi_i : A_i \to A$ such that

(1) $\phi_j \circ \phi_{i,j} = \phi_i$, whenever $i \leq j$;
(2) If B_S is a right S-poset and $\alpha_i : A_i \to B$ are right S-poset morphisms such that $\alpha_j \circ \phi_{i,j} = \alpha_i$ whenever $i \leq j$, then there exists a unique S-poset morphism $\psi : A \to B$ such that the diagram

$$\begin{tikzcd} A_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_i"'] & A \arrow[dl, "\phi"] \\ B & \end{tikzcd}$$

commutes for all $i \in I$.

Further, if the indexing set I satisfies the property that for all $i, j \in I$ there exists $k \in I$ such that $k \geq i, j$, then we say that I is *directed*. In this case, we call the colimit (A, ϕ_i) is a *directed colimit*.

From the following lemma, we remark that in Pos-S, directed colimits of directed systems of S-posets exist.

Lemma 3.1 ([5, Proposition 2.5]). *The directed colimit of any directed system $((A_i)_{i\in I}, (\phi_{i,j})_{i\leq j})$ of right S-posets exists, and may be represented as $(A/\theta, (\phi_i)_{i\in I})$, where*
(1) $A = \coprod_{i\in I} A_i$;
(2) $a \,\theta\, a'$ $(a \in A_i, a' \in A_j)$ *if and only if* $\phi_{i,k}(a) = \phi_{j,k}(a')$ *in* A_k *for some* $k \geq i, j$;
(3) $[a]_\theta \leq [a']_\theta$ $(a \in A_i, a' \in A_j)$ *if and only if* $\phi_{i,k}(a) \leq \phi_{j,k}(a')$ *for some* $k \geq i, j$;
(4) *for each* $i \in I$ *and* $a \in A_i$, $\phi_i(a) = [a]_\theta$.

The next result is often useful in dealing with directed colimits.

Lemma 3.2 ([5, Proposition 2.6]). *Let $(A_i)_{i\in I}$, $(\phi_{i,j})_{i\le j}$ be a direct system of right S-posets. Then the directed colimit $(A, (\phi_i)_{i\in I})$ is characterized up to isomorphism by the conditions*
(1) *each $\phi_i : A_i \to A$ is an S-poset morphism;*
(2) *$\phi_j \circ \phi_{i,j} = \phi_i$, whenever $i \le j$;*
(3) *$A = \bigcup_{i\in I} \phi_i[A_i]$;*
(4) *$\phi_i^{-1}[R] \subseteq \bigcup_{j\ge i} \phi_{i,j}^{-1}[R_j]$, where R (resp. R_j) denotes the graph of the relation $\le$ in the S-poset A (resp. A_j).*

Actually, the property (4) of Lemma 3.2 states the order relation on A_S, that is, $\phi_i(a) \le \phi_j(a')$ if and only if $\phi_{i,k}(a) \le \phi_{j,k}(a')$ for some $k \ge i, j$. Thereby, we obtain the fact that ϕ_i is an order-embedding if and only if $\phi_{i,j}$ is an order-embedding for all $i \le j$.

Using these properties of directed colimits, the following result is obtained.

Lemma 3.3. *Let $(A_i, \phi_{i,j})$ be a direct system of right S-posets with directed index set I and directed colimit (A, ϕ_i). For every family $a_1, \cdots, a_m \in A$ and relations*

$$a_{\alpha_i} s_i \le a_{\beta_i} t_i \qquad (i = 1, \cdots, n),$$

there exist some $l \in I$ and $a_1', \cdots, a_m' \in A_l$ such that $\phi_l(a_j') = a_j$ for all $1 \le j \le m$, and $a_{\alpha_i}' s_i \le a_{\beta_i}' t_i$ for all $1 \le i \le n$.

Proof. The technique used in [3, Lemma 2.3] will be employed. □

In order to prepare for our main results, we will introduce the concept of directed colimits of S-poset morphisms.

Suppose that $(A_i, \phi_{i,j})$ and $(B_i, \varphi_{i,j})$ are direct systems of S-posets and S-poset morphisms. Suppose that for each $i \in I$ there exists an S-poset morphism $\psi_i : A_i \to B_i$ and suppose that (A, ϕ_i) and (B, φ_i), the directed colimits of these systems, are such that the diagrams

$$\begin{tikzcd} A_i \arrow[r, "\psi_i"] \arrow[d, "\phi_i"'] & B_i \arrow[d, "\varphi_i"] \\ A \arrow[r, "\psi"'] & B \end{tikzcd} \qquad \begin{tikzcd} A_i \arrow[r, "\phi_{i,j}"] \arrow[d, "\psi_i"'] & A_j \arrow[d, "\psi_j"] \\ B_i \arrow[r, "\varphi_{i,j}"'] & B_j \end{tikzcd}$$

commute for all $i \le j \in I$. Then we shall refer to ψ as the *directed colimit of the ψ_i*.

In light of Lemma 3.2, the following easily proved result is probably well-known.

Proposition 3.4. *Let S be a pomonoid. Directed colimits of (order-embeddings) epimorphisms of S-posets are (order-embeddings) epimorphisms.*

We now establish one of our main results.

Proposition 3.5. *Let S be a pomonoid. Directed colimits of pure epimorphisms of right S-posets are pure.*

Proof. Suppose that $(A_i, \phi_{i,j})$ and $(B_i, \varphi_{i,j})$ are direct systems of right S-posets and S-poset morphisms. Suppose that for each $i \in I$ there exists a pure epimorphism $\psi_i : A_i \to B_i$ and suppose that (A, ϕ_i) and (B, φ_i), the directed colimits of these systems, are such that the diagrams

$$\begin{tikzcd} A_i \arrow[r, "\psi_i"] \arrow[d, "\phi_i"'] & B_i \arrow[d, "\varphi_i"] \\ A \arrow[r, "\psi"'] & B \end{tikzcd} \qquad \begin{tikzcd} A_i \arrow[r, "\phi_{i,j}"] \arrow[d, "\psi_i"'] & A_j \arrow[d, "\psi_j"] \\ B_i \arrow[r, "\varphi_{i,j}"'] & B_j \end{tikzcd}$$

commute for all $i \le j \in I$.

Suppose that there are $b_1, \cdots, b_m \in B$, $s_1, \cdots, s_n, t_1, \cdots, t_n \in S$ and relations

$$b_{\alpha_i} s_i \leq b_{\beta_i} t_i \qquad (i = 1, \cdots, n).$$

In view of Lemma 3.3, there exist $l \in I$ and $b'_1, \cdots, b'_m \in B_l$ such that $\varphi_l(b'_j) = b_j$ for all $1 \leq j \leq m$, and

$$b'_{\alpha_i} s_i \leq b'_{\beta_i} t_i \qquad (i = 1, \cdots, n).$$

Since ψ_l is a pure epimorphism, there exist $a_1, \cdots, a_m \in A_l$ with $\psi_l(a_j) = b'_j$ for all $1 \leq j \leq m$, and $a_{\alpha_i} s_i \leq a_{\beta_i} t_i$ for all $1 \leq i \leq n$. Then we can calculate that

$$\psi(\phi_l(a_j)) = (\psi\phi_l)(a_j) = (\varphi_l\psi_l)(a_j) = \varphi_l(\psi_l(a_j)) = \varphi_l(b'_j) = b_j$$

for all j, and

$$\phi_l(a_{\alpha_i})s_i \leq \phi_l(a_{\beta_i})t_i$$

for all i. Hence ψ is pure. □

By the same approach, one can get

Proposition 3.6. *Let S be a pomonoid. Directed colimits of 1-pure ($1'$-pure) epimorphisms of right S-posets are 1-pure ($1'$-pure).*

In what follows, we consider the situation for quasi-2-pure epimorphisms.

Proposition 3.7. *Let S be a pomonoid. Directed colimits of quasi-2-pure epimorphisms of right S-posets are quasi-2-pure.*

Proof. Suppose that $(A_i, \phi_{i,j})$ and $(B_i, \varphi_{i,j})$ are direct systems of right S-posets and S-poset morphisms. Suppose that for each $i \in I$ there exists a quasi-2-pure epimorphism $\psi_i : A_i \to B_i$ and suppose that (A, ϕ_i) and (B, φ_i), the directed colimits of these systems, are such that the diagrams

$$\begin{tikzcd} A_i \arrow[r, "\psi_i"] \arrow[d, "\phi_i"'] & B_i \arrow[d, "\varphi_i"] \\ A \arrow[r, "\psi"'] & B \end{tikzcd} \qquad \begin{tikzcd} A_i \arrow[r, "\phi_{i,j}"] \arrow[d, "\psi_i"'] & A_j \arrow[d, "\psi_j"] \\ B_i \arrow[r, "\varphi_{i,j}"'] & B_j \end{tikzcd}$$

commute for all $i \leq j \in I$.

Suppose that $b_1, b_2 \in B$, $s, t \in S$ are such that $b_1 s \leq b_2 t$. By Lemma 3.2(3), there exist $i, j \in I$, $b_i \in B_i$ and $b_j \in B_j$ with $b_1 = \varphi_i(b_i)$ and $b_2 = \varphi_j(b_j)$. According to the order relation on B_S, we obtain $k \geq i, j$ with $\varphi_{i,k}(b_i)s \leq \varphi_{j,k}(b_j)t$ in B_k. Since ψ_k is quasi-2-pure, there exist $a_1, a_2 \in A_k$ such that $\psi_k(a_1) = \varphi_{i,k}(b_i)$, $\psi_k(a_2) = \varphi_{j,k}(b_j)$ and $a_1 s \leq a_2 t$. Now we can compute that $\phi_k(a_1)s \leq \phi_k(a_2)t$ and

$$b_1 = \varphi_i(b_i) = \varphi_k(\varphi_{i,k}(b_i)) = \varphi_k(\psi_k(a_1)) = (\varphi_k\psi_k)(a_1) = (\psi\phi_k)(a_1) = \psi(\phi_k(a_1)).$$

In the same way, $b_2 = \psi(\phi_k(a_2))$, and so ψ is quasi-2-pure. □

Note that, the situation for quasi-1-pure (resp., quasi-$1'$-pure, quasi-$2'$-pure, weakly quasi-2-pure, weakly quasi-$2'$-pure, quasi-w-2-pure, weakly quasi-w-2-pure, quasi-pw-2-pure and weakly quasi-pw-2-pure) epimorphisms, is similar in nature to Proposition 3.7, and so here will be omitted.

We have already seen in Section 2 that some flatness properties are equivalent to certain purity conditions of epimorphisms. And then, using Propositions 3.5, 3.6, 3.7 and the above note, the following result is immediately established.

Corollary 3.8. *Let S be a pomonoid. Every directed colimit of a direct system of right S-posets that are strongly flat (resp., have Conditions* (E), (E′), (P), (P′), (P_w), (P'_w), (WP), $(WP)_w$, (PWP) *and* $(PWP)_w$) *has again these properties.*

Observing the result above, by a new way we have obtained that many flatness properties of S-posets are preserved under directed colimits.

However, as for acts, the situation for projective S-posets is slightly different.

Proposition 3.9. *Let S be a pomonoid. Every directed colimit of a direct system of projective right S-posets is projective if and only if S is right (po-)perfect.*

Proof. **Necessity.** In light of [14, Theorem 6.3], it suffices to show that every strongly flat right S-poset is projective, so now suppose that A_S is a strongly flat S-poset. It then follows from [5, Proposition 4.4] that A_S is isomorphic to a directed colimit of a family of finitely generated, free S-posets. So by assumption, A_S is projective since free S-posets are projective.

Sufficiency. Let (A, ϕ_i) be a directed colimit of a direct system of projective right S-posets. In view of Corollary 3.8, A_S is strongly flat because projective S-posets are strongly flat. Also, since S is right (po-)perfect, by [14, Theorem 6.3], A_S is projective, and the proof is complete. □

From [5, Proposition 4.6], it follows that an S-poset epimorphism $\varphi : B_S \to A_S$ is pure if and only if, for every finitely presented S-poset C_S and morphism $\eta : C_S \to A_S$ there exists a morphism $\mu : C_S \to B_S$ such that $\varphi\mu = \eta$. It is easy to check that split epimorphisms are pure. But the following example from [3] shows that the converse is false. Let $S = (\mathbb{N}, \max)$ with the discrete order. Consider the one-element S-poset Θ_S and note that $S_S \to \Theta_S$ is a pure epimorphism. On the other hand, since S does not contain a fixed point then it does not split.

But, the following result is a small improvement. Its straightforward proof is omitted.

Proposition 3.10. *Let $\psi : A_S \to B_S$ be a surjective S-poset morphism with B_S is finitely presented. Then ψ is pure if and only if it is split.*

For future use, we record

Lemma 3.11. *Let S be a pomonoid, let*

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\psi"'] & D \end{tikzcd}$$

be a pullback diagram of S-posets and suppose that ψ is a pure epimorphism. Then ϕ is also a pure epimorphism.

Proof. It is routine. □

Note that in the above result, if we replace “pullback diagram" by “subpullback diagram", then this result is also valid.

As previously discussed, not every pure epimorphism splits, but every pure epimorphism is a directed colimit of split epimorphisms. The strategy for the proof the following proposition is taken from the unordered case in [3].

Proposition 3.12. *Let S be a pomonoid and let $\psi : A_S \to B_S$ be a surjective S-poset morphism. Then ψ is pure if and only if it is a directed colimit of split epimorphisms.*

Proof. Suppose that $\psi : A_S \to B_S$ is a pure epimorphism. From [5, Proposition 4.4] it follows that B_S is a directed colimit of finitely presented S-posets $(B_i, \phi_{i,j})$ and so let $\phi_i : B_i \to B$ be the canonical morphisms.

For each B_i let

$$\begin{tikzcd} A_i \arrow[r, "\psi_i"] \arrow[d, "\varphi_i"'] & B_i \arrow[d, "\phi_i"] \\ A \arrow[r, "\psi"'] & B \end{tikzcd}$$

be a pullback diagram so that by Lemma 3.11 ψ_i is pure. Because B_i is finitely presented, it follows from Proposition 3.10 that ψ_i splits. Notice that

$$A_i = \{(b_i, a) \in B_i \times A \mid \phi_i(b_i) = \psi(a)\},$$

$\psi_i(b_i, a) = b_i$ and $\varphi_i(b_i, a) = a$, and that since ψ is onto then $A_i \neq \varnothing$.

For $i \leq j$ define $\varphi_{i,j} : A_i \to A_j$ by $\varphi_{i,j}(b_i, a) = (\phi_{i,j}(b_i), a)$ and notice that $\varphi_j\varphi_{i,j} = \varphi_i$ and $\psi_j\varphi_{i,j} = \phi_{i,j}\psi_i$. Next we show that (A, φ_i) is the directed colimit of $(A_i, \varphi_{i,j})$. So assume there exist an S-poset C_S and S-poset morphisms $\alpha_i : A_i \to C$ with $\alpha_j\varphi_{i,j} = \alpha_i$ for all $i \leq j$. We now define $\alpha : A \to C$ by $\alpha(a) = \alpha_i(b_i, a)$ where i and b_i are chosen so that $\varphi_i(b_i) = \psi(a)$. Then α is well-defined since if $\psi(a) = \phi_j(b_j)$ then there exists $k \geq i, j$ with $\phi_{i,k}(b_i) = \phi_{j,k}(b_j)$ and

$$\alpha_i(b_i, a) = \alpha_k\varphi_{i,k}(b_i, a) = \alpha_k(\phi_{i,k}(b_i, a)) = \alpha_k(\phi_{j,k}(b_j, a)) = \alpha_k\varphi_{j,k}(b_j, a) = \alpha_j(b_j, a).$$

Then α is an S-poset morphism and clearly $\alpha\varphi_i = \alpha_i$. Finally, if $\beta : A \to C$ is such that $\beta\varphi_i = \alpha_i$ for all i, then $\beta(a) = \beta\varphi_i(b_i, a) = \alpha_i(b_i, a) = \alpha(a)$ and so α is unique. We therefore see that we have reached the desired conclusion.

The converse holds by Proposition 3.5. □

4 Directed colimits of flatness properties

From previous section, we find some flatness properties that are closed under directed colimits. But for the other flatness properties, we need to characterize the behavior of subpullbacks diagrams under directed colimits. In this section, we first prove that every directed colimit of a direct system of subequalizer flat S-posets is subequalizer flat. Then, we consider subpullbacks in order to prove that every class of S-posets having a flatness property is closed under directed colimits. First let us to recall the concept of subpullbacks and subequalizers in the category S-Pos which are defined in [5] as follows.

The categories S-Pos and Pos are poset-enriched concrete categories, where the order relation on morphism sets is defined pointwise (i.e. $f \leq g$ for $f, g : A \to B$ if and only if $f(a) \leq g(a)$ for every $a \in A$). In such categories, a diagram

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[d, "\alpha"] \\ N \arrow[r, "\beta"'] & Q \end{tikzcd}$$

is called *a subpullback diagram for* α *and* β if

(1) $\alpha p_1 \leq \beta p_2$ and
(2) for any diagram

$$\begin{tikzcd} P' \arrow[r, "p'_1"] \arrow[d, "p'_2"'] & M \arrow[d, "\alpha"] \\ N \arrow[r, "\beta"'] & Q \end{tikzcd}$$

with $\alpha p_1' \leq \beta p_2'$, there exists a unique morphism $\varphi : P' \to P$ such that $p_i\varphi = p_i'$ for $i = 1, 2$. In S-Pos or Pos, P may in fact be realized as

$$P = \{(m, n) \in M \times N | \ \alpha(m) \leq \beta(n)\}.$$

The first subpullback diagram is denoted by $P(M, N, \alpha, \beta, Q)$ and tensoring it by any right S-poset A_S one gets the diagram

$$\begin{tikzcd} A_S \otimes {}_S P \arrow[r, "1_A \otimes p_1"] \arrow[d, "1_A \otimes p_2"'] & A_S \otimes {}_S M \arrow[d, "\alpha"] \\ A_S \otimes {}_S N \arrow[r, "\beta"'] & A_S \otimes {}_S Q \end{tikzcd}$$

in Pos. Take

$$P' = \{(a \otimes m, a \otimes n) \in (A_S \otimes {}_S M) \otimes (A_S \otimes {}_S N) | \ a \otimes \alpha(m) \leq a \otimes \beta(n)\}$$

with p_1', p_2' being the restrictions of the projections. From the definition of subpullbacks it follows the existence of a unique monotonic mapping $\phi : A_S \otimes {}_S P \to P'$ such that $p_i'\phi = 1_A \otimes p_i$ for $i = 1, 2$. This mapping is called *the ϕ corresponding to the subpullback diagram $P(M, N, \alpha, \beta, Q)$ for A_S*. It can be checked that $\phi(a \otimes (m, n)) = (a \otimes m, a \otimes n)$.

A subequalizer diagram for α and β

$$\begin{tikzcd} E \arrow[r, "l"] & M \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & N \end{tikzcd}$$

is defined similarly, where $E = \{m \in M | \ \alpha(m) \leq \beta(m)\}$. As we mentioned earlier, an S-poset A_S is called *subpullback flat* (*subequalizer flat*) if the functor $A_S \otimes -$ takes subpullbacks (subequalizers) in S-Pos to subpullbacks (subequalizers) in Pos. We begin by proving the result for subequalizer flat. Before that we need some preparations.

If A_S is an S-subposet of B_S and x, y are different elements not belonging to B_S, then

$$B_S \coprod^A B_S = \{(a, x) | \ a \notin A_S\} \dot\cup A_S \dot\cup \{(a, y) | \ a \notin A_S\}.$$

Define an S-action

$$(a, w)s = \begin{cases} (as, w) & \text{if} \quad as \notin A_S \\ as & \text{otherwise,} \end{cases}$$

for every $a \in B_S \setminus A_S$, $s \in S$ and $w \in \{x, y\}$.

The order on $B_S \coprod^A B_S$ is given by

$$(a, w_1) \leq (b, w_2) \Leftrightarrow (w_1 = w_2, \ a \leq b) \text{ or } (w_1 \neq w_2, \ a \leq a' \leq b \text{ for some } a' \in A_S),$$

where $\{w_1, w_2\} = \{x, y\}$ and $a, b \notin A_S$. For $w \in \{x, y\}$, $b \notin A_S$ and $a \in A$,

$$(b, w) \leq a \Leftrightarrow b < a \ \text{ and } \ a \leq (b, w) \Leftrightarrow a < b.$$

It is easily checked $B_S \coprod^A B_S$ is a right S-poset.

Lemma 4.1. *Every regular monomorphism $h : A_S \to B_S$ in Pos-S can be consider as the subequalizer of the diagram*

$$\begin{tikzcd} A_S \arrow[r, "h"] & B_S \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & B_S \coprod^{h(A)} B_S. \end{tikzcd}$$

Proof. Let $h : A_S \to B_S$ be a regular monomorphism. Define $\alpha, \beta : B_S \to B_S \coprod^{h(A)} B_S$ as

$$\alpha(b) = \begin{cases} (b, x) & \text{if} \quad b \notin h(A) \\ b & \text{otherwise} \end{cases}$$

and

$$\beta(b) = \begin{cases} (b, y) & \text{if} \quad b \notin h(A) \\ b & \text{otherwise.} \end{cases}$$

To show that (A_S, h) is a subequalizer diagram for α and β, clearly $\alpha h \leq \beta h$, and consider the diagram

$$\begin{tikzcd} C_S \arrow[r, "f"] & B_S \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & B_S \coprod^{h(A)} B_S \\ A_S \arrow[ur, "h"] & & \end{tikzcd}$$

where C_S and f are such that $\alpha f \leq \beta f$. By the construction of $B_S \coprod^{h(A)} B_S$ we see that $f(C_S) \subseteq h(A_S)$. Now, since h is a regular monomorphism, one can define a mapping $\overline{f} : C_S \to A_S$ by $\overline{f}(c) = h^{-1}(f(c))$ for $c \in C_S$. Then $\overline{f}$ is a well-defined S-morphism and unique with the property $\overline{f} = hf$. □

From the previous lemma, it follows that every subequalizer flat S-poset is po-flat. Now we are ready to consider directed colimit of a direct system of subequalizer flat S-posets.

Proposition 4.2. *Let S be a pomonoid. Every directed colimit of a direct system of subequalizer flat S-posets is subequalizer flat.*

Proof. Let $(A_i, \phi_{i,j})$ be a direct system of S-psets and S-morphisms with directed index set I and directed colimit (A, ϕ_i). If every A_i is subequalizer flat, we will prove that A is subequalizer flat. Let the pair (E, l) be a subequalizer in the following diagram

$$E \xrightarrow{l} M \overset{\alpha}{\underset{\beta}{\rightrightarrows}} N.$$

Since every subequalizer flat S-poset is po-flat, from po-flatness of A_S we have the following diagram

$$A \otimes E \xrightarrow{1_A \otimes l} A \otimes M \overset{1_A \otimes \alpha}{\underset{1_A \otimes \beta}{\rightrightarrows}} A \otimes N$$

and $1_A \otimes l$ is a regular monomorphism. By the definition, the subequalizer of $1_A \otimes \alpha$ and $1_A \otimes \beta$ is

$$E' = \{a \otimes m \in A \otimes M | \, (1_A \otimes \alpha)(a \otimes m) \leq (1_A \otimes \beta)(a \otimes m), a \in A, m \in M\}.$$

By the definition of the subequalizer, it is easily checked that $A \otimes E \subseteq E'$, next we want to show that $E' \subseteq A \otimes E$. Let $a \in A$, $m \in M$ be such that $a \otimes m \in E'$. So $a \otimes \alpha(m) \leq a \otimes \beta(m)$ in $A \otimes N$ implies that the existence of $u_1, v_1, \ldots, u_n, v_n \in S$, $a_1, \ldots, a_n \in A_S$, $y_2, \ldots, y_n \in_S N$, $n \in \mathbb{N}$ such that

$$\begin{array}{ll} a \leq a_1 u_1 & \\ a_1 v_1 \leq a_2 u_2 & u_1 \alpha(m) \leq v_1 y_2 \\ \vdots & \vdots \\ a_n v_n \leq a & u_n y_n \leq v_n \beta(m). \end{array}$$

Denote a by a_0, then there exist $a_{i_j} \in A_{i_j}$ such that $a_j = \phi_{i_j}(a_{i_j})$, $j = 0, 1, .., n$. Then we get

$$\begin{array}{ll} \phi_{i_0}(a_{i_0}) \leq \phi_{i_1}(a_{i_1}) u_1 & \\ \phi_{i_1}(a_{i_1}) v_1 \leq \phi_{i_2}(a_{i_2}) u_2 & u_1 \alpha(m) \leq v_1 y_2 \\ \vdots & \vdots \\ \phi_{i_n}(a_{i_n}) v_n \leq \phi_{i_0}(a_{i_0}) & u_n y_n \leq v_n \beta(m). \end{array}$$

Since I is directed, there exists $l \geq i_1, i_2, \ldots, i_n$ such that

$$\begin{array}{ll} \phi_{i_0,l}(a_{i_0}) \leq \phi_{i_1,l}(a_{i_1})u_1 & \\ \phi_{i_1,l}(a_{i_1})v_1 \leq \phi_{i_2,l}(a_{i_2})u_2 & u_1\alpha(m) \leq v_1 y_2 \\ \vdots & \vdots \\ \phi_{i_n,l}(a_{i_n})v_n \leq \phi_{i_0,l}(a_{i_0}) & u_n y_n \leq v_n\beta(m). \end{array}$$

This means that $\phi_{i_0,l}(a_{i_0}) \otimes \alpha(m) \leq \phi_{i_0,l}(a_{i_0}) \otimes \beta(m)$ in $A_l \otimes N$. Now, from the fact that A_l is subequalizer flat, and $A_l \otimes E$ is the subequalizer of $1_{A_l} \otimes \alpha$ and $1_{A_l} \otimes \beta$, it follows that $\phi_{i_0,l}(a_{i_0}) \otimes m \in A_l \otimes E$. So $m \in E$, $a \otimes m \in A \otimes E$ and we are done. □

In the rest of this section we concentrate on subpullback diagrams. We now present two fundamental propositions that yield the main result of this section.

Lemma 4.3. *Let S be a pomonoid, and $(A_i, \phi_{i,j})$ be a direct system of S-posets with directed index set I and let (A, ϕ_i) be the directed colimit. Suppose that for each A_i the mapping ϕ corresponding to the subpullback diagram $P(M, N, \alpha, \beta, Q)$ is surjective, then for A the mapping ϕ corresponding to the subpullback diagram $P(M, N, \alpha, \beta, Q)$ is surjective.*

Proof. Suppose that $(x \otimes m, y \otimes n)$ belongs to the subpullback of $P(A \otimes M, A \otimes N, 1_A \otimes \alpha, 1_A \otimes \beta, A \otimes Q)$, where $x, y \in A$, $m \in M$, $n \in N$. Then $x \otimes \alpha(m) \leq y \otimes \beta(n)$ in $A \otimes Q$. We should find $a \in A$, $(m, n) \in P(M, N, \alpha, \beta, Q)$ such that $\phi(a \otimes (m, n)) = (x \otimes m, y \otimes n)$. Since $x \otimes \alpha(m) \leq y \otimes \beta(n)$, there exist $k \in \mathbb{N}$ and elements $a_1, \ldots, a_k \in A_S$, $q_2, \ldots, q_k \in {}_SQ$, $u_1, v_1, \ldots, u_k, v_k \in S$ such that

$$\begin{array}{ll} x \leq a_1 u_1 & \\ a_1 v_1 \leq a_2 u_2 & u_1\alpha(m) \leq v_1 q_2 \\ \vdots & \vdots \\ a_k v_k \leq y & u_k q_k \leq v_k \beta(n). \end{array}$$

Denote x by a_0 and y by a_{k+1}, so there exist $a_{i_j} \in A_{i_j}$ such that $a_j = \phi_{i_j}(a'_{i_j})$, where $i_j \in I$ and $j = 0, 1, \ldots, k, k+1$. Hence we have

$$\begin{array}{ll} \phi_{i_0}(a'_{i_0}) \leq \phi_{i_1}(a'_{i_1})u_1 & \\ \phi_{i_1}(a'_{i_1})v_1 \leq \phi_{i_2}(a'_{i_2})u_2 & u_1\alpha(m) \leq v_1 q_2 \\ \vdots & \vdots \\ \phi_{i_k}(a'_{i_k})v_k \leq \phi_{i_{k+1}}(a'_{i_{k+1}}) & u_k q_k \leq v_k \beta(n). \end{array}$$

Since I is directed, there exists $l \geq i_0, i_1, \ldots, i_{k+1}$ such that

$$\begin{array}{ll} \phi_{i_0}(a'_{i_0}) \leq \phi_{i_1}(a'_{i_1})u_1 & \\ \phi_{i_1}(a'_{i_1})v_1 \leq \phi_{i_2}(a'_{i_2})u_2 & u_1\alpha(m) \leq v_1 q_2 \\ \vdots & \vdots \\ \phi_{i_k}(a'_{i_k})v_k \leq \phi_{i_{k+1}}(a'_{i_{k+1}}) & u_k q_k \leq v_k \beta(n). \end{array}$$

Then $\phi_{i_0,l}(a'_{i_0}) \otimes \alpha(m) \leq \phi_{i_{k+1},l}(a'_{i_{k+1}}) \otimes \beta(n)$ in $A_l \otimes Q$. That is $(\phi_{i_0,l}(a'_{i_0}) \otimes m, \phi_{i_{k+1},l}(a'_{i_{k+1}}) \otimes n)$ belongs to the subpullback of $P(A_l \otimes M, A_l \otimes N, 1_{A_l} \otimes \alpha, 1_{A_l} \otimes \beta, A_l \otimes Q)$. By the assumption, for A_l the mapping ϕ corresponding to the subpullback diagram $P(M, N, \alpha, \beta, Q)$ is surjective, so there exist $a'' \in A_l$, $m' \in M$ and $n' \in N$ such that $\phi(a'' \otimes (m', n')) = (\phi_{i_0,l}(a'_{i_0}) \otimes m, \phi_{i_{k+1},l}(a'_{i_{k+1}}) \otimes n)$, where $\alpha(m') \leq \beta(n')$. That is $\phi_{i_0,l}(a'_{i_0}) \otimes m = a'' \otimes m'$ and $\phi_{i_{k+1},l}(a'_{i_{k+1}}) \otimes n = a'' \otimes n'$. Since $\phi_{i_0,l}(a'_{i_0}) \otimes m = a'' \otimes m'$, there exist elements $c_i, c'_j \in A_l$, $m_2, \ldots, m_p, m'_2, \ldots, m'_{p'} \in {}_SM$, $s_i, t_i, s'_j, t'_j \in S$, $1 \leq i \leq p$, $1 \leq j \leq p'$ for $p, p' \in \mathbb{N}$ such that

$$\phi_{i_0,l}(a'_{i_0}) \leq c_1 s_1$$

$$\begin{array}{rl} c_1t_1 \le c_2s_2 & s_1m \le t_1m_2 \\ \vdots & \vdots \\ c_pt_p \le a'' & t_pm_p \le t_pm', \\ a'' \le c'_1s'_1 & \\ c'_1t'_1 \le c'_2s'_2 & s'_1m' \le t'_1m'_2 \\ \vdots & \vdots \\ c'_{p'}t'_{p'} \le \phi_{i_0,l}(a'_{i_0}) & t'_{p'}m'_{p'} \le t'_{p'}m. \end{array}$$

Acting ϕ_l on left column inequations and since $x = \phi_{i_0}(a'_{i_0}) = \phi_l\phi_{i_0,l}(a'_{i_0})$, we get

$$\begin{array}{rl} x \le \phi_l(c_1)s_1 & \\ \phi_l(c_1)t_1 \le \phi_l(c_2)s_2 & s_1m \le t_1m_2 \\ \vdots & \vdots \\ \phi_l(c_p)t_p \le \phi_l(a'') & t_pm_p \le t_pm', \\ \phi_l(a'') \le \phi_l(c'_1)s'_1 & \\ \phi_l(c'_1)t'_1 \le \phi_l(c'_2)s'_2 & s'_1m' \le t'_1m'_2 \\ \vdots & \vdots \\ \phi_l(c'_{p'})t'_{p'} \le x & t'_{p'}m'_{p'} \le t'_{p'}m. \end{array}$$

Thus $x \otimes m = \phi_l(a'') \otimes m'$ in $A \otimes M$. Since $\phi_{i_{k+1},l}(a'_{i_{k+1}}) \otimes n = a'' \otimes n'$, by a similar way we can prove $y \otimes n = \phi_l(a'') \otimes n'$ in $A \otimes N$. Therefore, $\phi(\phi_l(a'') \otimes (m', n')) = (\phi_l(a'') \otimes m', \phi_l(a'') \otimes n') = (x \otimes m, y \otimes n)$, as desired. □

Lemma 4.4. *Let S be a pomonoid, and $(A_i, \phi_{i,j})$ be a direct system of S-posets with directed index set I and let (A, ϕ_i) be the directed colimit. Suppose that for each A_i the mapping ϕ corresponding to the subpullback diagram $P(M, N, \alpha, \beta, Q)$ is an order-embedding, then for A the mapping ϕ corresponding to the subpullback diagram $P(M, N, \alpha, \beta, Q)$ is an order-embedding.*

Proof. Suppose that $\phi(a \otimes (m, n)) \le \phi(a' \otimes (m', n'))$ belongs to the subpullback of $P(A \otimes M, A \otimes N, 1_A \otimes \alpha, 1_A \otimes \beta, A \otimes Q)$, where $a, a' \in A$ and $(m, n), (m', n') \in {}_SP$. So we have $\alpha(m) \le \beta(n)$, $\alpha(m') \le \beta(n')$, and

$$a \otimes m \le a' \otimes m' \text{ in } A_S \otimes {}_SM, \ a \otimes n \le a' \otimes n' \text{ in } A_S \otimes {}_SN.$$

We will show that $a \otimes (m, n) \le a' \otimes (m', n')$ in $A_S \otimes {}_SP$. Since $a \otimes m \le a' \otimes m'$ in $A_S \otimes {}_SM$ and $a \otimes n \le a' \otimes n'$ in $A_S \otimes {}_SN$, there exist $p, p' \in \mathbb{N}$ and elements $a_i, c_j \in A_S$, $m_2, \ldots, m_p \in {}_SM$, $n_2, \ldots, n_{p'} \in {}_SN$, $u_i, v_i, u'_j, v'_j \in S$, $1 \le i \le p$, $1 \le j \le p'$ such that

$$\begin{array}{rl} a \le a_1u_1 & \\ a_1v_1 \le a_2u_2 & u_1m \le v_1m_2 \\ \vdots & \vdots \\ a_pv_p \le a' & u_pm_p \le v_mm', \\ a \le c_1u'_1 & \\ c_1v'_1 \le c_2u'_2 & u'_1n \le v'_1n_2 \\ \vdots & \vdots \\ c_{p'}v'_{p'} \le a' & u'_{p'}n_{p'} \le v'_{p'}n'. \end{array}$$

By Lemma 3.3, there exist $l \in I$ and $a'_0, a'_1, \cdots, a'_p, c'_1, \ldots, c'_{p'}, c'_0 \in A_l$ such that $\phi_l(a'_0) = a$, $\phi_l(a'_i) = a_i$, $\phi_l(c'_j) = c_j$, $\phi_l(c'_0) = a'$ for all $1 \le i \le p$, $1 \le j \le p'$, and

$$a'_0 \le a'_1u_1$$

$$\begin{array}{ll} a'_1 v_1 \leq a'_2 u_2 & u_1 m \leq v_1 m_2 \\ \vdots & \vdots \\ a_p v_p \leq c'_0 & u_p m_p \leq v_m m', \\ a'_0 \leq c'_1 u'_1 & \\ c'_1 v'_1 \leq c'_2 u'_2 & u'_1 n \leq v'_1 n_2 \\ \vdots & \vdots \\ c'_{p'} v'_{p'} \leq c'_0 & u'_{p'} n_{p'} \leq v'_{p'} n'. \end{array}$$

This means that

$$a'_0 \otimes m \leq c'_0 \otimes m' \text{ in } A_l \otimes {}_S M,\ \alpha(m) \leq \beta(n),$$

$$a'_0 \otimes n \leq c'_0 \otimes n' \text{ in } A_l \otimes {}_S N,\ \alpha(m') \leq \beta(n').$$

Hence $\phi(a'_0 \otimes (m,n)) \leq \phi(c'_0 \otimes (m',n'))$. Since for A_l the mapping ϕ corresponding to the subpullback diagram $P(M, N, \alpha, \beta, Q)$ is an order-embedding, we imply $a'_0 \otimes (m,n) \leq c'_0 \otimes (m',n')$ in $A_l \otimes {}_S P$. So there exist $d_1, \ldots, d_r \in A_l$, $(m_2, n_2), \ldots, (m_r, n_r) \in {}_S P$, $x_1, y_1, \ldots, x_r, y_r \in S$, $r \in \mathbb{N}$ such that

$$\begin{array}{ll} a'_0 \leq d_1 x_1 & \\ d_1 y_1 \leq d_2 x_2 & x_1(m,n) \leq y_1(m_2,n_2) \\ \vdots & \vdots \\ d_r y_r \leq c'_0 & x_r(m_r,n_r) \leq y_r(m',n'). \end{array}$$

Acting ϕ_l on left column inequations and since $a = \phi_l(a'_0)$ and $a' = \phi_l(c'_0)$, we get

$$\begin{array}{ll} a \leq \phi_l(d_1) x_1 & \\ \phi_l(d_1) y_1 \leq \phi_l(d_2) x_2 & x_1(m,n) \leq y_1(m_2,n_2) \\ \vdots & \vdots \\ \phi_l(d_r) y_r \leq a' & x_r(m_r,n_r) \leq y_r(m',n'). \end{array}$$

Therefore, $a \otimes (m,n) \leq a' \otimes (m',n')$ in $A_S \otimes {}_S P$ and the result follows. □

It is shown in [7, Theorems 2.1-2.4, 3.2, 4.1, 5.3] and [16, Definition 2.2] that most of flatness properties of S-posets over a pomonoid S are equivalent to the surjectivity or bijectivity of mappings corresponding to the subpullback diagrams in special cases. Using that results and two previous propositions we conclude the main result of this section.

Theorem 4.5. *Every class of S-posets having a flatness property such as torsion freeness, principal weak flatness, weak flatness, flatnes, pullback flatness, subpullback flatness, principal weak kernel po-flatness, weak kernel po-flatness, translation kernel po-flatness, and satisfying Conditions* (P), (WP) *or* (PWP) *is closed under directed colimits.*

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