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Xiaozhou Feng*, Yi Song, and Xiaomin An

Dynamic behavior analysis of a prey-predator model with ratio-dependent Monod-Haldane functional response

Abstract: This paper investigates the dynamic behavior analysis on the prey-predator model with ratio-dependent Monod-Haldane response function under the homogeneous Dirichlet boundary conditions, which is used to simulate a class of biological system. Firstly, the sufficient and necessary conditions on existence and non-existence of coexistence states of this model are discussed by comparison principle and fixed point index theory. Secondly, taking a as a main bifurcation parameter, the structure of global bifurcation curve on positive solutions is established by using global bifurcation theorem and properties of principal eigenvalue. Finally, the stability of coexistence states is obtained by the eigenvalue perturbation theory ; the multiplicity of coexistence states is investigated when a satisfies some condition by the fixed point index theory.

1 Introduction

Population ecology is an important branch of ecology. Due to the complexity of ecological relations, mathematical methods and results have been increasingly used in ecology and population ecology. In recent years, due to the widespread application of biological models, such as predator-prey model in population ecology, this field of research has gained increasing interest. However, the predator-prey model is an important branch of reaction-diffusion equations. The dynamic relationship between predator and their prey is one of the dominant themes in ecology and mathematical ecology. During the last thirty years, the investigation on the prey-predator models has been developed, and more realistic models have been derived in view of laboratory experiments. Moreover, the research on the prey-predator models [1–4] has been studied from various views and interesting results have been obtained (see [5–8] and the references therein). For every specific prey-predator model, we know that the functional response of the predator to the prey density is very important, which represents the specific transformation rule of the two organisms. For example, the Holling I type functional response in a predator-prey model was considered by Cheng et al. [9] and Zhang et al. [10, 11], the Holling II type functional response was investigated by Wang and Wu [12], Zhu et al. [13] and Cui et al. [14], the Holling Tanner type functional response was studied by Casal et al. [15] and Du et al. [16], the Beddington-DeAngelis functional response was proposed by [17] and Guo and Wu [18]. In this paper, we are

*Corresponding Author: Xiaozhou Feng: College of Science, Xi'an Technological University, Xi'an 710032, China,
E-mail: flxzfxz8@163.com
Yi Song: College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590,
China, E-mail: songyi_sy@sina.com
Xiaomin An: College of Science, Xi'an Technological University, Xi'an 710032, China, E-mail: lyxaxm@sina.com

concerned with the prey-predator model with ratio-dependent Monod-Haldane response function [19, 20] under the homogeneous Dirichlet boundary conditions as follows:

$$\begin{cases} -\Delta u = u(a - u - \frac{bv}{1+mu+lu^2+kv}) & x \in \Omega, \\ -\Delta v = v(c - v + \frac{du}{1+mu+lu^2+kv}) & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^N ($N \geq 1$) with smooth boundary $\partial\Omega$. u and v stand for the densities of prey and predators, respectively. The parameters a, b, c, d are assumed to be only positive constants. a and c denote the intrinsic growth rate of prey u and predator v, respectively. b and d stand for capturing rate to predator and conversion rate of prey captured by predator, respectively. Where $f(u, v) = \frac{uv}{1+mu+lu^2+kv}$ stands for the Monod-Haldane response function with ratio-dependent, which not only has the characteristics of Monod-Haldane reaction, but also has the characteristics of Beddington-DeAngelis functional response. Thus, this response function can better simulate the transformation law of two species. In this paper, we are concerned with the coexistence states on the prey-predator model with the response in this case $m > 0$, $k > 0$ and $l = 1 > 0$; the specific model is as follows:

$$\begin{cases} -\Delta u = u(a - u - \frac{bv}{1+mu+u^2+kv}), & x \in \Omega, \\ -\Delta v = v(c - v + \frac{du}{1+mu+u^2+kv}), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

where the coexistence state is a solution (u, v) of (2) satisfying $u(x) > 0$ and $v(x) > 0$ for all $x \in \Omega$.

The rest of this paper is arranged as follows. In Section 2, the sufficient and necessary conditions on existence and non-existence of coexistence states of (2) are discussed by the fixed point index theory. In Section 3, taking a as a main bifurcation parameter, the structure of global bifurcation curve on coexistence states is established by using global bifurcation theorem. In Section 4, the stability of coexistence states is obtained by the eigenvalue perturbation theory; the multiplicity of coexistence stats of (2) is investigated when a satisfies some condition by the fixed point index theory.

2 Coexistence

The goal of this section is to discuss the condition on existence and non-existence of coexistence states by the fixed point index theory and properties of principal eigenvalue. In order to present the main results, we introduce some basic conclusion and notations which refer to [21–23].

Suppose $\lambda_1(q) < \lambda_2(q) \leq \lambda_3(q) \leq \cdots$ are all eigenvalues of the equation

$$-\Delta\phi + q(x)\phi = \lambda\phi, \quad \phi|_{\partial\Omega} = 0,$$

and $q(x) \in C(\overline{\Omega})$. And $\lambda_1(q)$ is simple and $\lambda_1(q)$ is strictly increasing which implies that $q_1 \leq q_2$ and $q_1 \not\equiv q_2$ can deduce $\lambda_1(q_1) < \lambda_1(q_2)$. Denote $\lambda_i(0)$ by λ_i, and the eigenfunction corresponding to λ_1 by ϕ_1 with normalization $\|\phi_1\|_\infty = 1$ and $\phi_1 > 0$ in Ω.

Consider the following problem

$$-\Delta u = \rho u - u^2, \quad u|_{\partial\Omega} = 0,$$

let θ_ρ ($\theta_\rho < \rho$) be unique positive solution as $\rho > \lambda_1$. It is easy to see that the mapping $\rho \to \theta_\rho$ is strictly increasing, continuously differentiable on $(\lambda_1, \infty) \to C^2(\Omega) \cap C_0(\overline{\Omega})$ such that $\theta_\rho \to 0$ uniformly on $\overline{\Omega}$ as $\rho \to \lambda_1$.

Hence, in the system (2) there exist two semi-trivial solutions $(\theta_a, 0)$ and $(0, \theta_c)$ when $a, c > \lambda_1$. Next, we present some a priori estimate by maximum principle (see [12, 24–28]). It's proof will be omitted.

Lemma 2.1. *Any coexistence state (u, v) of (2) has a priori boundary, i.e.,*

$$u \le a, \quad v \le B := c + \frac{ad}{1 + am}.$$

Next, we give the fixed point index theory (see [29]). Let E be a Banach space. $W \subset E$ is named a wedge if W is the closed convex set and $\beta W \subset W$ for all $\beta \ge 0$. For $y \in W$, let $W_y = \{x \in E : \exists r = r(x) > 0, \text{s.t. } y + rx \in W\}$, $S_y = \{x \in \overline{W}_y : -x \in \overline{W}_y\}$, and $E = \overline{W - W}$. Let $T : W_y \to W_y$ be a compact linear operator on E. We claim that T has property α on $\overline{W}_y$ if there exists $t \in (0, 1)$ and $\omega \in \overline{W}_y \setminus S_y$ such that $\omega - tT\omega \in S_y$. Define $B_\delta^+(y) = B_\delta(y) \cap W$, $\forall \delta > 0$, $y \in W$. Let $F : B_\delta^+(y) \to W$ be a compact operator and y be an isolated fixed point of F. Denote F' as the Fréchet differentiable at y, then $F'(y) : \overline{W}_y \to \overline{W}_y$. The fixed point index of F at y relative to W is denoted by $\text{index}_W(F, y)$ throughout this paper. Now we give a general result on the fixed point index theory with respect to the positive cone W (see [22, 23]).

Lemma 2.2 ([29, 30]). *Suppose that $I - L$ is invertible on $\overline{W}_y$. Then the following conclusions hold.*
(i) If L has property α on $\overline{W}_y$, then $\text{index}_W(F, y) = 0$.
(ii) If L does not have property α on $\overline{W}_y$, then $\text{index}_W(F, y) = (-1)^\sigma$, where σ is the sum of algebra multiplicities of the eigenvalue of L which are greater than 1.

Let $X = C_0^1(\overline{\Omega}) \oplus C_0^1(\overline{\Omega})$, where $C_0^1(\overline{\Omega}) = \{\omega \in C^1(\overline{\Omega}) : \omega|_{\partial\Omega} = 0\}$; $W = K \oplus K$, where $K = \{\varphi \in C(\overline{\Omega}) : \varphi(x) \ge 0\}$; $D := \{(u, v) \in X : u \le a + 1, v \le B + 1\}$; $D' := (\text{int}D) \cap W$.

Define $F_t : D' \to W$ as the following form:

$$F_t(u, v) = (-\Delta + P)^{-1} \begin{pmatrix} tu(a - u - \dfrac{bv}{1 + mu + u^2 + kv}) + Pu \\ tv(c - v + \dfrac{du}{1 + mu + u^2 + kv}) + Pv \end{pmatrix},$$

where $t \in [0, 1]$, $P > \max\{a + bB, c + \frac{da}{1+ma}\}$. By Maximum Principle, we obtain that $(-\Delta + P)^{-1}$ is a compact positive operator, F_t is complete continuous and Fréchet differentiable. Let $F_1 = F$, then (2) has a coexistence state in W if and only if $F_t = F_1$ has a positive fixed point in D'.

If $a > \lambda_1$ and $c > \lambda_1$, $(0, 0)$, $(\theta_a, 0)$ and $(0, \theta_c)$ are the non-negative fixed points of F. Hence, $\text{index}_W(F, (0, 0))$, $\text{index}_W(F, (\theta_a, 0))$ and $\text{index}_W(F, (0, \theta_c))$ are well defined. By calculating, we can get the Fréchet operator of F as follows:

$$(-\Delta + P)^{-1} \begin{pmatrix} a - 2u - \dfrac{bv(1 - u^2 + kv)}{(1 + mu + u^2 + kv)^2} + P & -\dfrac{bu(1 + mu + u^2)}{(1 + mu + u^2 + kv)^2} \\ \dfrac{dv(1 - u^2 + kv)}{(1 + mu + u^2 + kv)^2} & c - 2v + \dfrac{du(1 + mu + u^2)}{(1 + mu + u^2 + kv)^2} + P \end{pmatrix}.$$

Applying similar methods as in the proof of Lemma 2.1 and Lemma 2.2 in [12], we can easily establish the following Lemmas by the fixed point index theory (see [24–28]). We omit the proof procedure.

Lemma 2.3. *Assume $a > \lambda_1$.*
(i) $\deg_W(I - F, D') = 1$, here $\deg_W(I - F, D')$ is the degree of $T - F$ in D' on W.
(ii) If $c \ne \lambda_1$, then $\text{index}_W(F, (0, 0)) = 0$.
(iii) If $c > \lambda_1(-\frac{d\theta_a}{1+m\theta_a+\theta_a^2})$, then $\text{index}_W(F, (\theta_a, 0)) = 0$.
(iv) If $c < \lambda_1(-\frac{d\theta_a}{1+m\theta_a+\theta_a^2})$, then $\text{index}_W(F, (\theta_a, 0)) = 1$.

Lemma 2.4. *Assume $c > \lambda_1$.*
(i) If $a > \lambda_1(\frac{b\theta_c}{1+k\theta_c})$, then $\text{index}_W(F, (0, \theta_c)) = 0$.
(ii) If $a < \lambda_1(\frac{b\theta_c}{1+k\theta_c})$, then $\text{index}_W(F, (0, \theta_c)) = 1$.

In the following, some conditions on existence and non-existence of coexistence states of (2) are established by comparison principle.

Theorem 2.5. *(i) If $a \leq \lambda_1$, then (2) does't have any coexistence state; if $a \leq \lambda_1$ and $c \leq \lambda_1$, then (2) does't exist non-negative non-zero solution.*
(ii) If $c \leq \lambda_1$ and for (2) there exists a coexistence state, then $a > \lambda_1$, $c + \frac{da}{1+ma} > \lambda_1$.
(iii) If $c > \lambda_1$ and for (2) there exists a coexistence state, then $a > \lambda_1\left(\frac{b\theta_c}{1+m\theta_a+\theta_a^2+k\theta_c}\right)$.

Proof. (i) Suppose (u,v) is a coexistence state to (2), then (u,v) satisfies

$$-\Delta u = u\left(a - u - \frac{bv}{1+mu+u^2+kv}\right) \ x \in \Omega, \ u = 0 \ x \in \partial\Omega,$$

and $a = \lambda_1\left(u + \frac{bv}{1+mu+u^2+kv}\right)$. With the aid of the comparison principle, we get $a > \lambda_1$, a contradiction. We suppose that (u,v) is a non-negative non-zero solution of (2). Obviously, $u \not\equiv 0$ and $v \equiv 0$, then $a > \lambda_1$ by the above proof. Similarly, we can deduce $c > \lambda_1$ as $u \equiv 0$ and $v \not\equiv 0$, which derives a contradiction again.

(ii) Let (u,v) be a coexistence state of (2). By (i), we know that $a > \lambda_1$, and (2) has the positive semi-trivial solution θ_a. Due to

$$-\Delta u = u\left(a - u - \frac{bv}{1+mu+u^2+kv}\right) \leq u(a-u) \ x \in \Omega, \ u = 0 \ x \in \partial\Omega,$$

u is a lower solution to (2). Owing to the uniqueness of θ_a, $u \leq \theta_a$. It follows that v meets

$$-\Delta v = v\left(c - v + \frac{du}{1+mu+u^2+kv}\right) \ x \in \Omega, \ v = 0 \ x \in \partial\Omega,$$

then $0 = \lambda_1\left(-c + v - \frac{du}{1+mu+u^2+kv}\right) > \lambda_1\left(-c - \frac{da}{1+ma}\right)$, which gives the result.

(iii) Suppose (u,v) is a coexistence state of (2), then (2) has the unique positive solution θ_a with $u \leq \theta_a$. Similarly, $c > \lambda_1$ implies the existence of positive solution θ_c of (2) with $\theta_c \leq v$. According to the same method of (i), we directly obtain

$$a = \lambda_1\left(u + \frac{bv}{1+mu+u^2+kv}\right) > \lambda_1\left(\frac{b\theta_c}{(1+m\theta_a+\theta_a^2+k\theta_c)}\right).$$

Since the function $\frac{bv}{1+mu+u^2+kv}$ has a minimum at $u = \theta_a$ and $v = \theta_c$ (for $u \leq \theta_a$ and $v \geq \theta_c$). So the result holds. $\square$

Theorem 2.6. *(i) If $c > \lambda_1$ and $a > \lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right)$. Then (2) has at least a coexistence state.*
(ii) Suppose $c < \lambda_1$. Then (2) has a coexistence state if and only if $a > \lambda_1$ and $c > \lambda_1\left(-\frac{d\theta_a}{1+m\theta_a+\theta_a^2}\right)$.

Proof. (i) By Lemma 2.3-2.4 and the fixed point index theory, we have

$$\deg_W(I - F, D) - \text{index}_W(f, (0,0)) - \text{index}_W(f, (\theta_a, 0)) - \text{index}_W(f, (0, \theta_c)) = 1.$$

Then (2) has at least a coexistence state.

(ii) Firstly, we demonstrate the sufficiency. Due to $c < \lambda_1$, (2) does't have the solution of the form $(0, v)$ with $v > 0$. If $a > \lambda_1$ and $c > \lambda_1\left(-\frac{d\theta_a}{1+m\theta_a+\theta_a^2}\right)$, since $c < \lambda_1$, by Lemma 2.4 and the fixed point index theory, we get

$$\deg_W(I - F, D) - \text{index}_W(f, (0,0)) - \text{index}_W(f, (\theta_a, 0)) = 1.$$

Hence (2) has at least a coexistence state.

Conversely, we suppose that (u,v) is a coexistence state of (2). Then $a > \lambda_1$, and $u < \theta_a$. Thanks to (u,v) satisfies

$$-\Delta v = v\left(c - v + \frac{du}{1+mu+u^2+kv}\right) \ x \in \Omega, \ v = 0 \ x \in \partial\Omega.$$

Hence, $0 = \lambda_1\left(-c + v - \frac{du}{1+mu+u^2+kv}\right) > \lambda_1\left(-c - \frac{da}{1+ma}\right)$. $\square$

Theorem 2.7. *If one of the following two conditions holds, then (2) does't have any non-negative non-zero solution:*
(i) $b \geq 1 + ma + a^2 + kB$ and $a \leq c$;
(ii) $b < 1 + ma + a^2 + kB$ and $c - a \geq \left(1 - \frac{b}{(1+ma+a^2+kB)}\right)B$.

Proof. (i) Assume that (u, v) is a coexistence state of (2) as $b \geq 1 + ma + a^2 + kB$ and $a \leq c$. Note that $u(x) \leq a$ by Lemma 2.1 and $1 + mu(x) + u^2(x) + kv(x) > 0$ in $\bar{\Omega}$. we get

$$0 = \lambda_1 \left(-a + u + \frac{bv}{1 + mu + u^2 + kv} \right)$$

$$\geq \lambda_1 \left(-c + v - \frac{du}{1 + mu + u^2 + kv} - v + \frac{bv}{1 + mu + u^2 + kv} + u + \frac{du}{1 + mu + u^2 + kv} \right)$$

$$> \lambda_1 \left(-c + v - \frac{du}{1 + mu + u^2 + kv} - (1 - \frac{b}{1 + mu + u^2 + kv})v \right)$$

$$> \lambda_1 \left(-c + v - \frac{du}{1 + mu + u^2 + kv} - (1 - \frac{b}{1 + ma + a^2 + kB})v \right)$$

$$\geq \lambda_1 \left(-c + v - \frac{du}{1 + mu + u^2 + kv} \right).$$

Which deduces a contradiction.

(ii) Based on the proof of (i), thanks to the fact $b < 1 + ma + a^2 + kB$ and $c - a \geq (1 - \frac{b}{(1+ma+a^2+kB)})B$, we can obtain the following inequality:

$$0 = \lambda_1 \left(-a + u + \frac{bv}{1 + mu + u^2 + kv} \right)$$

$$> \lambda_1 \left(-c + v - \frac{du}{1 + mu + u^2 + kv} - a + c - (1 - \frac{b}{1 + ma + a^2 + kB})B \right)$$

$$\geq \lambda_1 \left(-c + v - \frac{du}{1 + mu + u^2 + kv} \right),$$

which is a contradiction. The proof is completed. □

3 Global bifurcation

The purpose of this section is to investigate a coexistence state bifurcates from the semi-trivial non-negative branch $\{(0, \theta_c, a)\}$ depending on the change of the parameter a and $c > \lambda_1$. Particularly, the structure of global bifurcation curve on coexistence states is discussed by using global bifurcation theorem and property of principal eigenvalue.

Throughout this paper, the principal eigenvalue of the following problem is denoted by $\tilde{a}$,

$$-\Delta\phi + \frac{b\theta_c}{1 + k\theta_c}\phi = a\phi, \quad \phi|_{\partial\Omega} = 0, \tag{3}$$

and the corresponding eigenfunction is denoted by Φ with $\|\Phi\|_\infty = 1$.

For convenience of the calculation, we do the following variable substitution, suppose $\omega = u$, $\chi = v - \theta_c$, then $0 \leq \omega \leq \theta_a$, $\chi \geq 0$, and ω, χ meet

$$\begin{cases} -\Delta\omega = (a - \frac{b\theta_c}{1+k\theta_c})\omega + F_1(\omega, \chi), & x \in \Omega, \\ -\Delta\chi = (c - 2\theta_c)\chi + \frac{d\theta_c}{1+k\theta_c}\omega + F_2(\omega, \chi), & x \in \Omega, \\ \omega = \chi = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

here

$$F_1(\omega, \chi) = \frac{b\omega\theta_c}{1 + k\theta_c} - \frac{b\omega(\chi + \theta_c)}{(1 + m\omega + \omega^2 + k(\chi + \theta_c))} - \omega^2,$$

$$F_2(\omega, \chi) = \frac{d\omega(\chi + \theta_c)}{(1 + m\omega + \omega^2 + k(\chi + \theta_c))} - \frac{d\omega\theta_c}{1 + k\theta_c} - \chi^2.$$

It is easy to see that $F = (F_1, F_2)$ is continuous, $F(0, 0) = 0$, and the Fréchet derivative $D_{(\omega,\chi)}F|_{(0,0)} = 0$. The inverse of $-\Delta$ is denoted by K. Then

$$\begin{cases} \omega = aK\omega - bK(\frac{\omega\theta_c}{1+k\theta_c}) + KF_1(\omega, \chi), & x \in \Omega, \\ \chi = cK\chi - 2K(\chi\theta_c) + dK(\frac{\omega\theta_c}{1+k\theta_c}) + KF_2(\omega, \chi), & x \in \Omega, \\ \omega = \chi = 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

Now we introduce the operator $T : R^+ \times X \to X$ as follows:

$$T(a; \omega, \chi) = \begin{pmatrix} aK\omega - bK(\frac{\omega\theta_c}{1+k\theta_c}) + KF_1(\omega, \chi) \\ cK\chi - 2K(\chi\theta_c) + dK(\frac{\omega\theta_c}{1+k\theta_c}) + KF_2(\omega, \chi) \end{pmatrix}.$$

Obviously, $T(a; \omega, \chi)$ is a compact operator on X. Define $G(a; \omega, \chi) = (\omega, \chi)^T - T(a; \omega, \chi)$, then G is continuous, and $G(a; 0, 0) = 0$. $G(a; \omega, \chi) = 0$ $(0 \le \omega \le \theta_a, \chi \ge 0)$ if and only if $(\omega, \chi + \theta_c, a)$ is a positive solution of (2).

Theorem 3.1. *Suppose $c > \lambda_1$. Then there exists a coexistence state of (4) (or (2)) bifurcate from the point $(\tilde{a}; 0, \theta_c)$, and $\tilde{a} = \lambda_1(\frac{b\theta_c}{1+k\theta_c})$.*

Remark 3.2. *The proof of Theorem 3.1 refers to Theorem 2 in [19], more precisely, the bifurcation branch near $\tilde{a} = \lambda_1(\frac{b\theta_c}{1+k\theta_c})$ is determined by C^1 continuous curve $(a(s); \phi(s), \varphi(s)) : (-\delta, \delta) \to R \times Z$, for some $\delta > 0$ such that*

$$a(0) = \tilde{a}, \phi(0) = 0, \varphi(0) = 0, \text{ and } (a(s); \omega(s), \chi(s)) = (a(s); (\Phi + \phi(s)), s(\Psi + \varphi(s)))$$

meets $G(a(s); \omega(s), \chi(s)) = 0$, here $X = Z \oplus \text{span}\{(\Phi, \Psi)\}$. Thus $(a(s); U(s), V(s))$ $(|s| < \delta)$ is a bifurcation solution of (4) (or (2)), here $U(s) = s(\Phi + \phi(s))$, $V(s) = \theta_c + s(\Psi + \varphi(s))$, $\Psi = (-\Delta - c + 2\theta_c)^{-1}(\frac{d\theta_c}{1+k\theta_c}\Phi)$.

If $0 < s < \delta$, we can deduce that the positive solution of (2) nearby $(\tilde{a}; 0, \theta_c)$ lies either on the branch $\{(a; 0, \theta_c) : a \in R^+\}$ or on the branch $\{(a(s); U(s), V(s)) : 0 < s < \delta\}$.

Define $T : X \times R \to X$ as the compact continuously differentiable operator, and $T(0, a) = 0$. Let T be $T(u, a) = K(a)u + R(u, a)$, here $K(a)$ is a linear compact operator and the Fréchet derivative $R_u(0, 0) = 0$. Suppose x_0 is an isolated fixed point of T, we denote the index of T at x_0 by $\text{index}(T, x_0) = \deg(I - T, U_\delta(x_0), x_0)$, here $U_\delta(x_0)$ is a ball with center at x_0, and x_0 is the unique fixed point of T in $U_\delta(x_0)$. If $I - T'(x_0)$ is invertible, then x_0 is an isolated fixed point of T, meanwhile, $\text{index}(T, x_0) = \deg(I - T, U_\delta(x_0), x_0) = \deg(I - T'(x_0), U_\delta(x_0), 0)$.

If $x_0 = 0$, then $\deg(I - K(a), U_\delta(x_0), 0) = (-1)^\sigma$, where σ is equal to the algebraic multiplicities of the eigenvalue of K which is greater than one.

Next, we study the structure of global bifurcation curve on coexistence states, we will extend the local bifurcation solution $\{(a(s); U(s), V(s)) : 0 < s < \delta\}$ established by Theorem 3.1 to the global bifurcation branch. For this purpose, we define the following notation: $P_1 = \{u \in C_0^1(\bar{\Omega}) : u(x) > 0, x \in \Omega, \frac{\partial u}{\partial n} < 0, x \in \partial\Omega\}$, $P = \{(u, v, a) \in X \times R^+ : u, v \in P_1\}$.

Theorem 3.3. *If $c > \lambda_1$, then the local bifurcation solution $\{(a(s); U(s), V(s)) : 0 < s < \delta\}$ can be extended to the global solution which is denoted by C and unbounded by going to infinity in P.*

Proof. Define

$$T'(a) \cdot (\omega, \chi) = D_{(\omega, \chi)} T(a; 0, 0) \cdot (\omega, \chi)$$
$$= (aK\omega - bK\frac{\omega\theta_c}{1+k\theta_c}), cK\chi - 2K(\chi\theta_c) + dK\frac{\omega\theta_c}{1+k\theta_c}).$$

Let $\mu \ge 1$ be an eigenvalue of $T'(a)$. Then

$$\begin{cases} -\mu\Delta\omega = (a - \frac{b\theta_c}{1+k\theta_c})\omega, & x \in \Omega, \\ -\mu\Delta\chi = (c - 2\theta_c)\chi + \frac{d\theta_c}{1+k\theta_c}\omega, & x \in \Omega, \\ \omega = \chi = 0, & x \in \partial\Omega. \end{cases}$$

Obviously, $\omega \not\equiv 0$, otherwise, $\omega \equiv 0$, due to the fact that all eigenvalues of $(-\mu\Delta - c + 2\theta_c)$ are greater than 0, so $\chi \equiv 0$, a contradiction. Thus, $a = a_i(\mu)$ is the eigenvalue of the following problem

$$-\mu\Delta\omega + \frac{b\theta_c}{1+k\theta_c}\omega = a\omega, \quad \omega|_{\partial\Omega} = 0.$$

So $a_i(\mu)$ is increasing along with μ on $[1, +\infty)$ and can be ordered by

$$0 < a_1(\mu) < a_2(\mu) \le a_3(\mu) \le \cdots \to \infty, \quad a_1(1) = \tilde{a}.$$

Conversely, if $\mu \ge 1$, we demonstrate that all eigenvalues of $(-\mu\Delta - c + 2\theta_c)$ are greater than 0, then $\chi = (-\mu\Delta - c + 2\theta_c)^{-1}(\frac{d\theta_c}{1 + k\theta_c}\omega)$. Hence, $\mu \ge 1$ is an eigenvalue of $T'(a)$ if and only if $a = a_i(\mu)$ for some $i = 1, 2, \ldots$.

If $a < \tilde{a}$, then $a < a_1(1) \le a_i(\mu)$ for any $\mu \ge 1, i \ge 1$. It follows that $T'(a)$ has no eigenvalue greater than 1, meanwhile, index$(T(a; \cdot), 0) = 1$ for $a < \tilde{a}$.

If $\tilde{a} < a < a_2(1)$, then $a < a_i(\mu)$. Since $a_1(1) = \tilde{a}$ for any $\mu \ge 1, i \ge 2$, $\lim\limits_{\mu \to \infty} a_1(\mu) = +\infty$, and $a_1(\mu)$ is increasing along with μ. Hence, there exists a unique $\mu_1 > 1$ such that $a = a_1(\mu_1)$. Thus $N(\mu_1 I - T'(a)) = \text{span}\{(\overline{\omega}, \overline{\chi})\}$, $\dim N(\mu_1 I - T'(a)) = 1$, here $\overline{\omega} > 0$ is the principal eigenvalue of the following equation

$$\mu_1 \Delta \overline{\omega} + (a - \frac{b\theta_c}{1 + k\theta_c})\overline{\omega} = 0, \quad \overline{\omega}|_{\partial\Omega} = 0, \tag{6}$$

here $\overline{\chi} = (-\mu_1\Delta - c + 2\theta_c)^{-1}(\frac{d\theta_c}{1 + k\theta_c}\overline{\omega})$.

Next, we shall demonstrate that $R(\mu_1 I - T'(a)) \cap N(\mu_1 I - T'(a)) = 0$. As a matter of fact, suppose the assertion is false , we may assume that $(\overline{\omega}, \overline{\chi}) \in R(\mu_1 I - T'(a))$. Then there exists $(\omega, \chi) \in X$ such that $(\mu_1 I - T'(a))(\omega, \chi) = (\overline{\omega}, \overline{\chi})$, i.e.,

$$\mu_1 \Delta \omega + (a - \frac{b\theta_c}{1 + k\theta_c})\omega = \Delta\overline{\omega}, \quad \omega|_{\partial\Omega} = 0.$$

Multiplying the above equation by $\overline{\omega}$, and integrating over Ω, by Green's formula, we have

$$\int_\Omega \overline{\omega}\Delta\overline{\omega} = \int_\Omega (\mu_1\Delta\omega + a\omega - \frac{b\omega\theta_c}{1 + k\theta_c})\overline{\omega}$$

$$= \int_\Omega (\mu_1\Delta\overline{\omega} + a\overline{\omega} - \frac{b\overline{\omega}\theta_c}{1 + k\theta_c})\omega = 0.$$

Thus $\int_\Omega \frac{1}{\mu_1}(a - \frac{b\theta_c}{1 + k\theta_c})\overline{\omega}^2 = 0$, a contradiction, which has proved the claim. Then the multiplicity of μ_1 is one and index$(T(a; \cdot), 0) = -1$ for $\tilde{a} < a < a_2(1)$. With the aid of global bifurcation theory [23], we deduce that there exists a continuum C_0 of zero points of $G(a; \omega, \chi) = 0$ in $R^+ \times X$ bifurcating from $(\tilde{a}; 0, 0)$ and all zero points of $G(a; \omega, \chi)$ nearby $(\tilde{a}; 0, 0)$ lie on the curve for which the existence was established by Theorem 3.1.

Define the maximal continuum C_1 as $C_1 = C_0 - \{(a(s); s(\Phi + \phi(s)), s(\Psi + \varphi(s))) : -\delta < s < 0\}$, then C_1 consists of the curve $\{(a(s); s(\Phi + \phi(s)), s(\Psi + \varphi(s))) : -\delta < s < 0\}$ in the neighborhood of $(\tilde{a}; 0, 0)$. Suppose $\tilde{C} = \{(a; u, v) : U = \omega, V = \theta_c + \chi, (\omega, \chi) \in C_1\}$. It follows that $\tilde{C}$ is the solution branch of (2) which bifurcates from $(\tilde{a}; 0, \theta_c)$ and keeps the positive near $(\tilde{a}; 0, \theta_c)$ and $\tilde{C} \subset P$. Thus, the continuum $\tilde{C} - \{(\tilde{a}; 0, \theta_c)\}$ must satisfy one of the three alternatives:

(i) joining up with a bifurcation point of the form $(\bar{a}; 0, \theta_c)$, which $I - T'(\bar{a})$ is not invertible, $\tilde{a} \ne \bar{a}$.
(ii) joining up from $(\tilde{a}; 0, \theta_c)$ to ∞ in $R \times X$.
(iii) containing points of the form $(a; u, \theta_c + v)$ and $(a; -u, \theta_c - v)$, here $(u, v) \ne (0, 0)$.

Now we claim that $\tilde{C} - \{(\tilde{a}; 0, \theta_c)\} \subset P$. Suppose $\tilde{C} - \{(\tilde{a}; 0, \theta_c)\} \not\subset P$, it follows that there exists $(\hat{a}; \hat{u}, \hat{v}) \in (\tilde{C} - \{(\tilde{a}; 0, \theta_c)\}) \cap \partial P$ and the sequence $\{(a_n; u_n, v_n)\} \subset \tilde{C} \cap P$, $u_n > 0$, $v_n > 0$ such that $(a_n; u_n, v_n) \to (\hat{a}; \hat{u}, \hat{v})$ as $n \to \infty$. We can obtain that $\hat{u} \in \partial P_1$ or $\hat{v} \in \partial P_1$. If $\hat{u} \in \partial P_1$, then $\hat{u} \ge 0, x \in \overline{\Omega}$. Thus, we find either $x_0 \in \Omega$ such that $\hat{u}(x_0) = 0$ or $x_0 \in \partial\Omega$ such that $\frac{\partial \hat{u}}{\partial n}|_{x_0} = 0$. Since $\hat{u}$ satisfies

$$-\Delta\hat{u} = (\hat{a} - \hat{u} - \frac{b\hat{v}}{(1 + m\hat{u} + \hat{u}^2 + k\hat{v})})\hat{u}, \quad \hat{u}|_{\partial\Omega} = 0.$$

By maximum principle, we can get $\hat{u} \equiv 0$. Similarly, we can deduce that $\hat{v} \equiv 0$ for $\hat{v} \in \partial P_1$.

Thus, we will investigate the following three cases:

$$\text{(i) } (\hat{u}, \hat{v}) \equiv (\theta_{\hat{a}}, 0); \quad \text{(ii) } (\hat{u}, \hat{v}) \equiv (0, \theta_c); \quad \text{(iii) } (\hat{u}, \hat{v}) \equiv (0, 0).$$

(i) If $(\hat{u}, \hat{v}) \equiv (\theta_{\hat{a}}, 0)$, then $(a_n; u_n, v_n) \to (\hat{a}; \theta_{\hat{a}}, 0)$ as $n \to \infty$. Set $V_n = \frac{v_n}{\|v_n\|_\infty}$, then V_n meets

$$-\Delta V_n = (c - v_n + \frac{du_n}{(1 + mu_n + u_n^2 + kv_n)})V_n, \quad V_n|_{\partial\Omega} = 0. \tag{7}$$

With the aid of L^P estimates and Sobolev embedding theorem, we deduce that there exists a convergent subsequence of V_n, which be still denoted by V_n, it follows that $V_n \to V$ in $C_0^1(\overline{\Omega})$ as $n \to \infty$, and $V \geq 0, \not\equiv 0, x \in \Omega$. Taking the limit in (7) as $n \to \infty$, we get

$$-\Delta V = (c + \frac{d\theta_{\hat{a}}}{1 + m\theta_{\hat{a}} + \theta_{\hat{a}}^2})V, \quad V|_{\partial\Omega} = 0.$$

By the maximum principle, we get $V > 0, x \in \Omega$, which implies $c = \lambda_1\left(-\frac{d\theta_{\hat{a}}}{1+m\theta_{\hat{a}}+\theta_{\hat{a}}^2}\right)$. A contradiction to $c > \lambda_1$.

(ii) If $(\hat{u}, \hat{v}) \equiv (0, \theta_c)$. Then $(a_n; u_n, v_n) \to (\hat{a}; 0, \theta_c)$ as $n \to \infty$. Set $U_n = \frac{u_n}{\|u_n\|_\infty}$, hence, U_n meets

$$-\Delta U_n = (a_n - u_n - \frac{bv_n}{(1 + mu_n + u_n^2 + kv_n)})U_n, \quad U_n|_{\partial\Omega} = 0. \tag{8}$$

Similarly, using L^P estimates and Sobolev embedding theorem, we can get a convergent sub-sequence of U_n, which be still denoted by U_n, then $U_n \to U$ in $C_0^1(\overline{\Omega})$ as $n \to \infty$, and $U \geq 0, \not\equiv 0, x \in \Omega$. Taking limit in (8) by $n \to \infty$, we get

$$-\Delta U = (\hat{a} - \frac{b\theta_c}{1 + k\theta_c})U, \quad U|_{\partial\Omega} = 0.$$

Due to maximum principle, we get $U > 0, x \in \Omega$. Thus $\hat{a} = \lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right)$. A contradiction to $\hat{a} \neq \tilde{a}$.

(iii) If $(\hat{u}, \hat{v}) \equiv (0, 0)$, then similarly as in the method from the case above, we also arrive at a contradiction. Hence, $\tilde{C} - \{(\tilde{a}; 0, \theta_c)\} \subset P$. It follows from Lemma 2.1 that $0 \leq U \leq a$, $\theta_c \leq V \leq c + \frac{da}{1+\alpha a}$. With the aid of L^P estimates and Sobolev embedding theorem, we prove that there exists a constant $M > 0$ such that $\|U\|_{C^1}, \|V\|_{C^1} \leq M$. Then, the global bifurcation solution branch $\tilde{C}$ of positive solutions of (2) bifurcating at $(\tilde{a}; 0, \theta_c)$ contains points with a arbitrarily large in P. □

4 Stability and multiplicity

The purpose of this section is to investigate the stability and multiplicity of coexistence states of (2) by means of eigenvalue perturbation theory and the fixed point index theory.

Set $X_1 = [C^{2,\alpha}(\bar{\Omega}) \times C^{2,\alpha}(\bar{\Omega}) \cap X]$, $Y = [C^\alpha(\bar{\Omega}) \times C^\alpha(\bar{\Omega})]$, here $0 < \alpha < 1$. $i : X_1 \to Y$ is the inclusion mapping. Since L_1 is the linearized operator at $(\tilde{a}; 0, \theta_c)$ of (2), according to the proof of Theorem 3.1, we get $N(L_1) = \text{span}\{(\Phi, \Psi)\}$, Codim $R(L_1) = 1$, and $R(L_0) = \{(u, v) \in X : \int_\Omega u\Phi dx = 0\}$. Due to $i(\Phi, \Psi) \in R(L_1)$, it follows from [31] that 0 is an i-simple eigenvalue of L_1.

Lemma 4.1. 0 *is the eigenvalue of L_1 with the largest real part, and all the other eigenvalues of L_1 lie in the left half complex plane.*

Proof. The proof of Lemma 4.1 can be found in [9, 18, 24, 26], here we omit it. □

Let $L(u(s), v(s), a(s))$ and $L(a; 0, \theta_c)$ be the linearized operators of (2) at $(u(s), v(s), a(s))$, $(a; 0, \theta_c)$, respectively. Applying the linearized stability theory (see [31, 32]), we can obtain the following result.

Lemma 4.2. *There exists C^1-function: $a \to (M(a), \gamma(a))$ and $s \to (N(s), \pi(s))$, which are defined by the mapping from the neighborhood of $\tilde{a}$ and 0 into $X_1 \times R$, respectively, satisfying the forms $\gamma(\tilde{a}) = \pi(0) = 0$, $M(\tilde{a}) = N(0) = (\Phi, \Psi)$ and*

$$L(a; 0, \theta_c)M(a) = \gamma(a)M(a), \text{ for } |a - \tilde{a}| \ll 1,$$

$$L(u(s), v(s), a(s))N(s) = \pi(s)N(s), \text{ for } |s| \ll 1,$$

where $M(a) = (\phi_1(a), \phi_2(a))$, $N(s) = (\varphi_1(a)\varphi_2(a))$. *Meanwhile,* $\gamma'(\tilde{a}) \neq 0$, *furthermore, for* $|s| \ll 1$, $\pi(s) \neq 0$, $\pi(s)$ *and* $-sa'(s)\gamma'(\tilde{a})$ *have the same sign, where* $\gamma'(\tilde{a})$ *is the derivative of* $\gamma(a)$ *at* $a = \tilde{a}$, *and* $a'(s)$ *is the derivative of* $a(s)$ *on the point s.*

Lemma 4.3. $\gamma'(\tilde{a}) > 0$.

Proof. By calculating $L(a; 0, \theta_c)M(a) = \gamma(a)M(a)$, for $|a - \tilde{a}| \ll 1$, we have

$$\begin{cases} \Delta\phi_1 + (a - \frac{b\theta_c}{1+k\theta_c})\phi_1 = \gamma(a)\phi_1, & x \in \Omega, \\ \Delta\phi_2 + (c - 2\theta_c)\phi_2 + \frac{d\theta_c}{1+k\theta_c}\phi_1 = \gamma(a)\phi_2, & x \in \Omega, \\ \phi_1 = \phi_2 = 0, & x \in \partial\Omega. \end{cases}$$

Note that $|a - \tilde{a}| \ll 1$, then $|\gamma(a)| \ll 1$. Obviously, $\phi_1 \not\equiv 0$, otherwise, $\phi_1 \equiv 0$, then $\phi_2 \equiv 0$, a contradiction. Thus $\gamma(a)$ is an eigenvalue of $(\Delta + (a - \frac{b\theta_c}{1+k\theta_c})I)$. So $\Phi > 0$, implies $\phi_1 = \phi_1(a) > 0$ as $|a - \tilde{a}| \ll 1$. It follows that $\gamma(a)$ is the principal eigenvalue of $(\Delta + (a - \frac{b\theta_c}{1+k\theta_c})I)$, and $\gamma(a)$ is increasing along with a for $|a - \tilde{a}| \ll 1$. Meanwhile, $\gamma'(\tilde{a}) \neq 0$. Thus $\gamma'(\tilde{a}) > 0$. □

Lemma 4.4. $a'(0)$ *satisfies*

$$a'(0) \int_\Omega \Phi^2 dx = \int_\Omega (1 - \frac{bm\theta_c}{(1+k\theta_c)^2})\Phi^3 dx + \int_\Omega \frac{b\Psi}{(1+k\theta_c)^2}\Phi^2 dx.$$

Proof. By substituting $(u(s), v(s), a(s))$ into (2), differentiating on s, and let $s = 0$, we get

$$-\Delta\phi'(0) = (\tilde{a} - \frac{b\theta_c}{1+k\theta_c})\phi'(0) + [a'(0) - \Phi - b\frac{\Psi - m\theta_c\Phi}{(1+k\theta_c)^2}]\Phi,$$

where $\phi'(0)$ is the derivative of ϕ on the point $s = 0$.

Multiplying the above equation by Φ, and applying Green's formula and the definition of Φ, we obtain

$$a'(0) \int_\Omega \Phi^2 dx = \int_\Omega (1 - \frac{bm\theta_c}{(1+k\theta_c)^2})\Phi^3 dx + \int_\Omega \frac{b\Psi}{(1+k\theta_c)^2}\Phi^2 dx. \qquad \square$$

Taking the advantage of Lemma 4.1-Lemma 4.4, we directly derive the following theorem.

Theorem 4.5. *Let* $\sigma = \int_\Omega (1 - \frac{bm\theta_c}{(1+k\theta_c)^2})\Phi^3 dx + \int_\Omega \frac{b\Psi}{(1+k\theta_c)^2}\Phi^2 dx$. *If* $\sigma > 0$, *then local bifurcation solution* $(u(s), v(s))$ *is stable; if* $\sigma < 0$, *then local bifurcation solution* $(u(s), v(s))$ *is unstable.*

Remark 4.6. *In section 2, by the sufficient condition and necessary condition on coexistence states of* (2) *established in Theorem 2.5-2.6, we find that there exists a gap between* $a > \lambda_1(\frac{b\theta_c}{1+k\theta_c})$ *and* $a > \lambda_1(\frac{b\theta_c}{(1+m\theta_a+\theta_a^2+k\theta_c)})$ *as* $c > \lambda_1$.

In the following, we shall investigate the multiplicity of coexistence states in the gap.

Theorem 4.7. *Suppose that* $c > \lambda_1$ *and* $\int_\Omega (1 - \frac{bm\theta_c}{(1+k\theta_c)^2})\Phi^3 dx < 0$. *If there exists a sufficiently small* $\varepsilon > 0$ *and* $d \ll 1$, *then coexistence state* $(u(s), v(s))$ *is non-degenerate and unstable for* $a \in (\tilde{a} - \varepsilon, \tilde{a})$, *moreover,* (2) *has at least two coexistence states.*

Proof. Firstly, we will demonstrate the coexistence state $(u(s), v(s))$ is non-degenerate and unstable. To this end, we only need to prove that there exists a sufficiently small $\varepsilon > 0$ such that any coexistence state $(u(s), v(s))$ of (2) is non-degenerate for $a \in (\tilde{a} - \varepsilon, \tilde{a})$, and the linearized eigenvalue problem

$$\begin{cases} -\Delta\xi - [a(s) - 2u(s) - \frac{bv(s)(1-u(s)^2+kv(s))}{(1+mu(s)+u(s)^2+kv(s))^2}]\xi + \frac{bu(s)(1+mu(s)+u(s)^2)}{(1+mu(s)+u(s)^2+kv(s))^2}\eta = \mu\xi, & x \in \Omega, \\ -\Delta\eta - [c - 2v(s) + \frac{du(s)(1+mu(s)+u(s)^2)}{(1+mu(s)+u(s)^2+kv(s))^2}]\eta - \frac{dv(s)(1-u(s)^2+kv(s))}{(1+mu(s)+u(s)^2+kv(s))^2} = \mu\eta, & x \in \Omega, \qquad (9) \\ \xi = \eta = 0, & x \in \partial\Omega \end{cases}$$

has a unique eigenvalue μ_* and $\text{Re}(\mu_*) < 0$ with algebra multiplicity one.

Let $\{\varepsilon_i > 0\}$ and $\{d_i > 0\}$ be sequences which converge to 0 by $i \to \infty$. Owing to $a = \tilde{a} + a'(0)s + O(s^2)$, we can get the sequence $\{\varepsilon_i > 0\}$ and $\{a_i\}$ yield $a_i \in (\tilde{a} - \varepsilon_i, \tilde{a})$ and $s_i \to 0$ as $i \to \infty$. It is easy to see that (u_i, v_i) is one solution of (2). Hence, the corresponding linearized problem (9) can be written in the following form:

$$L_i \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} = \mu_i \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} \text{ and } L_i = \begin{pmatrix} M_i^{11} & M_i^{12} \\ M_i^{21} & M_i^{22} \end{pmatrix},$$

where $(\xi_i, \eta_i) \not\equiv (0, 0)$ and

$$M_i^{11} = -\Delta - [a_i - 2u_i - \frac{bv_i(1-u_i^2+kv_i)}{(1+mu_i+u_i^2+kv_i)^2}], \qquad M_i^{12} = \frac{bu_i(1+mu_i+u_i^2)}{(1+mu_i+u_i^2+kv_i)^2},$$

$$M_i^{21} = -\frac{dv_i(1-u_i^2+kv_i)}{(1+mu_i+u_i^2+kv_i)^2}, \qquad M_i^{22} = -\Delta - [c - 2v_i + \frac{du_i(1+mu_i+u_i^2)}{(1+mu_i+u_i^2+kv_i)^2}].$$

Observe that, as $i \to \infty$, $L_i \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$ converges to

$$L_0 \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -\Delta\xi - (\tilde{a} - \frac{b\theta_c}{1+k\theta_c})\xi & 0 \\ 0 & -\Delta\eta - (c - 2\theta_c)\eta \end{pmatrix}.$$

It follows that 0 is a simple eigenvalue of L_0 with the corresponding eigenfunction $(\xi, \eta)^T = (\Phi, 0)^T$. Meanwhile, all the other eigenvalues of L_0 are positive and keep away from 0. Furthermore, using perturbation theory [31, 33], for large i, the operator L_i has a unique eigenvalue μ_i which is close to zero. Moreover, all other eigenvalues of L_i have positive real parts and keep away from 0. Because μ_i is simple real eigenvalue which tends to zero, we denote the corresponding eigenfunction by $(\xi_i, \eta_i)^T$ which satisfies $(\xi_i, \eta_i) \to (\Phi, 0)$ as $i \to \infty$.

Now we claim that $\text{Re}\mu_i < 0$ for large i. Multiplying Φ on the first equation of $L_i(\xi_i, \eta_i)^T = \mu_i(\xi_i, \eta_i)^T$ and integrating over Ω, we obtain

$$-\int_\Omega \Phi\Delta\xi_i - \int_\Omega (a_i - 2u_i - \frac{bv_i(1 - u_i^2 + kv_i)}{(1 + mu_i + u_i^2 + kv_i)^2})\Phi\xi_i + \int_\Omega \frac{bu_i(1 + mu_i + u_i^2)\Phi\eta_i}{(1 + mu_i + u_i^2 + kv_i)^2} = \int_\Omega \mu_i\Phi\xi_i. \quad (10)$$

Multiplying two sides of the first equation of (2) with $(a, u, v) = (a_i, u_i, v_i)$ by ξ_i and integrating, we obtain

$$-\int_\Omega \xi_i\Delta u_i - \int_\Omega (a_i - u_i - \frac{bv_i}{1 + mu_i + u_i^2 + kv_i})u_i\xi_i = 0.$$

Taking $u_i = s_i\Phi + O(s_i^2)$ into the above equation we have

$$-\int_\Omega \Phi\Delta\xi_i - \int_\Omega \xi_i\Phi(a_i - u_i - \frac{bv_i}{1 + mu_i + u_i^2 + kv_i}) + O(s_i^2) = 0. \quad (11)$$

By combining (10) and (11), we obtain

$$\int_\Omega [1 - \frac{bv_i(m + 2u_i)}{(1 + mu_i + u_i^2 + kv_i)^2}]u_i\Phi\xi_i + \int_\Omega \frac{bu_i(1 + mu_i + u_i^2)\Phi\eta_i}{(1 + mu_i + u_i^2 + kv_i)^2} = \int_\Omega \mu_i\Phi\xi_i.$$

Notice that $(u_i, v_i) = (s_i\Phi + O(s_i^2), \theta_c + s_i\Psi_{d_i} + O(s_i^2))$, where Ψ_{d_i} stands for Ψ defined in Remark 3.2. Thus, dividing the above equation by s_i and taking the limit, we get

$$\lim_{i\to\infty} \frac{\mu_i}{s_i} = \int_\Omega (1 - \frac{bm\theta_c}{(1 + k\theta_c)^2})\Phi^3 / \int_\Omega \Phi^2 < 0,$$

which results in $\text{Re}\mu_i < 0$ for large i. Hence, our claim is illustrated.

Finally, using similar method as in [25], we will demonstrate the remaining part of Theorem 4.7. In order to apply reduction to absurdity, we suppose that there exists a unique coexistence state $(\tilde{u}, \tilde{v})$ for (2), then $(\tilde{u}, \tilde{v})$ must be bifurcated from $(0, \theta_c)$, since there exists a coexistence state near $\tilde{a}$. Thus, $(\tilde{u}, \tilde{v})$ is non-degenerate,

and the corresponding linearized problem has a unique eigenvalue $\tilde{\mu}$ with algebra multiplicity one which satisfies $Re\tilde{\mu} < 0$. The above fact implies that $I - F'(\tilde{u}, \tilde{v})$ is invertible and does not have property α on $\overline{W}_{(\tilde{u},\tilde{v})}$, hence index$(F, (\tilde{u}, \tilde{v})) = (-1)^1 = -1$ by Lemma 2.2(ii). Thus, according to Lemma 2.3-2.4 and the fixed point index theory, we get

$$1 = \text{index}_W(F, D)) = \text{index}_W(F, (0, 0)) + \text{index}_W(F, (\theta_a, 0))$$

$$+\text{index}_W(F, (0, \theta_c)) + \text{index}_W(F, (\tilde{u}, \tilde{v})) = 0 + 0 + 1 - 1 = 0,$$

which leads to a contradiction. Thus, the proof is completed. □

Remark 4.8. *The proof of Theorem 4.7 implies that the multiplicity of* (2) *can be obtained easily as $m \geq 0$. Due to $a'(0) < 0$ for a sufficiently small d, and $\int_\Omega (1 - \frac{bm\theta_c}{(1+k\theta_c)^2})\Phi^3 < 0$, it is easy to see that $a = a(s) \in (\lambda_1, \tilde{a})$. From Theorem 2.5(i), we know that there is no coexistence state of* (2) *if $a \leq \lambda_1(\frac{b\theta_c}{(1+m\theta_a+\theta_a^2+k\theta_c)})$ as $m \geq 0$, $c > \lambda_1$. As a result, we have demonstrated that there exist at least two coexistence states for $a \in (a^*, \tilde{a})$ and some $a^* \in (\lambda_1(\frac{b\theta_c}{(1+m\theta_a+\theta_a^2+k\theta_c)}), \lambda_1(\frac{b\theta_c}{1+k\theta_c}))$.*

Finally, we study the stability of any positive solutions (if exists) as the parameter a, b, c belong to some domain.

Theorem 4.9. *If $a > \lambda_1$, $c > \lambda_1$, then there exists some sufficiently small $\tilde{B} > 0$ such that any coexistence state of* (2) *(if exists) is non-degenerate and stable for $b \leq \tilde{B}$.*

Proof. Let $\tilde{v}$ be a positive solution of the following equation

$$-\Delta v = v(b - v - \frac{d\theta_a}{1 + m\theta_a + l\theta_a^2 + kv}) \, x \in \Omega, \quad v|_{\partial\Omega} = 0.$$

Hence, we need only demonstrate that the linearized problem of (2) has no the real part of eigenvalue. By reduction to absurdity, we suppose that the positive solution (u_i, v_i) of (2) is either non-degenerate or unstable. Suppose the sequence b_i meets $b_i \to 0$ with $i \geq 1$, then there exists μ_i such that $Re\mu_i \leq 0$, and the corresponding eigenfunction $(\xi_i, \eta_i) \neq (0, 0)$ with $\|\xi_i\|^2 + \|\eta_i\|^2 = 1$ satisfying the following linearized problem

$$\begin{cases} -\Delta\xi_i - [a - 2u_i - \frac{bv_i(1-u_i^2+kv_i)}{(1+mu_i+u_i^2+kv_i)^2}]\xi_i + \frac{bu_i(1+mu_i+u_i^2)}{(1+mu_i+u_i^2+kv_i)^2}\eta_i = \mu\xi_i, & x \in \Omega, \\ -\Delta\eta_i - [c - 2v_i + \frac{du_i(1+mu_i+u_i^2)}{(1+mu_i+u_i^2+kv_i)^2}]\eta_i - \frac{dv_i(1-u_i^2+kv_i)}{(1+mu_i+u_i^2+kv_i)^2} = \mu\eta_i, & x \in \Omega, \\ \xi_i = \eta_i = 0, & x \in \partial\Omega \end{cases} \quad (12)$$

Multiplying two sides of (12) with ξ_i, η_i), respectively, and integrating on Ω, it follows from Divergence theorem that

$$\mu_i = \int_\Omega |\nabla\xi_i|^2 dx - \int_\Omega [a - 2u_i - \frac{bv_i(1 - u_i^2 + kv_i)}{(1 + mu_i + u_i^2 + kv_i)^2}]|\xi_i|^2 dx + \int_\Omega \frac{bu_i(1 + mu_i + u_i^2)}{(1 + mu_i + u_i^2 + kv_i)^2}\eta_i\bar{\xi}_i dx$$

$$+ \int_\Omega |\nabla\eta_i|^2 dx - \int_\Omega [c - 2v_i + \frac{du_i(1 + mu_i + u_i^2)}{(1 + mu_i + u_i^2 + kv_i)^2}]|\eta_i|^2 dx - \int_\Omega \frac{dv_i(1 - u_i^2 + kv_i)}{(1 + mu_i + u_i^2 + kv_i)^2}\xi_i\bar{\eta}_i dx,$$

where $\bar{\xi}_i$ and $\bar{\eta}_i$ are the complex conjugates of ξ_i and η_i. According to the above equation, we can deduce that $Re(\mu_i)$ and $Im(\mu_i)$ are bounded, so we suppose that $\mu_i \to \mu$ with $Re(\mu) \leq 0$. Meanwhile, $(\xi_i, \eta_i) \to (\xi, \eta)$, $b_i \to 0$ as $i \to \infty$. Then (12) converges to

$$\begin{cases} -\Delta\xi - (a - 2\theta_a)\xi = \mu\xi, & x \in \Omega, \\ -\Delta\eta - [c - 2\tilde{v} + \frac{d\theta_a(1+m\theta_a+\theta_a^2)}{(1+m\theta_a+\theta_a^2+k\tilde{v})^2}]\eta_i - \frac{d\tilde{v}(1-\theta_a^2+k\tilde{v})}{(1+m\theta_a+\theta_a^2+k\tilde{v})^2} = \mu\eta, & x \in \Omega. \\ \xi = \eta = 0, & x \in \partial\Omega \end{cases} \quad (13)$$

It follows that μ must be real number with $\mu \leq 0$. Suppose $\xi \not\equiv 0$, one can deduce that μ is an eigenvalue of the equation

$$-\Delta\phi + (-a + 2\theta_a)\phi = \mu\phi, \; x \in \Omega, \; \phi|_{\partial\Omega} = 0,$$

Then $\mu \geq \lambda_1(-a + 2\theta_a)$, since $\lambda_1(-a + 2\theta_a) > \lambda_1(-a + \theta_a) = 0$, hence $\mu \leq 0$, a contradiction.

Suppose $\xi \equiv 0$, then $\eta \not\equiv 0$. By the second equation of (13), we get

$$-\Delta\eta - \Big[c - 2\tilde{v} + \frac{d\theta_a(1 + m\theta_a + \theta_a^2)}{(1 + m\theta_a + \theta_a^2 + k\tilde{v})^2}\Big]\eta_i = \mu\eta, x \in \Omega.\eta|_{\partial\Omega} = 0.$$

Observe that $\eta \not\equiv 0$, we obtain $\mu = \lambda_1\big[-c + 2\tilde{v} - \frac{d\theta_a(1+m\theta_a+\theta_a^2)}{(1+m\theta_a+\theta_a^2+k\tilde{v})^2}\big]$. Set $g(x, v) = c - v + \frac{d\theta_a}{(1+m\theta_a+\theta_a^2+k\tilde{v})}$, then $g_v < 0$. Hence,

$$\lambda_1(-g(x, \tilde{v})) < \lambda_1(-g(x, \tilde{v}) - \lambda_1(-\tilde{v}g_v(x, \tilde{v}))) = \lambda_1\Big[-c + 2\tilde{v} - \frac{d\theta_a(1 + m\theta_a + \theta_a^2)}{(1 + m\theta_a + \theta_a^2 + k\tilde{v})^2}\Big] = \mu.$$

Notice that $\lambda_1(-g(x, \tilde{v})) = 0$, so $\mu > 0$, a contradiction. □

5 Conclusion

This paper considers dynamic behavior of the prey-predator model with ratio-dependent Monod-Haldane response function under homogeneous Dirichlet boundary conditions. We come to the following conclusion. Firstly, the sufficient and necessary conditions on existence and non-existence of coexistence states of (2) are proved by the fixed point index theory, see Theorems 2.5, 2.6 and 2.7, which determine the conditions for the coexistence of two species of organisms. Secondly, taking a as a main bifurcation parameter, the structure of global bifurcation curve on coexistence states is established by global bifurcation theorem and property of principal eigenvalue, see Theorems 3.1 and 3.3, which show the global coexistence state of two species of organisms by controlling the change of the parameter a. Finally, the stability of coexistence states is obtained by the eigenvalue perturbation theory, see Theorems 4.5 and 4.9; the multiplicity of coexistence states to (2) is obtained when $a \in (a^*, \tilde{a})$ by the fixed point index theory, see Theorem 4.7, which show that two species of organisms have two coexistence states when the parameter satisfying some condition.

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