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Shaohui Liang*

The points and diameters of quantales

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Abstract: In this paper, we investigate some properties of points on quantales. It is proved that the two sided prime elements are in one to one correspondence with points. By using points of quantales, we give the concepts of p-spatial quantales, and some equivalent characterizations for P-spatial quantales are obtained. It is shown that two sided quantale Q is a spatial quantale if and only if Q is a P-spatial quantale. Based on a quantale Q, we introduce the definition of diameters. We also prove that the induced topology by diameter coincides with the topology of the point spaces.

Keywords: Quantale, Point, P-spatial, Diameter, Topology

MSC: 06F07, 54A05

1 Introduction

Quantale was proposed by Mulvey in 1986 to study the foundations of quantum logic and non-commutative C*-algebras. The term quantale was coined as a combination of quantum logic and locale by Mulvey in [1]. Since quantale theory provides a powerfull tool in studying non-commutative structure, it has a wide range of applications, especially in studying linear logic which supports part of the foundation of theoretical computer science [2-5]. It is known that quantales are one of the semantics of linear logic, a logic system developed by Girard [6]. A systematic introduction of quantale theory can be found in book [7] written by Rosenthal in 1990. Following Mulvey, quantale theory has had a wide range of applications in computer science, logic, topological, category, C*-algebras, fuzzy theory, roughness theory and so on [8-27].

A point in topological space is considered to be a point space embedded in a space, which corresponds to a mapping from local to binary chain. The analogous term in commutative C*-algebras is a non-trivial representation onto C, which is a pure state of the algebra. The points of quantales are studied mainly in the context of C*-algebras [28]. A very precise way how to transfer properties of C*-algebras into terms of quantales was introduced by Mulvey and Pelletier. In paper [29-31] Pultr extended the metric structure to pontless spaces (frames or locales). The classical distance function is being replaced by the notion of diameter satisfying certain properties. Some good properties of diameter in locales are given. In this paper, we discuss the points and diameters of quantales. For the notions and concepts not explained in this paper, refer to [32].

This paper is organized as follows: In Section 2, we review some facts about quantales and category theory, which are needed in the sequel. In Section 3, we discuss some properties of points in quantales. We prove that the set of all completed files is isomorphic to the set of all points of quantales. The definition of P-spatial is given and some equivalent characterizations for P-spatial quantales are obtained. In Section 4, the definition of diameters of quantales is given. We prove that the induced topology by diameter coincides with the topology of the point spaces. In Section 5, finally, we give a summary of the paper.

***Corresponding Author: Shaohui Liang:** Department of Mathematics, Xi'an University of Science and Technology, Xi'an 710054, China, E-mail: Liangshaohui1011@163.com

2 Preliminaries

Definition 2.1 ([2]). *A quantale is a complete lattice Q with an associative binary operation "&" satisfying: $a\&(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a\&b_i)$ and $(\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a)$ for all $a, b_i \in Q$, where I is a set, 0 and 1 denote the smallest element and the greatest element of Q, respectively. An element $r \in Q$ is called right-sided if $r\&1 \le r$. Similarly, $l \in Q$ is called left-sided if $1\&l \le l$. $q \in Q$ is called two-sided if $q\&1 \le q$, $1\&q \le q$. The sets of right and left-sided elements will be denoted by $R(Q), L(Q)$.*

A subset K of Q is called a subquantale if K is closed under $\vee$ and &. Let Q and P, the function $f : Q \to P$ is called a homomorphism of quantales if f preserves $\vee$ and &.

Definition 2.2 ([13,14,26]). *Let Q be a quantale. A nonempty subset $I \subseteq Q$ is called an ideal of Q if it satisfies the following conditions:*
(i) $a \vee b \in I$ for all $a, b \in I$;
(ii) for all $a, b \in Q$, if $a \in I$ and $b \le a$, then $b \in I$;
(iii) for all $a \in Q$ and $x \in I$, we have $a\&x \in I$ and $x\&a \in I$.
Let Idl(Q) denote the set of all ideals of Q.

Definition 2.3 ([13,14,26]). *Let Q be a quantale. An ideal I of Q is called:*
(i) a prime ideal if $I \neq Q$ and for all $a, b \in Q$, $a\&b \in I$ implies $a \in I$ or $b \in I$;
(ii) a semi-prime ideal if $a\&a \in I$ implies $a \in I$ for all $a \in Q$;
(iii) a primary ideal if $I \neq Q$, and for all $a, b \in Q$, $a\&b \in I$, $a \notin I$ imply $b^n \in I$ for some $n > 0$.

Definition 2.4 ([13,14]). *Let Q be a quantale. A nonempty subset $F \subseteq Q$ is called a file of Q if it satisfies the following conditions:*
(i) $0 \notin F$;
(ii) for all $a, b \in Q$, if $a \in F$ and $a \le b$, then $b \in F$;
(iii) for all $a, b \in F$, we have $a\&b \in F$.
Let Fil(Q) denote the set of all files of Q.

Definition 2.5 ([7]). *Let Q be a quantale. A file F of Q is called a prime file if $F \neq Q$ and for all $a, b \in Q$, $a \vee b \in F$ implies $a \in F$ or $b \in F$.*

Definition 2.6 ([7]). *Let Q be a quantale. An element $p \in Q$, $p \neq 1$ is said to be a prime if $r\&l \le p \Rightarrow r \le p$ or $l \le p$ for all $r, l \in Q$. The set of all primes elements is denoted by $P_r(Q)$.*

Theorem 2.7 ([7]). *A quantale is spatial iff every element is a meet of primes.*

Theorem 2.8 ([30]). *If A and B are categories, then a functor F from A to B is a function that assigns to each A-object A and a B-object $F(A)$, to each A-morphism $f : A \to A'$ and a B morphism $F(f) : F(A) \to F(A')$, in such a way that:*
(i) $F(f \circ g) = F(f) \circ F(g)$; (ii) $F(id_A) = id_{F(A)}$.

3 The points of quantales

Definition 3.1 ([3,7]). *Let Q and $2 = (\{0, 1\}, \&)$ be quantales, a binary multiplication "&" of 2 satisfying: $1\&1 = 1$, $0\&1 = 1\&0 = 0\&0 = 0$. A mapping $f : Q \to 2$ is said to be a point if f is a epimorphism of quantales. Let Pt(Q) denote the set of all points of Q.*

Remark 3.2. *Let $x \in Q$, $\Sigma_x = \{p \in Pt(Q) \mid p(a) = 1\}$.*
(1) $\Sigma_{a\&b} = \Sigma_a \cap \Sigma_b$, $\Sigma_0 = \varnothing$, $\Sigma_1 = Pt(Q)$, $\Sigma_{\bigvee_{i\in I} a_i} = \bigcup_{i\in I} \Sigma_{a_i}$.
(2) $\forall\ p \in Pt(Q)$, then $p^{-1}(1)$ is a prime filter and $p^{-1}(0)$ is a prime ideal of Q. It is easy to prove that $\bigvee p^{-1}(0) \in Pr(Q)$, $p^{-1}(0) = \downarrow(\bigvee p^{-1}(0))$.
(3) Define a mapping $f : TPr(Q) \longrightarrow Pt(Q)$, $r \longmapsto p_r(x) = \begin{cases} 0, & x \le r, \\ 1, & otherwise, \end{cases}$ where $TPr(Q)$ denotes the set of all two sided prime elements of Q. It is easy to prove that f is a bijection.
(4) Let $x \in Q$, and $x \in p^{-1}(1)$, then $x^T = \bigvee\{y \in Q \mid x\&y = 0\} \in p^{-1}(0)$.

Theorem 3.3. *Let Q be a quantale, then:*
(1) $\psi : Q \longrightarrow P(Pt(Q))$, $\forall\ x \in Q$, $\psi(x) = \Sigma_x = \{p \in Pt(Q) \mid p(x) = 1\}$ is a morphism of quantale.
(2) The set $\{\Sigma_x \mid x \in Q\}$ is a topology of $Pt(Q)$, denoted by Σ_Q.

Proof. (1) It is easy to verify that $(P(Pt(Q)), \cap)$ is a quantale and the function ψ preserves $\vee$.

Let $\varnothing \neq S \subseteq Q$, $p \in Pt(Q)$, then $p \in \psi(\bigvee S) \Longleftrightarrow p(\bigvee S) = 1 \Longleftrightarrow \bigvee\{p(s) \mid s \in S\} = 1 \Longleftrightarrow \exists\ s_0 \in S,\ p(s_0) = 1 \Longleftrightarrow p \in \cup\{\psi(s) \mid s \in S\}$, we have $\psi(\vee S) = \cup\{\psi(s) \mid s \in S\}$.

Let $p \in Pt(Q)$, $\forall\ a, b \in Q$, then $p \in \psi(a\&b) \Longleftrightarrow p(a\&b) = 1 \Longleftrightarrow p(a)\&p(b) = 1 \Longleftrightarrow p(a) = 1,\ p(b) = 1 \Longleftrightarrow p \in \psi(a),\ p \in \psi(b) \Longleftrightarrow p \in \psi(a) \cap \psi(b)$, We have $\psi(a\&b) = \psi(a) \cap \psi(b)$.

(2) It is easy to prove by definition of topology. □

Theorem 3.4. *Let Q be a quantale, then:*
(1) $\varphi : Q \longrightarrow P(TPr(Q))$, $\forall\ x \in Q$, $\varphi(x) = \{r \in Pr(Q) \mid x \nleq r\}$ is a homomorphism of quantale.
(2) The set $\{\varphi(x) \mid x \in Q\}$is a topology of $TPr(Q)$, denoted by $\Omega(TPr(Q))$.

Proof. Obviously, $(P(Pr(Q)), \cap)$ is a quantale, the function φ preserves the empty set.

Let $\varnothing \neq S \subseteq Q$, $r \in Pr(Q)$, then $r \in \varphi(\bigvee S) \Longleftrightarrow \bigvee S \nleq r \Longleftrightarrow \exists\ s \in S,\ s \nleq r \Longleftrightarrow r \in \cup\{\varphi(s) \mid s \in S\}$, that is $\varphi(\bigvee S) = \cup\{\varphi(s) \mid s \in S\}$.

Let $r \in Pr(Q)$, $\forall\ a, b \in Q$, we have $r \in \varphi(a\&b) \Longleftrightarrow a\&b \nleq r \Longleftrightarrow a \nleq r, b \nleq r \Longleftrightarrow r \in \varphi(a),\ r \in \varphi(b) \Longleftrightarrow r \in \varphi(a) \cap \varphi(b)$, that is $\varphi(a\&b) = \varphi(a) \cap \varphi(b)$. Therefore φ is a quantale homorphism.

(2) It is easy to prove by definition of topology. □

Definition 3.5. *Let Q be a quantale, the topology space $(Pt(Q), \Sigma_Q)$ is called points space on Q.*

Lemma 3.6. *Let $h : Q \longrightarrow K$ be a quantale epimorphism, the map $\Sigma(h) : (Pt(K), \Sigma_K) \longrightarrow (Pt(Q), \Sigma_Q)$, $\forall\ f \in Pt(K)$, $\Sigma(h)(f) = f \circ h$, then $\forall\ x \in Q$, such that $(\Sigma(h))^{-1}(\Sigma_x) = \Sigma_{h(x)}$.*

Proof. Since $(\Sigma(h))(f) \in \Sigma_x \Leftrightarrow (\Sigma(h))(f)(x) = 1 \Leftrightarrow f \in \Sigma_{h(x)}$, then $f \circ h(x) = 1$. □

Lemma 3.7. *Let $h : Q \longrightarrow K$ be a quantale epimorphism, then $\Sigma(h) : (Pt(K), \Sigma_K) \longrightarrow (Pt(Q), \Sigma_Q)$ is a continuous map.*

Let $Quant_e$ denote the category of quantales and epimorphisms, Top^{op} denote the dual category of topology spaces and continuous maps.

Define a map

$$\begin{aligned} &\Sigma : Quant_e \longrightarrow Top^{op} \\ &Q \longrightarrow (Pt(Q), \Sigma_Q) \\ &h : Q \longrightarrow K \longrightarrow \Sigma(h) : (Pt(K), \Sigma_K) \longrightarrow (Pt(Q), \Sigma_Q) \\ &f \longmapsto f \circ h. \end{aligned}$$

Proof. It is easy to be verified that Σ is well defined by Lemmas 3.6 and 3.7.

(1) $\forall\ f \in Pt(Q)$, $\forall\ x \in Q$, we have $\Sigma(id_Q)(f)(x) = f \circ id_Q(x) = f(x) = id_{\Sigma_{Pt(Q)}}(f)(x)$;

(2) Let $s : Q \longrightarrow P$ and $t : P \longrightarrow H$ be quantale epimorphisms, then $\forall f \in \Sigma_H$, $\Sigma(t \circ s)(f) = f \circ t \circ s = (\Sigma(s) \circ \Sigma(t))(f) = (\Sigma(t) \circ^{op} \Sigma(s))(f)$. □

Definition 3.8. *Let Q be a quantale, F is a file of Q. The file F is called completed file of Q if it satisfies the following condition: $\bigvee A \in F \Rightarrow A \cap F \neq \varnothing$. Let $CFil(Q)$ denote the set of all completed files of Q.*

Theorem 3.9. *Let Q be a two sided quantale. We define a map $\varphi : CFil(Q) \longrightarrow Pt(Q)$ such that $\forall F \in CFil(Q)$, $\forall x \in Q$, $\varphi(F)(x) = \begin{cases} 0, x \notin F, \\ 1, x \in F, \end{cases}$ and $\psi : Pt(Q) \longrightarrow CFil(Q)$ such that $\forall f \in Pt(Q)$, $\psi(f) = \{x \in Q \mid f(x) = 1\}$, then $\varphi \circ \psi = id_{CFil(Q)}$, $\psi \circ \varphi = id_{Pt(Q)}$.*

Proof. $\forall x, y \in Q$, $\forall F \in CFil(Q)$. Obviously, $\varphi(F)$ is onto map.

(i) $\varphi(F)(x\&y) = \begin{cases} 0, & x\&y \notin F, \\ 1, & x\&y \in F, \end{cases} = \begin{cases} 0, & x \notin F, y \notin F, \\ 1, & x \in F, \ y \in F, \end{cases} = \varphi(F)(x)\&\varphi(F)(y)$.

(ii) $\varphi(F)(\bigvee\limits_{i\in I} x_i) = \begin{cases} 0, & \bigvee\limits_{i\in I} x_i \notin F, \\ 1, & \bigvee\limits_{i\in I} x_i \in F, \end{cases} = \begin{cases} 0, & \forall\, i \in I, x_i \notin F, \\ 1, & \exists\, x_i \in F, \end{cases} = \bigvee\limits_{i\in I} \varphi(F)(x_i)$.

Therefore the map φ is well defined.

$\forall f \in Pt(Q)$, $\psi(f) = \{x \in Q \mid f(x) = 1\}$, we have

(i) $0 \notin \psi(f)$;

(ii) $\forall x_1, x_2 \in \psi(f)$, then $f(x_1\&x_2) = f(x_1)\&f(x_2) = 1\&1 = 1$, thus $x_1\&x_2 \in \psi(f)$;

(iii) Let $x \in \psi(f)$, $y \in Q$, and $x \leq y$, then $1 = f(x) \leq f(y)$, that is $f(y) = 1$;

(iv) $\forall A \subseteq \psi(f)$, and $\bigvee\limits_{a\in A} a \in \psi(f)$, then $f(\bigvee\limits_{a\in A} a) = \bigvee\limits_{a\in A} f(a) = 1$, there exists $a_0 \in A$, such that $f(a_0) = 1$, thus $a_0 \in \psi(f)$, that is $A \cap \psi(f) \neq \varnothing$.

Hence, $\psi(f) \in CFil(Q)$, the map $\psi(f)$ is well defined.

In the following, we will prove $\psi \circ \varphi = id_{CFil(Q)}$ and $\varphi \circ \psi = id_{Pt(Q)}$.

$\forall F' \in CFil(Q)$, then $\psi \circ \varphi(F') = \psi(\varphi(F')) = \{x \in Q \mid \varphi(F')(x) = 1\}$, $\forall x \in \{x \in Q \mid \varphi(F')(x) = 1\}$, that is $\varphi(F')(x) = 1$, then $x \in F'$, therefore $\psi \circ \varphi(F') \subseteq F'$. $\forall x \in F'$, then $\varphi(F')(x) = 1$, that is $F' \subseteq \psi \circ \varphi(F')$, therefore $\psi \circ \varphi = id_{CFil(Q)}$.

Conversely, $\forall f' \in Pt(Q)$, $\forall x \in Q$, then $\varphi\circ\psi(f')(x) = \begin{cases} 0, x \notin \psi(f'), \\ 1, x \in \psi(f'), \end{cases} = \begin{cases} 0, & f'(x) = 0, \\ 1, & f'(x) = 1, \end{cases} = f'(x)$. Therefore $\varphi \circ \psi = id_{Pt(Q)}$. □

Theorem 3.10. *Let Q be a two sides quantale. $\forall x \in Q$. Define $\Sigma'_x = \{F \in CFil(Q) \mid x \in F\}$, then $(CFil(Q), \Sigma'_Q = \{\Sigma'_x \mid x \in Q\})$ is a topological spaces.*

Proof. (i) Since $\Sigma'_0 = \varnothing$, $\Sigma'_1 = CFil(Q)$, then $\varnothing, CFil(Q) \in \{\Sigma'_x \mid x \in Q\}$;

(ii) $\Sigma'_{x\&y} = \{F \in CFil(Q) \mid x\&y \in F\} = \{F \in CFil(Q) \mid x \in F, \ y \in F\} = \Sigma'_x \cap \Sigma'_y$;

(iii) $\Sigma'_{\bigvee\limits_{i\in I} x_i} = \{F \in CFil(Q) \mid \bigvee\limits_{i\in I} x_i \in F\} = \bigvee\limits_{i\in I} \bigcup\{F \in CFil(Q) \mid x_i \in F\} = \bigcup\limits_{i\in I} \Sigma'_{x_i}$.

Therefore $\Sigma'_Q = \{\Sigma'_x \mid x \in Q\}$ is a topology on $CFil(Q)$.

Let $TQuant$ denote the category of the two sides quantales and quantale homomorphisms.

Define

$$\begin{aligned} &\Sigma' : TQuant \longrightarrow Top^{op} \\ &Q \longrightarrow (CFil(Q), \Sigma'_Q) \\ &f : Q \longrightarrow K \longrightarrow \Sigma'(f) : (CFil(K), \Sigma'_K) \longrightarrow (CFil(Q), \Sigma'_Q) \\ &F \longmapsto f^{-1}(F). \quad \square \end{aligned}$$

Theorem 3.11. *The map $\Sigma' : Quant \longrightarrow Top^{op}$ is a functor.*

Proof. Obviously, the map Σ' is well defined. In what follows, we will prove that the map $\Sigma'(f) : \Sigma'(Y) \to \Sigma'(X)$ is a quantale homomorphism with f is a contionous map.

(i) $\forall\ U_1, U_2 \in \Omega(Y)$, then $\Omega(f)(U_1 \cap U_2) = f^{-1}(U_1 \cap U_2) = f^{-1}(U_1) \cap f^{-1}(U_2) = \Omega(f)(U_1) \cap \Omega(f)(U_2)$;

(ii) $\forall\ \{U_i\}_{i\in I} \subseteq \Omega(Y)$, then $\Omega(f)(\underset{i\in I}{\cup} U_i) = f^{-1}(\underset{i\in I}{\cup} U_i) = \underset{i\in I}{\cup} f^{-1}(U_i) = \underset{i\in I}{\cup} \Omega(f)(U_i)$,

Next, we check that the map Σ' preserves unit element and composition.

(i) $\forall\ X \in ob(Top)$, then $\Omega(id_X)(U) = id_X^{-1}(U) = U = id_{\Omega(X)}(U)$;

(ii) For any continous map $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, $\forall\ U \in \Omega(X)$, then $\Omega(g \circ f)(U) = (g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U) = \Omega(f) \circ \Omega(g)(U) = \Omega(g) \circ^{op} \Omega(f)(U)$. □

Definition 3.12. *Let Q be a quantale. Q is called P-spatial quantale or sufficient points if the quantale epimorphisms $\psi : Q \longrightarrow \Sigma_Q,\ x \longmapsto \Sigma_x$ is a injective function for all $x \in Q$.*

Now, we shall give some charaterizations of P-spatial quantales. It is easy to prove that two sided quantale is spatial if and only if Q is p-spatial.

Theorem 3.13. *Let Q be a quantale. Then the following statements are equivalent:*
(1) Q is a P-spatial quantale;
(2) if $a, b \in Q$, $a \nleq b$, then there exists an element $p \in Pt(Q)$, such that $p(a) = 1,\ p(b) = 0$;
(3) if $a, b \in Q$, $a \nleq b$, then there exists an elemen $r \in Pr(Q)$, such that $a \nleq r,\ b \leq r$;
(4) for all $a \in Q$, $a = \wedge\{p \in Q \mid a \leq p, p \in Pr(Q)\}$.

Proof. (1)⇒ (2) Let $a, b \in Q$, $a \nleq b$. Since Q is a P-spatial quantale, that is the map $\psi : Q \longrightarrow \Sigma_Q$ is a one to one map, then $a \nleq b$ implies $\psi(a) \nsubseteq \psi(b)$, there exists $p \in Pt(Q)$ such that $p \in \psi(a)$, $p \notin \psi(b)$. Thus $p(a) = 1, p(b) = 0$.

(2) ⇒ (3) Let $a, b \in Q$, $a \nleq b$. By (2), we have that there exists $p \in Pt(Q)$, that is epimorphisms $p : Q \longrightarrow 2$ such that $p(a) = 1,\ p(b) = 0$. By Remark 3.2, we have that $\vee p^{-1}(0) \in Pr(Q)$, and $p^{-1}(0) = \downarrow (\vee p^{-1}(0))$, then $a \notin p^{-1}(0),\ b \in p^{-1}(0)$, therefore $a \nleq \vee p^{-1}(0),\ b \leq \vee p^{-1}(0)$.

(3) ⇒ (1) Let $a, b \in Q$, $a \neq b$. By (3), we have that there exists $r \in Pr(Q)$, such that $a \nleq r,\ b \leq r$. By Remark 3.2, we have that $\vee p_r^{-1}(0) = r$, and $p_r^{-1}(0) = \downarrow (\vee p_r^{-1}(0)) = \downarrow r$. Since $a \nleq r,\ b \leq r$, then $a \notin p_r^{-1}(0)$, $b \in p_r^{-1}(0)$, that is $p_r(a) = 1,\ p_r(b) = 0$. Therefore $p_r \in \psi(a),\ p_r \notin \psi(b)$, that is $\psi(a) \neq \psi(b)$, then $\psi : Q \longrightarrow \Sigma_Q$ is a injection map. Thus Q is a P-spatial quantale.

(3) ⇒ (4) Let $a \in Q$, denote $a^* = \wedge\{r \in Pr(Q) \mid a \leq r\}$, then $a \leq a^*$. Suppose $a \neq a^*$, then there exists $r^* \in TPr(Q)$, such that $a^* \nleq r^*$, $a \leq r^*$ by (3). Since $r^* \in TPr(Q)$, and $a \leq r^*$, then $a^* \leq r^*$, this contradicts the assumption that $a^* \nleq r^*$. Thus $a = a^*$.

(4) ⇒ (3) Let $a, b \in Q$, $a \neq b$, by (4) we have $b = \wedge\{r \in Pr(Q) \mid b \leq r\}$, then there exists $r^* \in \{r \in Pr(Q) \mid b \leq r\}$, such that $a \nleq r^*$, otherwise $\forall\ r \in \{r \in Pr(Q) \mid b \leq r\}$, then $a \leq r$, thus $a \leq \{r \in Pr(Q) \mid b \leq r\} = b$, this contradicts the assumption that $a \nleq b$, therefore $a \nleq r^*,\ b \leq r^*$. □

Theorem 3.14. *Let Q be a quantale, K is a frame, $f : Q \longrightarrow K$ is left adjoint of $g : K \longrightarrow Q$, $(f \dashv g)$. If K is a spatial frame (P-spatial quantale) and f is a onto map, the map f is a quantale homomorphism if and only if g preserve the prime elements.*

Proof. It is easy to verify that f preserves arbitrary sups. Next we check if f preserves the operation &. Since K is spatial frame, then for all $x, y \in K$, we have

$$\begin{aligned} f(x\&y) &= \bigwedge\{r \in Pr(k) \mid f(x\&y) \leq r\}, \\ &= \bigwedge\{r \in Pr(k) \mid x\&y \leq g(r)\}, \\ &= \bigwedge(\{r \in Pr(k) \mid x \leq g(r)\} \cup \{r \in Pr(k) \mid y \leq g(r)\}), \\ &= \bigwedge(\{r \in Pr(k) \mid f(x) \leq r\} \cup \{r \in Pr(k) \mid f(y) \leq r\}), \\ &= f(x) \wedge f(y). \end{aligned}$$

Hence f is a quantale homomorphism.

Conversely, let f be a quantale homomorphism, $r \in Pr(K)$, then $g(r) \neq 1$. Otherwise, $g(r) = 1$, then $r \geq f \circ g(r) = f(1) = 1$. Therefore $r = 1$, this is in contradiction with $r \in Pr(K)$.

Let $x, y \in Q$, and $x\&y \leq g(r)$. Since $f \dashv g$, then $f(x)\&f(y) = f(x\&y) \leq r$. Because K is a prime element of K, then $f(x) \leq r$ or $f(y) \leq r$. Therefore $g(r)$ is a prime element of Q, that is the map g preserves prime elements. □

4 The diameters of quantale

In this section, we introduce the concept of diametric quantale, which extends the metric structure to quantale. The classical distance function is here being replaced by the notion of diameter satisfying certain properties. We give the respective characterizations in the terms of diametric quantales.

Definition 4.1. *The set C is called a conver of q quantale Q provided that $C \subseteq Q$ and $\vee C = 1$.*

Definition 4.2. *A diameter on a quantale Q is a mapping $d : Q \longrightarrow R^+ \cup \{0\}$ (where $R^+ \cup \{0\}$ is the set of nonegative reals) satisfying:*
(i) $d(0) = 0$;
(ii) $a \leq b \Rightarrow d(a) \leq d(b)$;
(iii) $a\&b \neq 0 \Rightarrow d(a \vee b) \leq d(a) + d(b)$;
(iv) $\forall\ \epsilon > 0$, the set $U_\epsilon^d = \{x \in Q \mid d(x) < \epsilon\}$ is a cover of Q.

Sometimes we will drop condition (iv). In such a case, we will note that d is a prediameter.

Definition 4.3.
($*$) A diameter is said to be a star diameter if for each $x \in Q$, and $Y \subseteq Q$ such that $\forall\ y \in Y$, $x\&y \neq 0$, then $d(x \vee \bigvee Y) \leq d(x) + Sup\{d(s) + d(t) \mid s, t \in Y,\ s \neq t\}$.
(M) A diameter is said to be a metric diameter if for each $x \in Q$, $\forall\ \epsilon > 0$, there are $y, z \in Q$ such that $y, z \leq x$, and $d(y), d(z) < \epsilon,\ d(x) < d(y \vee z) + \epsilon$.

It is easy to check that (M) implies ($*$). If C is a cover of a quantale Q and $x \in Q$, we denote $Cx = \bigvee\{c \in C \mid c\&x \neq 0\}$.

Remark 4.4. *(1) Let $\{x_i\}_{i\in I}$ be a family of quantale Q, C is a cover of a quantale Q, then $C_{\bigvee\limits_{i\in I} x_i} = \bigvee\limits_{i\in I} C_{x_i}$.*
(2) Let C be a cover of a quantale Q. Defined $C : Q \longrightarrow Q$ for $\forall\ x \in Q,\ C(x) = Cx$, then C preserves $\vee$.

Definition 4.5. *Let Q be a quantale, d be a diameter of Q, $\mu(d) = \{U_\epsilon^d \mid \epsilon > 0,\ \epsilon \in R\}$ is a family covers of Q. Let α_ϵ^d be the right adjoin of U_ϵ^d. If $\forall\ x \in Q$, $x = \bigvee\limits_{\epsilon>0} \alpha_\epsilon^d x$. We say that the diameter d is compatible.*

If $x \geq \bigvee\limits_{\epsilon>0} \alpha_\epsilon^d x$ for $x \in Q$, we say that d is week compatile.

Let $f : Q \longrightarrow P$ be a quantale epimorphism, d is prediameter of Q. We define $\overline{d}(y) = inf\{d(x) \mid y \leq f(x)\}$.

Theorem 4.6. *Let $f : Q \longrightarrow P$ be a quantale epimorphism, d is a prediameter (diameter, $*$-diameter), then $\overline{d}$ is a prediameter (diameter, $*$-diameter).*

Proof. Firstly, we prove $\overline{d}$ is a prediameter of P.

(i) By the definition of $\overline{d}$, we have that $\overline{d}(0) = inf\{d(x) \mid 0 \leq f(x)\} = 0$;

(ii) Let $a, b \in Q$, and $a \leq b$, then $\overline{d}(a) = inf\{d(x) \mid a \leq f(x)\}$, $\overline{d}(b) = inf\{d(y) \mid b \leq f(y)\}$. Since $a \leq b \leq f(y)$, then $\overline{d}(a) \leq \overline{d}(b)$.

(iii) Let $a, b \in P$, and $a\&b \neq 0$, then $\overline{d}(a \vee b) = inf\{d(x) \mid a \vee b \leq f(x)\}$, $\overline{d}(a) = inf\{d(y) \mid a \leq f(y)\}$, $\overline{d}(b) = inf\{d(z) \mid b \leq f(z)\}$.

Since $inf\{d(y) \mid a \le f(y)\} + inf\{d(z) \mid b \le f(z)\} = inf\{d(y) + d(z) \mid a \le f(y),\ b \le f(z)\}$, d is a prediameter of P, and $a \le f(y), b \le f(z)$. $0 \neq a\&b$, $a\&b \le f(y\&z)$, then $f(y\&z) \neq 0$, and $y\&z \neq 0$. We have $d(y \vee z) \le d(y) + d(z)$, $a \vee b \le f(y) \vee f(z) = f(y \vee z)$. Thus $\overline{d}(a \vee b) \le \overline{d}(a) + \overline{d}(b)$.

If d is a diameter of Q, we can prove that $U_\epsilon^d = \{x \in Q \mid \overline{d}(x) < \epsilon\}$ for $\epsilon > 0$ is a cover of Q. Since $f(1) = 1$, then $f(\bigvee U_\epsilon^d) = 1$, and $d(y) < \epsilon$, $\overline{d}(f(y)) < \epsilon$. Hence $1 = f(\bigvee U_\epsilon^d) = \bigvee f(U_\epsilon^d) \le \bigvee U_\epsilon^{\overline{d}}$, that is $\bigvee U_\epsilon^{\overline{d}} = 1$.

Secondly, we can show that $\overline{d}$ is a $*$-diameter when d is a $*$-diameter of P.

Let $a \in P$, $B \subseteq P$, and $\forall\ b \in B$, $a\&b \neq 0$, then $\overline{d}(a) = inf\{d(x) \mid a \le f(x)\}$, $\overline{d}(b) = inf\{d(y) \mid b \le f(y)\}$. By the definition of infimum, for all $\epsilon > 0$, there exist $x_a, x_b \in Q$, such that $d(x_a) < \overline{d}(a) + \epsilon$, $d(x_b) < \overline{d}(b) + \epsilon$, $a \le f(x_a)$, $b \le f(x_b)$, then $0 \neq a\&b \le f(x_a\&x_b)$, $a \vee \bigvee B \le f(x_a \vee (\bigvee_{b\in B} x_b))$, Therefore $\overline{d}(a \vee \bigvee B) \le d(x_a \vee (\bigvee_{b\in B} x_b)) \le d(x_a) + Sup\{d(x_s) + d(x_t) \mid s \neq t,\ s, t \in B\} < \overline{d}(a) + Sup\{\overline{d}(s) + \overline{d}(t) \mid s \neq t,\ s, t \in B\} + 3\epsilon$. Thus $\overline{d}$ is a $*$-diameter of P. □

Theorem 4.7. *Let Q be a quantale, d is a diameter on Q, $f : Q \longrightarrow P$ is a epimorphism. For all $p \in P$, $\forall\ \epsilon > 0$, we define $\Phi_\epsilon(p) = \{q \in Q \mid q \in U_\epsilon^d,\ f(q)\&p \neq 0\}$, $\phi_\epsilon(p) = \bigvee \Phi_\epsilon(p)$. We have:*
(1) Let Q be a idempotent quantale, and $\forall\ q \in U_\epsilon^d, f(q) \neq 0$, then $\phi_\epsilon(f(q)) \ge q$;
(2) $\phi_\epsilon(A) = \bigvee_{a\in A} \phi_\epsilon(a)$ for all $A \subseteq P$;
(3) If $p \le f(q)$, then $\phi_\epsilon(p) \le U_\epsilon^d(q)$.

Proof. (1) $\forall\ q \in U_\epsilon^d, f(q) \neq 0$, then $\phi_\epsilon(f(q)) = \bigvee \Phi_\epsilon(f(q)) = \bigvee\{x \in U_\epsilon^d \mid f(x)\&f(q) \neq 0\}$. Since Q is a idempotent quantale, and $f(q) \neq 0$, then $\phi_\epsilon(f(q)) \ge q$.

(2) $\forall\ A \subseteq P$, then

$$\begin{aligned}
\phi_\epsilon(\bigvee A) &= \bigvee \Phi_\epsilon(\vee A) = \bigvee\{q \in Q \mid f(q)\&(\vee A) \neq 0\} \\
&= \bigvee\{q \in Q \mid (\bigvee_{a\in A}(f(q)\&a)) \neq 0\} \\
&= \bigvee\{q \in Q \mid \exists\ a \in A,\ f(q)\&a \neq 0\} \\
&= \bigvee_{a\in A} \bigvee\{q \in Q \mid f(q)\&a \neq 0\} \\
&= \bigvee_{a\in A} \bigvee\{q \in Q \mid f(q)\&a \neq 0\} \\
&= \bigvee_{a\in A} \Phi_\epsilon(a).
\end{aligned}$$

(3) Since $\phi_\epsilon(p) = \bigvee \Phi_\epsilon(p) = \bigvee\{q \in U_\epsilon^d \mid f(q)\&p \neq 0\}$, and $U_\epsilon^d q = \bigvee\{x \in Q \mid d(x) < \epsilon,\ x\&q \neq 0\}$. let $A = \{q \in Q \mid f(q)\&p \neq 0\}$, $B = \{x \in Q \mid d(x) < \epsilon,\ x\&q \neq 0\}$. $\forall\ q \in A$, and $q \in U_\epsilon^d$, $f(q)\&p \neq 0$, then $0 \neq f(q)\&p \le f(q\&q)$. We can see that $q\&q \neq 0$, thus $q \in B$, that is $\phi_\epsilon(p) \le U_\epsilon^d q$. □

Theorem 4.8. *Let Q and P be quantales, $f : Q \longrightarrow P$ be an epimorphism. $\overline{d}$ be induced by a compatible diameter d, then $\overline{d}$ is compatible.*

Proof. Let $\overline{d}(u) = inf\{d(a) \mid u \le f(a)\} < \epsilon$, $u\&f(\alpha_\epsilon^d)(x) \neq 0$, then there exists $v \in Q$, such that $d(v) < \epsilon$, $u \le f(v)$. Since $0 \neq u\&f(\alpha_\epsilon^d)(x) \le f(v)\&f(\alpha_\epsilon^d)(x) = f(v\&\alpha_\epsilon^d(x))$, then $f(v\&\alpha_\epsilon^d(x)) \neq 0$, so that $v\&\alpha_\epsilon^d(x) \neq 0$. Thus $v\&(\bigvee\{y \in Q \mid U_\epsilon^d y \le x\}) \neq 0$, that is $(\bigvee\{v\&y \mid y \in Q,\ U_\epsilon^d y \le x\}) \neq 0$, then $\exists\ y_0 \in Q$, $U_\epsilon^d y_0 \le x$, $v\&y_0 \neq 0$, therefore $v \le x$. Thus $u \le f(v) \le f(x)$, $U_\varepsilon^{\overline{d}}(f\alpha_\epsilon^d(x)) \le f(x)$, then $f\alpha_\epsilon^d(x) \le \alpha_\epsilon^{\overline{d}}(f(x))$. We have that $\forall\ p \in P$, there exists $x \in Q$ such that $p = f(x)$, then $p = f(x) \le f(\bigvee_{\epsilon>0} \alpha_\epsilon^d x) = \bigvee_{\epsilon>0} f(\alpha_\epsilon^d x) \le \bigvee_{\epsilon>0} \alpha_\epsilon^{\overline{d}} f(x) = \bigvee_{\epsilon>0} \alpha_\epsilon^{\overline{d}} p$. □

Definition 4.9. *Let $(Q, d), (P, d')$ be prediametric quantales, and $f : Q \longrightarrow P$ be a quantale homomorphism. If $d'(p) < \epsilon$ for all $p \in P$, there exists an element $q \in Q$, such that $d(q) < \epsilon$, $p \le f(q)$, the quantale homomorphism f is called contractive.*

Theorem 4.10. *Let $(Q, d), (P, d'), (K, d'')$ be prediametric quantales and f be a quantale homomorphism. Consider the following statements:*

(1) $f : (Q, d) \longrightarrow (P, d')$ *is a contractive homomorphism;*
(2) $\forall\ \epsilon > 0$, $x \in Q$, *we have* $U_\epsilon^{d'} \circ f(x) \leq f(U_\epsilon^d x)$*;*
(3) $\forall\ \epsilon > 0$, $x \in Q$, *we have* $f \circ \alpha_\epsilon^d(x) \leq \alpha_\epsilon^{d'} f(x)$.
That $(1) \Longrightarrow (2) \Longleftrightarrow (3)$.

Proof. Firstly, we will prove the implication $(1) \Longrightarrow (2)$.

Let $p \in U_\epsilon^{d'}$ and $p\&f(x) \neq 0$. Since f is a contractive homomorphism, we have that there exists $q' \in Q$, such that $d(q') < \epsilon,\ p \leq f(q')$. But $p\&f(x) \neq 0$, then $0 \neq p\&f(x) \leq f(q')\&f(x) = f(q'\&x)$, which implies that $f(q'\&x) \neq 0$, $q'\&x \neq 0$. Thus

$$\begin{aligned} U_\epsilon^{d'} f(x) &= \bigvee\{p \in P \mid d'(p) < \epsilon\ p\&f(x) \neq 0\} \\ &\leq \{f(q) \in P \mid q \in Q,\ d(q) < \epsilon,\ q\&x \neq 0\} \\ &= f(\bigvee\{q \in Q \mid d(q) < \epsilon,\ q\&x \neq 0\}) = f(U_\epsilon^d(x)). \end{aligned}$$

$(2) \Longrightarrow (3)$ Let $\varepsilon > 0$, $x \in Q$, then $f \circ \alpha_\epsilon^d(x) = f(\bigvee\{q \in Q \mid U_\epsilon^d q \leq x\}) = \bigvee\{f(q) \mid U_\epsilon^d q \leq x,\ q \in Q\}$, and $\alpha_\epsilon^{d'} f(x) = \bigvee\{p \in P \mid U_\epsilon^{d'} p \leq f(x)\}$.

Let $p \in P$, and $d'(p) < \epsilon$, $p\&f(q) \neq 0$. By(2), we have $U_\epsilon^{d'} f(q) \leq f(U_\epsilon^d q)$, then $p \leq f(U_\epsilon^d q) = f(\bigvee\{y \in Q \mid d(y) < \epsilon,\ y\&q \neq 0\}) \leq f(x)$. Thus $\bigvee\{p \in P \mid d'(p) < \epsilon,\ p\&f(q) \neq 0\} \leq f(x)$, which implies $U_\epsilon^{d'} f(q) \leq f(x)$, therefore$f(\alpha_\epsilon^d(x)) \leq \alpha_\epsilon^{d'} f(x)$.

$(3) \Longrightarrow (2)$ Since $U_\epsilon^d \dashv \alpha_\epsilon^d$, then $U_\epsilon^{d'} f(x) \leq f(U_\epsilon^d x) \Longleftrightarrow f(x) \leq \alpha_\epsilon^d f(U_\epsilon^{d'}(x))$. But $f(x) \leq f(\alpha_\epsilon^d U_\epsilon^d x) \leq \alpha_\epsilon^{d'} f(U_\epsilon^d x)$, which implies $U_\epsilon^{d'} f(x) \leq f(U_\epsilon^d x)$. □

Lemma 4.11. *Let* (Q, d) *be a diametric quantale, and* d *be a compatible diameter,* $\beta : Q \longrightarrow 2$ *is a quantale epimorphism. Let* $a, b \in Q$, $\beta(a) = 1$*, then there is a* $\epsilon > 0$*, such that for all* $b \in Q$ *with* $d(b) < \epsilon$ *either* $\beta(b) = 0$ *or* $b \leq a$.

Proof. Since $1 = \beta(a) = \beta(\bigvee\limits_{\epsilon>0} \alpha_\epsilon^d(a)) = \bigvee\limits_{\epsilon>0} \beta(\alpha_\epsilon^d(a))$, then there exists $\epsilon > 0$, such that $\beta(\alpha_\epsilon^d(a)) = 1$. Let $d(b) < \epsilon$, and $b \nleq a$, then $b\&\alpha_\epsilon^d(a) = 0$. Otherwise, if $b\&\alpha_\epsilon^d(a) \neq 0$, that is $b\&(\bigvee\{x \in Q \mid U_\epsilon^d x \leq a\}) = \bigvee\{b\&x \in Q \mid U_\epsilon^d(x) \leq a\} \neq 0$, then there exists $x \in Q$, such that $U_\epsilon^d(x) \leq a$, $b\&x \neq 0$, then $b \leq a$, but this contradicts the assumption that $b \nleq a$. Thus $b\&\alpha_\epsilon^d(a) = 0$, and $0 = \beta(b\&\alpha_\epsilon^d(a)) = \beta(b)\&\beta(\alpha_\epsilon^d(a))$, then $\beta(b) = 0$. Since $\beta(b) \neq 0$, then $\beta(b) = 1$, that is $\beta(b)\&\beta(\alpha_\epsilon^d(a)) = \beta(b\&\alpha_\epsilon^d(a)) = 1$, then $b\&\alpha_\epsilon^d(a) \neq 0$, Therefore $b \leq a$. □

Let (Q, d) be a diametric quantale, d be a compatible diameter. $Pt(Q)$ denotes the collection of all points of Q.

Define: $\rho_d : Pt(Q) \times Pt(Q) \longrightarrow R^+ \cup \{0\}$, $\rho_d(\xi, \eta) = inf\{d(a) \mid a \in Q,\ \xi(a) = \eta(a) = 1\}$, for all $\xi, \eta \in Pt(Q)$.

Theorem 4.12. *Let* (Q, d) *be a diameteric quantale, and* $1\&1 \neq 0$*, d be a compatible diameter, then* ρ_d *is a metric on* $Pt(Q)$ *and the induced topology coincides with the topology* $\Sigma_Q = \{\Sigma_a \mid a \in Q\}$ *of* $Pt(Q)$, $\Sigma_a = \{\xi \mid \xi(a) = 1\}$.

Proof. (1) For every element $\xi, \eta \in Pt(Q)$, it is easy to verify that $\rho_d(\xi, \eta) \geq 0$.

(2) Since $\bigvee U_\epsilon^d = 1$ for all $\epsilon > 0$, then $\forall\ \xi \in Pt(Q)$, $\rho_d(\xi, \xi) = inf\{d(a) \mid \xi(a) = 1\}$, hence $\rho_d(\xi, \xi) = 0$. For all $\xi, \eta \in Pt(Q)$, $\epsilon > 0$, we have $\rho_d(\xi, \eta) = inf\{d(a) \mid \xi(a) = \eta(a) = 1\} = 0$. If $\xi(a) = 1$, there is an element $b \in Q$, such that $d(b) < \epsilon$, and $\xi(b) = \eta(b) = 1$. Thus $b \leq a$, therefore $\eta(a) = 1$. The symmetry is obvious. If $\eta(a) = 1$, then $\xi(a) = 1$. Thus $\xi = \eta$.

(3) For all $\xi, \eta \in Pt(Q)$, $\rho_d(\xi, \eta) = \rho_d(\eta, \xi)$ is obvious.

(4) For all $\xi, \eta, \delta \in Pt(Q)$, in the following, we will prove that $\rho_d(\xi, \eta) \leq \rho_d(\xi, \sigma) + \rho_d(\sigma, \eta)$.

Let $A = \{a \in Q \mid \xi(a) = \eta(a) = 1\}$, $B = \{b \in Q \mid \xi(b) = \delta(b) = 1\}$, $C = \{c \in Q \mid \delta(c) = \eta(c) = 1\}$,

$\forall\ b \in B$, $\forall\ c \in C$, $\xi(b \vee c) = \xi(b) \vee \xi(c) = 1 \vee \xi(c) = 1$, $\eta(b \vee c) = \eta(b) \vee \eta(c) = 1 \vee \eta(b) = 1$, then $b \vee c \in A$. Suppose $b\&c = 0$, then $0 = \xi(b\&c) = \xi(b)\&\xi(c) = 1\&1$, but this contradicts the fact $1\&1 \neq 0$. From

Definition 4.2(iii), we have $d(b \vee c) \leq d(b) + d(c)$, then $inf\{d(a) \mid \xi(a) = \eta(a) = 1\} \leq inf\{d(b) \mid \xi(b) = \delta(b) = 1\} + inf\{d(c) \mid \delta(c) = \eta(c) = 1\} = inf\{d(b) + d(c) \mid \xi(b) = \delta(b) = 1,\ \delta(c) = \eta(c) = 1\}$, that is $\rho_d(\xi, \eta) \leq \rho_d(\xi, \delta) + \rho_d(\delta, \eta)$. Thus ρ_d is a metric on $Pt(Q)$. Therefore $(Pt(Q), \rho_d)$ is a metric space.

In the following, we will prove the induced topology $(Pt(Q), \tau_{\rho_d})$ as ρ_d coincides with the topology $\Sigma_Q = \{\Sigma_a \mid a \in Q\}$ of $Pt(Q)$, $\Sigma_a = \{\xi \mid \xi(a) = 1\}$.

$\forall\ a \in A$, $\forall\ \xi \in \Sigma_a$, then $\xi(a) = 1$. By Lemma 4.11, there exists an $\epsilon > 0$. If $\rho_d(\xi, \eta) < \epsilon$, we have an element $b \in Q$, such that $\xi(b) = \eta(b) = 1$, and $d(b) \leq \epsilon$, then $b \leq a$. Thus $\eta(a) = 1$, that is $\eta \in \Sigma_a$, $\xi \in \{\eta \in Pt(Q) \mid \rho_d(\xi, \eta) < \epsilon\} \subseteq \Sigma_a$. Hence $\Sigma_Q \subseteq \tau_{\rho_d}$.

Conversely, for all $\xi \in Pt(Q)$, $\forall\ \epsilon > 0$, we put $a = \vee\{b \in Q \mid d(b) < \epsilon,\ \xi(b) = 1\}$. For all $\eta \in \Sigma_a$, there exists $b \in Q$, such that $d(b) < \epsilon$, and $\xi(b) = \eta(b) = 1$. Therefore $\rho_d(\xi, \eta) \leq d(b) < \epsilon$. If $\rho_d(\xi, \eta) < \epsilon$, then there exists an element $b \in B$, such that $\eta(b) = \xi(b) = 1$, $d(b) < \epsilon$. Thus $b \leq a$, $\eta(a) = 1$. Hence $\Sigma_a = \{\eta \mid \rho_d(\xi, \eta) < \epsilon\} = B_\epsilon^{\rho_d}(\xi)$. Since $U = \bigcup\{B_\epsilon^{\rho_d}(u) \mid u \in U\} = \bigvee\limits_{u \in U} \Sigma_{a_u} = \Sigma_{\bigvee\limits_{u \in U} a_u}$ for all $U \in \tau_{\rho_d}$, then $U \in \Sigma_Q$, therefore $\tau_{\rho_d} \subseteq \Sigma_Q$. □

5 Conclusion

The term quantale was coined as a combination of quantum logic and locale by Mulvey. Since quantale theory provides a powerful tool in noncommutative structures, it has a wide range of applications. In this paper, we discussed some properties of points on quantales. We proved that the set of all completed files is isomorphic to all points of quantales, and showed that two sided prime elements and points of quantales are in one to one correspondence. Furthermore, a functor from the category of the two sided quantales to the dual category of the topology was constructed. We introduced the definition of P-spatial quantales, and some equivalent characterizations for P-spatial quantales were given. The definition of diameter on frame was generalized to quantales. Finally, we proved that the topology induced by diameter coincides with the topology of the point spaces.

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