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### **Research Article**

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# Disjoint hypercyclicity equals disjoint supercyclicity for families of Taylor-type operators

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**Abstract:** We characterize disjointness of supercyclic operators which map a holomorphic function to a partial sum of the Taylor expansion. In particular, we show that disjoint hypercyclicity equals disjoint supercyclicity for families of Taylor-type operators. Moreover, we give a sufficient condition to yield the disjoint supercyclicity for families of Taylor-type operators.

Keywords: Disjoint supercyclic, Class, Compact set, Taylor-type, Operator

MSC: 47A16, 47B38, 30H99, 46E20

### 1 Introduction

Let X, Y be two topological vector space over  $\mathbb{R}$  or  $\mathbb{C}$ . A sequence of linear and continuous operators  $T_n: X \to Y$ ,  $n = 1, 2, \ldots$  is said to be hypercyclic if there exists a vector  $x \in X$  such that  $\{T_1x, T_2x, \ldots\}$  is dense in Y. Such a vector x is called a hypercyclic vector for  $\{T_n\}_{n\in\mathbb{N}}$ . If the sequence  $\{T_n\}_{n\in\mathbb{N}}$  comes from the iterates of a single operator  $T: X \to Y$ , *i.e.*  $T_n = T^n$ ,  $n = 1, 2, \ldots$ , then T is called hypercyclic.

In 1974, Hilden and Wallen introduced in [1] the notion of supercyclicity. They showed that all unilateral weighted backward shifts are supercyclic, but no vector is supercyclic for all unilateral weighted backward shifts. Recall that a sequence of linear and continuous operators  $T_n: X \to Y$ , n = 1, 2, ... is said to be supercyclic provided there exists a vector  $x \in X$  such that  $\{\alpha T_1 x, \alpha T_2 x, ... : \alpha \in \mathbb{C}\}$  is dense in Y. Such a vector x is called a supercyclic vector for  $\{T_n\}_{n\in\mathbb{N}}$ . Good sources of background information on hypercyclic and supercyclic operators include [2–4].

In 2007, Bernal [5] independently introduced the disjointness of operators. Bès et al. investigated disjoint hypercyclic operators in [6, 7], and disjoint mixing operators in [8]. For more results, see [9–11].

**Definition 1.1.** Let  $\sigma_0 \in \mathbb{N}$  and  $X, Y_1, Y_2, \ldots, Y_{\sigma_0}$  be topological vector space over  $\mathbb{K}$ . For each  $\sigma \in \{1, 2, \ldots, \sigma_0\}$  consider a sequence of linear and continuous operators  $T_{\sigma,n}: X \to Y_{\sigma}, n \in \mathbb{N}$ . We say that the sequence  $\{T_{\sigma,n}\}_{n\in\mathbb{N}}, \sigma=1,2,\ldots,\sigma_0$  are disjoint hypercyclic (respectively, disjoint supercyclic) if the sequence

$$(T_{1,n}(x), T_{2,n}(x), \dots, T_{\sigma_0,n}(x)): X \to Y_1 \times Y_2 \times \dots \times Y_{\sigma_0}$$

is hypercyclic (respectively, supercyclic), where  $Y_1 \times Y_2 \times \cdots \times Y_{\sigma_0}$  is assumed to be endowed with the product topology.

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Obviously, by the definition, the following diagram holds true in the disjoint setting:

Disjoint hypercyclicity  $\Rightarrow$  Disjoint supercyclicity.

First, we introduce some standard notations and terminology. The set of holomorphic functions on a simply connected domain  $\Omega \subset \mathbb{C}$ , to be denoted  $H(\Omega)$ , becomes a complete topological vector space under the topology inherited by the uniform convergence on all the compact subsets of  $\Omega$ . Moreover, for any compact set  $K \subset \mathbb{C}$ , we denote

$$\mathcal{A}(K) = \{ g \in H(K^0) : g \text{ is continuous on } K \},$$

 $\mathcal{M} = \{ K \subset \mathbb{C} : K \text{ is compact set and } K^c \text{ connected set} \},$ 

 $\mathcal{M}_{\Omega} = \{K \subset \mathbb{C} \setminus \Omega : K \text{ is compact set and } K^{c} \text{ connected set}\}.$ 

For a function g defined on K, we use the notation  $\|g\|_K = \sup_{z \in K} |g(z)|$ . Now for every  $K \in \mathcal{M}_{\Omega}$  and every sequence of natural numbers  $\{\lambda_n\}_{n \in \mathbb{N}}$  we consider the sequence of operators:

$$T_{\lambda_n}^{(\zeta_0)}: H(\Omega) \to \mathcal{A}(K), \quad n=1,2,\ldots$$

$$T_{\lambda_n}^{(\zeta_0)}(f)(z) = \sum_{k=1}^{\lambda_n} \frac{f^{(k)}(\zeta_0)}{k!} (z - \zeta_0)^k, \quad n = 1, 2, \dots$$

Let  $T_n^{(\zeta_0)}(f)(z) = \sum\limits_{k=1}^n \frac{f^{(k)}(\zeta_0)}{k!}(z-\zeta_0)^k$  denote the nth partial sum of the Taylor series of f with center  $\zeta_0$ . f is said to belong to the collection  $U(\Omega,\zeta_0)$  of functions with universal Taylor series expansions around  $\zeta_0$  whenever  $\{T_n^{(\zeta_0)}(f)(z): n=1,2,\ldots\}$  is dense in  $\mathcal{A}(K)$ , for every  $K\in\mathcal{M}$  disjoint from  $\Omega$ . Nestoridis [12, 13] had shown that the collection  $U(\Omega,\zeta_0)$  is a dense  $G_\delta$  subset of  $H(\Omega)$ , and  $U(\Omega,\zeta_0)\neq\emptyset$  for any simply connected domain  $\Omega$  and any point  $\zeta_0\in\Omega$ . Indeed, he proved that if the sequence  $\{\lambda_n\}_{n\in\mathbb{N}}$  is unbounded then the corresponding sequence of operators  $\{T_{\lambda_n}^{(\zeta_0)}\}_{n\in\mathbb{N}}$  is hypercyclic. Costakis and Tsirivas [14] provided a new strong notion of universality for Taylor series called doubly universal Taylor series. Chatzigiannakidou and Vlachou [15] dealt with the existence of doubly universal Taylor series defined on simply connected domains with respect to any center, which generalized the results of Costakis and Tsirivas for the unit disk. Moreover, Chatzigiannakidou [16] studied some approximation properties of doubly universal Taylor series defined on a simply connected domain  $\Omega$ .

In order to research the disjointness of hypercyclicy for families of Taylor-type operators directly, Vlachou [17] introduced a class.

**Definition 1.2** ([17]).  $\sigma = 1, 2, ..., \sigma_0$ , let  $\{\lambda_n^{(\sigma)}\}_{n \in \mathbb{N}}$  be a finite collection of sequences of natural numbers. If for every choice of compact sets  $K_1, K_2, ..., K_{\sigma_0} \in \mathcal{M}_{\Omega}$  the set

$$\left\{ \left( T_{\lambda_n^{(1)}}^{(\zeta_0)}(f), T_{\lambda_n^{(2)}}^{(\zeta_0)}(f), \dots, T_{\lambda_n^{(\sigma_0)}}^{(\zeta_0)}(f) \right) : n \in \mathbb{N} \right\}$$

is dense in  $\mathcal{A}(K_1) \times \mathcal{A}(K_2) \times \cdots \mathcal{A}(K_{\sigma_0})$ , we say that function  $f \in \mathcal{H}(\Omega)$  belongs to the class

$$U_{mult}^{(\sigma_0)}\left(\left\{\lambda_n^{(1)}\right\}_{n\in\mathbb{N}},\left\{\lambda_n^{(2)}\right\}_{n\in\mathbb{N}},\ldots,\left\{\lambda_n^{(\sigma_0)}\right\}_{n\in\mathbb{N}}\right).$$

As we all know, the functions of the above class are disjoint hypercyclic vectors, so if we want to research some characterizations of disjointness of hypercyclicity, we consider this class as empty or non-empty. It is clear that the sequences of natural numbers  $\{\lambda_n^{(\sigma)}\}_{n\in\mathbb{N}}$  play a key role in the study of this class. In this paper, we require a special definition of  $\{\lambda_n^{(\sigma)}\}_{n\in\mathbb{N}}$  called well ordered sequences.

**Definition 1.3.**  $\sigma = 1, 2, ..., \sigma_0$ , let  $\{\lambda_n^{(\sigma)}\}_{n \in \mathbb{N}}$  be a finite collection of sequences of natural numbers. We say that these sequences are well ordered if

$$\limsup_{n} \frac{\lambda_{n}^{(\sigma+1)}}{\lambda_{n}^{(\sigma)}} \geq \limsup_{n} \frac{\lambda_{n}^{(\sigma)}}{\lambda_{n}^{(\sigma+1)}}, \quad \sigma = 1, 2, \dots, \sigma_{0} - 1.$$

**Remark 1.4.** Vlachou [17] showed that there exists a rearrangement  $\{\lambda_n^{(\pi(\sigma))}\}_{n\in\mathbb{N}}$ , which is well ordered. Thus, in this paper we assume that we have a well ordered finite collection of sequences of natural numbers  $\{\lambda_n^{(\sigma)}\}_{n\in\mathbb{N}}$ .

Following the same path as [15], Vlachou [17] showed a necessary and sufficient condition for families of taylor-type operators to be disjoint hypercyclic as follows:

**Theorem 1.5.** The class  $U_{mult}^{(\zeta_0)}\left(\{\lambda_n^{(1)}\}_{n\in\mathbb{N}},\{\lambda_n^{(2)}\}_{n\in\mathbb{N}},\ldots,\{\lambda_n^{(\sigma_0)}\}_{n\in\mathbb{N}}\right)$  is nonempty if and only if there exists a strictly increasing sequence of natural numbers  $\{\mu_n\}_{n\in\mathbb{N}}$  such that  $\lim_{n\to\infty}\lambda_{\mu_n}^{(1)}=+\infty$  and  $\lim_{n\to\infty}\frac{\lambda_{\mu_n}^{(\sigma+1)}}{\lambda_{\mu_n}^{(\sigma)}}=+\infty$ ,  $\sigma=1,2,\ldots,\sigma_0-1$ .

Inspired by [17], we introduce another class to research the disjointness of supercyclicity for operators which map a holomorphic function to a partial sum of the Taylor expansion.

**Definition 1.6.**  $\sigma = 1, 2, ..., \sigma_0$ , let  $\{\lambda_n^{(\sigma)}\}_{n \in \mathbb{N}}$  be a finite collection of sequences of natural numbers. If for every choice of compact sets  $K_1, K_2, ..., K_{\sigma_0} \in \mathcal{M}_{\Omega}$  the set

$$\left\{ \left( \alpha T_{\lambda_n^{(1)}}^{(\zeta_0)}(f), \alpha T_{\lambda_n^{(2)}}^{(\zeta_0)}(f), \ldots, \alpha T_{\lambda_n^{(\sigma_0)}}^{(\zeta_0)}(f) \right) : n \in \mathbb{N}, \alpha \in \mathbb{C} \right\}$$

is dense in  $\mathcal{A}(K_1) \times \mathcal{A}(K_2) \times \cdots \mathcal{A}(K_{\sigma_0})$ , we say that  $f \in \mathcal{H}(\Omega)$  belongs to the class

$$V_{mult}^{(\zeta_0)}\left(\{\lambda_n^{(1)}\}_{n\in\mathbb{N}},\{\lambda_n^{(2)}\}_{n\in\mathbb{N}},\ldots,\{\lambda_n^{(\sigma_0)}\}_{n\in\mathbb{N}}\right).$$

The paper is organized in the following manner: In section 2, we obtain that *Disjoint hypercyclicity*  $\iff$  *Disjoint supercyclicity* for families of taylor-type operators. In section 3, we provide a sufficient condition to get the disjointness of supercyclic operators who map a holomorphic function to a partial sum of the Taylor expansion.

# 2 disjoint hypercyclicity equals disjoint supercyclicity

In this section, we prove that  $Disjoint\ hypercyclicity \iff Disjoint\ supercyclicity$  for families of taylor-type operators. In order to prove the main theorem, we need some fundamental knowledge about thinness.

**Definition 2.1** ([18, Chapter 5]). Let S be a subset of  $\mathbb C$  and  $\xi \in \mathbb C$ . Then S is non-thin at  $\xi$  if  $\xi \in \overline{S \setminus \{\xi\}}$  and if for every subharmonic function u defined on a neighbourhood of  $\xi$ ,

$$\limsup_{z\to\xi}u(z)=u(\xi),\ z\in S\setminus\{\xi\}.$$

Otherwise we say that S is thin at  $\xi$ .

Thinness is obviously a local property, i.e. S is non-thin at  $\xi$  if and only if  $U \cap S$  is non-thin at  $\xi$  for each open neighbourhood U of  $\xi$ . If two sets are both thin at a particular point, so is their union.

**Lemma 2.2** ([17, Lemma 2.2]). Let  $\Omega \subset \mathbb{C}$  be a simply connected domain. Then there exists an increasing sequence of compact sets  $E_k$ , k = 1, 2, ... with the following properties:

(i) 
$$E_k \in \mathcal{M}_{\Omega}, k = 1, 2, ...$$

(ii)  $\cup_k E_k$  is closed and non-thin at ∞.

**Theorem 2.3.** The following conditions are equivalent:

(i) The class 
$$U_{mult}^{(\zeta_0)}\left(\{\lambda_n^{(1)}\}_{n\in\mathbb{N}},\{\lambda_n^{(2)}\}_{n\in\mathbb{N}},\dots,\{\lambda_n^{(\sigma_0)}\}_{n\in\mathbb{N}}\right)$$
 is nonempty.

(ii) The class 
$$V_{mult}^{(\zeta_0)}\left(\{\lambda_n^{(1)}\}_{n\in\mathbb{N}},\{\lambda_n^{(2)}\}_{n\in\mathbb{N}},\ldots,\{\lambda_n^{(\sigma_0)}\}_{n\in\mathbb{N}}\right)$$
 is nonempty.

(iii) there exists a strictly increasing sequence of natural numbers  $\{\mu_n\}_{n\in\mathbb{N}}$  such that  $\lim_{n\to\infty}\lambda_{\mu_n}^{(1)}=+\infty$  and  $\lim_{n\to\infty}\frac{\lambda_{\mu_n}^{(\sigma+1)}}{\lambda_{\mu_n}^{(\sigma)}}=+\infty, \sigma=1,2,\ldots,\sigma_0-1.$ 

*Proof.* (i)⇒(ii): Noting that Disjoint Hypercyclicity⇒ Disjoint Supercyclicity, we obtain the result easily.

(ii) $\Rightarrow$ (iii): Choose an increasing sequence of compact sets  $E_k$  as stated in Lemma 2.2. Since  $V_{mult}^{(\zeta_0)}\left(\{\lambda_n^{(1)}\}_{n\in\mathbb{N}},\{\lambda_n^{(2)}\}_{n\in\mathbb{N}},\dots,\{\lambda_n^{(\sigma_0)}\}_{n\in\mathbb{N}}\right)\neq\varnothing$ , we may also fix a strictly increasing sequence of natural numbers  $\{n_k\}_{k\in\mathbb{N}}$  and  $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$  such that for  $f\in V_{mult}^{(\zeta_0)}(\{\lambda_n^{(1)}\}_{n\in\mathbb{N}},\{\lambda_n^{(2)}\}_{n\in\mathbb{N}},\dots,\{\lambda_n^{(\sigma_0)}\}_{n\in\mathbb{N}})$ ,

$$\left\|\alpha_{n_k} T_{\lambda_{n_k}^{(\sigma_0)}}^{(\sigma_0)}(f)\right\|_{E_k} < \frac{1}{k}, \quad \sigma \in \{1, 2, \dots, \sigma_0\} \quad odd, \tag{1}$$

$$\left\|\alpha_{n_k} T_{\lambda_{n_k}^{(\sigma_0)}}^{(\sigma_0)}(f) - 1\right\|_{E_k} < \frac{1}{k}, \quad \sigma \in \{1, 2, \dots, \sigma_0\} \quad even.$$
 (2)

Clearly  $\lim_{k\to\infty}\lambda_{n_k}^{(\sigma)}=+\infty$  for every  $\sigma\in\{1,2,\ldots,\sigma_0\}$ . If not,  $T_{\lambda_n^{(\sigma)}}^{(\zeta_0)}$  must have finite terms and  $V_{mult}^{(\zeta_0)}\Big(\{\lambda_n^{(1)}\}_{n\in\mathbb{N}},\{\lambda_n^{(2)}\}_{n\in\mathbb{N}},\ldots,\{\lambda_n^{(\sigma_0)}\}_{n\in\mathbb{N}}\Big)=\varnothing$ , which would be a contradiction,  $\lim_{n\to\infty}\lambda_{\mu_n}^{(\sigma)}=+\infty$  is proved.

Next, we prove the claim  $\lim_{n\to\infty}\frac{\lambda_{\mu_n}^{(\sigma+1)}}{\lambda_{\mu_n}^{(\sigma)}}=+\infty$ ,  $\sigma=1,2,\ldots,\sigma_0-1$ . Suppose on the contrary that there exists no such sequence. Thus, there exists  $m \in \{1, 2, \dots, \sigma_0\}$  such that  $\limsup_{k \to \infty} \frac{\lambda_{n_k}^{(m+1)}}{\lambda_{n_k}^{(m)}} < +\infty$ . As was mentioned in Remark 1.4,  $\{\lambda_n^{(\sigma)}\}_{k\in\mathbb{N}}$  is well-ordered, hence,

$$\limsup_{k\to\infty}\frac{\lambda_{n_k}^{(m+1)}}{\lambda_{n_k}^{(m)}}\geq \limsup_{k\to\infty}\frac{\lambda_{n_k}^{(m)}}{\lambda_{n_k}^{(m+1)}},$$

which implies that for some constant C > 0,  $\frac{\lambda_{n_k}^{(m+1)}}{\lambda_{n_k}^{(m)}} < C$  and  $\frac{\lambda_{n_k}^{(m)}}{\lambda_{n_k}^{(m+1)}} < C$ .

Now define two sets,  $I = \{k \in \mathbb{N} : \lambda_{n_k}^{(m+1)} \geq \lambda_{n_k}^{(m)} \}$ ,  $J = \{k \in \mathbb{N} : \lambda_{n_k}^{(m)} \geq \lambda_{n_k}^{(m+1)} \}$ . At least one of the above sets is infinite. Without loss of generality, we assume that I is infinite Let  $p_k(z)$  be defined by

$$p_k(z) = \left(\frac{R}{z - \zeta_0}\right)^{\lambda_{n_k}^{(m)}} \left(\alpha_{n_k} T_{\lambda_{n_k}^{(m+1)}}^{(\zeta_0)}(f)(z) - \alpha_{n_k} T_{\lambda_{n_k}^{(m)}}^{(\zeta_0)}(f)(z)\right),$$

where  $k \in I$ ,  $R = dist(\Omega^c, \zeta_0)$ . Obviously,  $\frac{\lambda_{n_k}^{(m+1)}}{\lambda_{n_k}^{(m)}} < C$  and  $\lambda_{n_k}^{(m+1)} \ge \lambda_{n_k}^{(m)}$  give that

$$deg(p_k) \leq \lambda_{n_k}^{(m+1)} < C\lambda_{n_k}^{(m)}$$
.

Let  $E = (\bigcup_{k \in \mathbb{N}} E_k) \cap D(\zeta_0, 2R)^c$ . Then E is closed and non-thin at  $\infty$ . Let  $z \in E$ . Then for large enough  $k, z \in E_k$ and  $|z - \zeta_0| \ge 2R$ . By (1) and (2), since *I* is infinite and *k* large enough, we obtain  $k \in I$  and thus

$$\begin{aligned} |p_{k}| &\leq \left| \frac{R}{z - \zeta_{0}} \right|^{\lambda_{n_{k}}^{(m)}} \left( |\alpha_{n_{k}}| \left\| T_{\lambda_{n_{k}}^{(m+1)}}^{(\zeta_{0})}(f) \right\|_{E_{k}} + |\alpha_{n_{k}}| \left\| T_{\lambda_{n_{k}}^{(m)}}^{(\zeta_{0})}(f) \right\|_{E_{k}} \right) \\ &\leq \left( \frac{1}{2} \right)^{\lambda_{n_{k}}^{(m)}} \left( 1 + \frac{2}{k} \right) \\ &< \frac{3}{2^{\lambda_{n_{k}}^{(m)}}}. \end{aligned}$$

Hence  $\limsup |p_k|^{\frac{1}{C\lambda_{n_k}^{(m)}}} \le (\frac{1}{2})^{\frac{1}{C}} < 1$ . Moreover, if  $\Gamma \subset E$  is a continuum (compact, connected but not a

singleton) we have  $\limsup_{k \in I, k \to \infty} \|p_k\|_{\Gamma}^{\frac{1}{C\lambda_{n_k}^{(m)}}} \le (\frac{1}{2})^{\frac{1}{C}} < 1$ . Therefore, by Theorem 1 of [19], we conclude that for  $k \in I$ ,

 $p_k \to 0$  compactly on  $\mathbb{C}$ . Let  $\xi \in \partial \Omega$  with  $|\xi - \zeta_0| = R$ , then from the above

$$\left(\frac{\xi - \zeta_0}{R}\right)^{\lambda_{n_k}^{(m)}} p_k(\xi) \to 0, k \in I. \tag{3}$$

But

$$\left(\frac{\xi-\zeta_0}{R}\right)^{\lambda_{n_k}^{(m)}}p_k(\xi)=\alpha_{n_k}T_{\lambda_{n_k}^{(m+1)}}^{(\zeta_0)}(f)(\xi)-\alpha_{n_k}T_{\lambda_{n_k}^{(m)}}^{(\zeta_0)}(f)(\xi), k\in I.$$

Thus

$$\left| \left( \frac{\xi - \zeta_0}{R} \right)^{\lambda_{n_k}^{(m)}} p_k(\xi) - 1 \right| \le \left\| \alpha_{n_k} T_{\lambda_{n_k}^{(m+1)}}^{(\zeta_0)}(f) - 1 \right\|_{E_k} + \left\| \alpha_{n_k} T_{\lambda_{n_k}^{(m)}}^{(\zeta_0)}(f) \right\|_{E_k} \\ \le \frac{2}{k} \to 0,$$

which contradicts (3).

Now we prove the case *J* is infinite, we set

$$p_k(z) = \left(\frac{R}{z - \zeta_0}\right)^{\lambda_{n_k}^{(m+1)}} \left(\alpha_{n_k} T_{\lambda_{n_k}^{(m)}}^{(\zeta_0)}(f)(z) - \alpha_{n_k} T_{\lambda_{n_k}^{(m+1)}}^{(\zeta_0)}(f)(z)\right), k \in J.$$

Then following the same argument as *I* is infinite, we arrive at contradiction.

(iii) $\Rightarrow$ (i): Obviously, this result is due to Theorem 1.5.

# 3 A sufficient condition for disjoint supercyclicity

In this section, we present a sufficient condition to imply the disjointness of supercyclicity for Taylor-type operators, which is different from Theorem 2.3.

**Definition 3.1.** Let  $h_n: U \to \mathbb{C}$ , n = 1, 2, ... be a sequence of continuous functions defined on an open set U and  $\sigma_n$  be a sequence of positive integers. If for every compact set  $K \subset U$  the sequence  $\|h_n\|_K^{\frac{1}{\sigma_n}}$  is bounded, we say that the sequence  $h_n$  is  $\{\sigma_n\}$ -locally bounded.

Any continuous function f can be approximated uniformly on a compact subset K of  $\mathbb{C}$  by polynomials provided that  $\mathbb{C} \setminus K$  is connected and f extends to be holomorphic on a neighbourhood of K. Ransford [18] gave a somewhat stonger version of this result called Bernstin-walsh Theorem. Vlachou [17] generalized Bernstin-walsh Theorem. On this basis, we give minor modifications. Though the proof is similar to the above two papers, for the convenience of the reader we give the details of the proof. Write deg(p) as the degree of a polynomial p. Note that  $d_{\tau_n}(f,K) = \inf\{\|f - p\|_K : deg(p) \le n\}$ .

**Proposition 3.2.** Let  $K \in \mathcal{M}$  and  $\{f_n\}_{n \in \mathbb{N}}$  be a  $\{\sigma_n\}$ -locally bounded sequence of holomorphic functions on an open neighbourhood U of K. If for any sequence of natural numbers  $\{\tau_n\}_{n \in \mathbb{N}}$  with  $\lim_{n \to \infty} \tau_n = +\infty$ , for some constant  $C_0 > 0$  such that  $\frac{\tau_n}{\sigma_n} > C_0$ , then there exists a constant C > 1 such that

$$\limsup_n d_{\tau_n}(f_n,K)^{\frac{1}{\tau_n}} \leq C\theta,$$

where

$$\theta = \begin{cases} \sup_{\mathbb{C}_{\infty} \setminus U} \exp(-g_{\mathbb{C}_{\infty} \setminus K}(z, \infty)), & \text{if } c(K) > 0; \\ 0, & \text{if } c(K) = 0. \end{cases}$$

*Proof.* The proof is divided into two cases.

**Case 1**: c(K) > 0.  $\Gamma$  is a closed contour in  $U \setminus K$  which winds once around each point of K and zero times

round each point of  $\mathbb{C} \setminus U$ . Since  $\lim_{n \to \infty} \tau_n = +\infty$ , we can choose n large enough to ensure  $\tau_n \geq 2$ . Thus we can consider a Fekete polynomial  $q_{\tau_n}$  of degree  $\tau_n$  for K, for  $\omega \in K$  define

$$p_n(\omega) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f_n(z)}{q_{\tau_n}(z)} \cdot \frac{q_{\tau_n}(\omega) - q_{\tau_n}(z)}{\omega - z} dz.$$

Obviously,  $deg(p_n) \le \tau_n - 1$ . Cauchy's integral formula gives

$$f_n(\omega) - p_n(\omega) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f_n(z)}{\omega - z} \cdot \frac{-q_{\tau_n}(\omega)}{q_{\tau_n}(z)} dz,$$

and hence,

$$d_{\tau_n}(f_n, K) \leq \|f_n - p_n\|_K \leq \frac{l(\Gamma)}{2\pi} \cdot \frac{\|f_n\|_{\Gamma}}{dist(\Gamma, K)} \cdot \frac{\|q_{\tau_n}\|_K}{\min_{z \in \Gamma} |q_{\tau_n}(z)|},$$

where  $l(\Gamma)$  is the length of  $\Gamma$  and  $dist(\Gamma, K)$  is the distance of  $\Gamma$  from K.

Since  $f_n$  is  $\{\sigma_n\}$ -locally bounded, there exists a positive constant A>1 such that  $\|f_n\|_{\Gamma}\leq A^{\sigma_n}$ . In addition, according to the proof of Theorem 6.3.1 in [18], we see that

$$\limsup_{n} \left( \frac{\|q_{\tau_n}\|_K}{\min_{z \in \Gamma} |q_{\tau_n}(z)|} \right) \leq \alpha^{\tau_n},$$

where  $\alpha = \sup_{z \in \Gamma} \exp(-g_{\mathbb{C}_{\infty} \setminus K}(z, \infty))$ .

For  $A^{\frac{\sigma_n}{\tau_n}} \le A^{\frac{1}{c_0}}$ , let  $C = A^{\frac{1}{c_0}}$ , clearly

$$\limsup_{n} d_{\tau_n}(f_n, K)^{\frac{1}{\tau_n}} \leq C\alpha.$$

**Case 2:** c(K) = 0. Let  $(K_k)_{k \ge 1}$  be a decreasing sequence of non-polar compact subsets of U, with connected complements, such that  $\lim_{k \to \infty} K_k = K$ . Let  $\theta_k$  denote the corresponding numbers defined in the theorem, as shown in case 1

$$\limsup_n d_{\tau_n}(f_n,K)^{\frac{1}{\tau_n}} \leq \limsup_n d_{\tau_n}(f_n,K_k)^{\frac{1}{\tau_n}} \leq C\theta_k.$$

Now we prove that  $\lim_{k\to\infty} \theta_k = 0$ . Define the function

$$h_k(z) = \begin{cases} g_{\mathbb{C}_{\infty} \setminus K_k}(z, \infty) - g_{\mathbb{C}_{\infty} \setminus K_1}(z, \infty), & \text{if } z \in \mathbb{C} \setminus K_1; \\ \log c(K_1) - \log c(K_k), & \text{if } z = \infty. \end{cases}$$

Thus  $(h_k)_{k\geq 1}$  is an increasing sequence of harmonic functions on  $z\in\mathbb{C}\setminus K_1$  and  $h_k(\infty)\to\infty$ , Harnack's Theorem implies that  $h_k\to\infty$  locally uniformly on  $\mathbb{C}\setminus K_1$ . In particular  $g_{\mathbb{C}_\infty\setminus K_k}(z,\infty)\to\infty$  uniformly on  $\mathbb{C}_\infty\setminus U$ , which shows that  $\lim_{k\to\infty}\theta_k=0$ .

For convenience, we define  $\gamma = C\theta$  in the following.

**Theorem 3.3.** For  $\sigma=1,2,\ldots,\sigma_0-1$ , let  $\lambda_n^{(\sigma)}$  be a finite collection of well-ordered sequences of natural numbers. If  $\lim_{n\to\infty}\lambda_n^{(1)}=+\infty$  and  $\lambda_n^{(\sigma+1)}>\lambda_n^{(\sigma)}$ , moreover,  $\lim_{n\to\infty}|\beta|^{\lambda_n^{(\sigma)}}|\gamma|^{\lambda_n^{(\sigma+1)}}=0$  for every  $|\beta|<1$ . then the class

$$V_{mult}^{(\zeta_0)}\left(\{\lambda_n^{(1)}\}_{n\in\mathbb{N}},\{\lambda_n^{(2)}\}_{n\in\mathbb{N}},\ldots,\{\lambda_n^{(\sigma_0)}\}_{n\in\mathbb{N}}\right)$$

is a  $G_{\delta}$  and dense subset of  $H(\Omega)$ .

*Proof.* Suppose  $\{f_j\}_{j\in\mathbb{N}}$  is an enumeration of polynomials with rational coefficients. In view of [13], there exists a sequence of compact sets  $\{K_m\}_{m\in\mathbb{N}}$  in  $\mathcal{M}_{\Omega}$ , such that for every  $K\in\mathcal{M}_{\Omega}$  is contained in some  $K_m$ . For  $\alpha\in\mathbb{C}$  and every choice of positive integers s, n,  $m_{\sigma}$  and  $j_{\sigma}$ , let

$$E\left(\left\{m_{\sigma}\right\}_{\sigma=1}^{\sigma_{0}},\left\{j_{\sigma}\right\}_{\sigma=1}^{\sigma_{0}},s,n,\alpha\right)=\left\{f\in H(\Omega):\left\|\alpha T_{\lambda_{n}^{(\sigma)}}^{(\zeta_{0})}-f_{j_{\sigma}}\right\|_{K_{m_{\sigma}}}<\frac{1}{s},\sigma=1,2,\ldots,\zeta_{0}\right\}.$$

An application of Mergelyan's Theorem shows that

$$V_{mult}^{(\zeta_0)}\left(\left\{\lambda_n^{(1)}\right\}_{n\in\mathbb{N}},\left\{\lambda_n^{(2)}\right\}_{n\in\mathbb{N}},\ldots,\left\{\lambda_n^{(\sigma_0)}\right\}_{n\in\mathbb{N}}\right)$$

$$=\bigcap_{\left\{m_\sigma\right\}_{\sigma=1}^{\sigma_0}}\bigcap_{\left\{j_\sigma\right\}_{\sigma=1}^{\sigma_0}}\bigcap_{s}\bigcup_{n}\bigcup_{\alpha\in\mathbb{C}}E\left(\left\{m_\sigma\right\}_{\sigma=1}^{\sigma_0},\left\{j_\sigma\right\}_{\sigma=1}^{\sigma_0},s,n,\alpha\right).$$

Therefore, by Baire's Category Theorem, it is sufficient to prove that

$$\bigcup_{n}\bigcup_{\alpha\in\mathbb{C}}E\left(\{m_{\sigma}\}_{\sigma=1}^{\sigma_{0}},\{j_{\sigma}\}_{\sigma=1}^{\sigma_{0}},s,n,\alpha\right) \text{ is dense in } H(\Omega).$$

Choose  $g \in H(\Omega)$ ,  $\varepsilon > 0$  and a compact subset L of  $\Omega$ . Without loss of generality, we may assume that L has connected complement,  $\zeta_0 \in L^0$ . For every  $|\beta| < 1$ , Runge's Theorem implies that we may fix a polynomial p such that:

$$\|g-p\|_L<\frac{\varepsilon}{2},\tag{4}$$

$$\|\beta p - f_{j_1}\|_{K_{m_1}} < \frac{1}{s}. \tag{5}$$

Moreover, we fix two open and disjoint sets  $U_1$ ,  $U_2$  with  $L \subset U_1$  and  $\bigcup_{\sigma=1}^{\sigma_0} K_{m_\sigma} \subset U_2$ . Let  $U = (U_1 - \zeta_0) \cup (U_2 - \zeta_0)$ ,  $K = (L - \zeta_0) \cup (K_{m_\sigma} - \zeta_0)$ .

Next, our proof is divided into two steps:

**Step 1.** For  $\sigma \geq 2$ , we will construct a sequence of polynomials  $\{Q_n^{(\sigma)}\}_{n\in\mathbb{N}}$  via a finite induction with the following properties:

$$(1) \operatorname{deg}(Q_n^{(\sigma)}) \leq \lambda_n^{(\sigma)}$$

(1) 
$$\deg(Q_n \cap) \leq \lambda_n$$
.  
(2)  $\{\beta Q_n^{(\sigma)}(z-\zeta_0)\} \underset{n\to\infty}{\to} 0 \text{ on } L$ .

(3) 
$$\beta p(z) + \sum_{k=2}^{\sigma} \beta Q_n^{(k)}(z-\zeta_0) - f_{j_{\sigma}}(z) \underset{n\to\infty}{\longrightarrow} 0 \text{ on } K_{m_{\sigma}}.$$

We define a function  $f_n$  as

$$f_n(z) = egin{cases} eta^{-\lambda_n^{(\sigma-1)}} g_n(z), & if & z \in U_2 - \zeta_0; \ 0, & if & z \in U_1 - \zeta_0. \end{cases}$$

where

$$g_{n} = \begin{cases} f_{j_{\sigma}}(z + \zeta_{0}) - \beta p(z + \zeta_{0}) - \sum_{k=2}^{\sigma-1} \beta Q_{n}^{(k)}(z), & \text{if } \sigma \geq 3; \\ f_{j_{2}}(z + \zeta_{0}) - \beta p(z + \zeta_{0}), & \text{if } \sigma = 2. \end{cases}$$

Moreover, let  $\sigma_n = \lambda_n^{(\sigma-1)}$  and  $\tau_n = \lambda_n^{(\sigma)}$  and a compact set  $\widetilde{K} \subset U_2 - \zeta_0$ .

First of all, we prove  $\sigma = 2$  case. Note that

$$\|f_n\|_{\widetilde{K}} = \|\beta^{-\lambda_n^{(1)}} (f_{j_2}(z+\zeta_0) - \beta p(z+\zeta_0))\|_{\widetilde{K}}$$

$$\leq \left(\frac{1}{|\beta|}\right)^{\sigma_n} c,$$

where  $c = \|f_{j_{\sigma}} - \beta p\|_{\widetilde{K} + \zeta_0}$ . So  $\{f_n\}_{n \in \mathbb{N}}$  is  $\{\sigma_n\}$ -locally bounded. By Proposition 3.2, it follows that there exists  $\gamma > 0$  such that

$$\limsup_{n} d_{\tau_n}(f_n, K)^{\frac{1}{\tau_n}} \leq \gamma.$$

This implies that we can fix a sequence of polynomials  $p_n$  with degree less or equal to  $\tau_n$  so that

$$||f_n - \beta p_n||_{K} \le (\gamma)^{\tau_n},\tag{6}$$

for n sufficiently large.

Define the function  $Q_n^{(2)}(z)$  by  $Q_n^{(2)}(z) = \beta^{\lambda_n^{(1)}} p_n(z)$ . Then  $deg(Q_n^{(2)}) \leq \lambda_n^{(2)}$ . Property (1) can be obtained. By (6), we have

$$\begin{split} \left\| \beta Q_{n}^{(2)}(z - \zeta_{0}) \right\|_{L} &\leq |\beta|^{\lambda_{n}^{(1)}} \|\beta p_{n}\|_{L - \zeta_{0}} \\ &\leq |\beta|^{\lambda_{n}^{(1)}} \|\beta p_{n} - f_{n}\|_{K} \\ &\leq |\beta|^{\lambda_{n}^{(1)}} (\gamma)^{\lambda_{n}^{(2)}}. \end{split}$$

Using  $\lim_{n\to\infty} |\beta|^{\lambda_n^{(\sigma)}} |\gamma|^{\lambda_n^{(\sigma+1)}} = 0$ , we obtain Property (2).

On the other hand,

$$\begin{split} \left\| \beta p(z) + \beta Q_n^{(2)}(z - \zeta_0) - f_{j_2}(z) \right\|_{K_{m_2}} &= \left\| f_{j_2}(z + \zeta_0) - \beta p(z + \zeta_0) - \beta Q_n^{(2)}(z) \right\|_{K_{m_2} - \zeta_0} \\ &= \left\| \beta^{\lambda_n^{(1)}} (f_n(z) - \beta p_n(z)) \right\|_{K_{m_2} - \zeta_0} \\ &\leq \left| \beta \right|^{\lambda_n^{(1)}} (\gamma)^{\lambda_n^{(2)}}. \end{split}$$

So Property (3) can be obtained.

Secondly, in the case  $\sigma - 1(\sigma \ge 3)$ , we assume there exist polynomials  $\{Q_n^{(\sigma-1)}\}_{n\in\mathbb{N}}$  with properties (1), (2) and (3).

Thirdly, we will show  $\sigma(\sigma \ge 3)$  case. Since  $\left\{\beta Q_n^{(\sigma-1)}(z-\zeta_0)\right\}_{n\to\infty} \to 0$ , it follows that for n large enough,  $\left\|\sum_{k=2}^{\sigma-1}\beta Q_n^{(k)}(z)\right\|_{L-\zeta_0} < 1$ . Let  $d_n = deg\left(\sum_{k=2}^{\sigma-1}Q_n^{(k)}(z)\right)$ , since for every n,  $d_n \le \lambda_n^{(\sigma-1)}$ , Bernstein's Lemma (a) of [18] yields

$$\left| \sum_{k=2}^{\sigma-1} \beta Q_n^{(k)}(z) \right|^{\frac{1}{d_n}} \leq e^{g_D(z,\infty)} \left\| \sum_{k=2}^{\sigma-1} \beta Q_n^{(k)}(z) \right\|_{L-C_0}^{\frac{1}{d_n}} < e^{g_D(z,\infty)},$$

for  $D = \mathbb{C}_{\infty} - (L - \zeta_0)$  and  $z \in D \setminus \{\infty\}$ . The compact set  $L - \zeta_0$  is non-polar since it contains an open disk of center 0. The function  $e^{g_D(z,\infty)}$  is bounded and continuous on  $\widetilde{K}$ . Define  $A = \max_{z \in \widetilde{K}} |e^{g_D(z,\infty)}| + 1$ , we obtain

$$\left\|\beta \sum_{k=2}^{\sigma-1} Q_n^{(k)}(z)\right\|_{\widetilde{K}} < A^{d_n} \le A^{\lambda_n^{(\sigma-1)}} = A^{\sigma_n}.$$

Hence, by the definition of  $f_n$ , we see that

$$||f_n||_{\widetilde{K}} \leq (|\beta|)^{\sigma_n} (c + A^{\sigma_n}),$$

where  $c = \|f_{j_{\sigma}} - p\|_{\widetilde{K} + \zeta_0}$ . Similarly to  $\sigma = 2$ , we can fix a sequence of polynomials  $p_n$  with degree less or equal to  $\tau_n$  such that:

$$||f_n - \beta p_n||_K \leq (\gamma)^{\tau_n}, n \geq n_0.$$
 (7)

We set  $Q_n^{(\sigma)}(z) = \beta^{\lambda_n^{(\sigma-1)}} p_n(z)$ , so the degree of the terms of  $Q_n^{(\sigma)}$  is at most  $\lambda_n^{(\sigma)}$ . Using inequality (7), we have

$$\begin{aligned} \left\| \beta Q_n^{(\sigma)}(z - \zeta_0) \right\|_L &\leq |\beta|^{\lambda_n^{(\sigma-1)}} \|\beta p_n\|_{L - \zeta_0} \\ &\leq |\beta|^{\lambda_n^{(\sigma-1)}} \|\beta p_n - f_n\|_K \\ &\leq |\beta|^{\lambda_n^{(\sigma-1)}} (\gamma)^{\lambda_n^{(\sigma)}} \end{aligned}$$

and

$$\left\|\beta p(z) + \sum_{k=2}^{\sigma} \beta Q_n^{(k)}(z-\zeta_0) - f_{j_{\sigma}}(z)\right\|_{K_{m_{\sigma}}}$$

$$= \left\| f_{j_{\sigma}}(z + \zeta_{0}) - \beta p(z + \zeta_{0}) - \sum_{k=2}^{\sigma-1} \beta Q_{n}^{(k)}(z) - \beta Q_{n}^{(\sigma)}(z) \right\|_{K_{m_{\sigma}} - \zeta_{0}}$$

$$= \left\| \beta^{\lambda_{n}^{(\sigma-1)}}(f_{n}(z) - \beta p_{n}(z)) \right\|_{K_{m_{\sigma}} - \zeta_{0}}$$

$$\leq \left| \beta \right|^{\lambda_{n}^{(\sigma-1)}}(\gamma)^{\lambda_{n}^{(\sigma)}}.$$

Applying  $\lim_{n\to\infty} |\beta|^{\lambda_n^{(\sigma)}} |\gamma|^{\lambda_n^{(\sigma+1)}} = 0$  to the above two inequalities, Property (2) and (3) can be obtained.

Thus, we have constructed a sequence of polynomials  $\{Q_n^{(\sigma)}\}_{n\in\mathbb{N}}$  via a finite induction with the above three

**Step 2.** Now, we will show that there exists  $f \in \bigcup_n \bigcup_{\alpha \in \mathbb{C}} E\left(\{m_\sigma\}_{\sigma=1}^{\sigma_0}, \{j_\sigma\}_{\sigma=1}^{\sigma_0}, s, n, \alpha\right)$  such that  $\|f - g\|_L < \varepsilon$ . If  $\sigma = 1$ , we define f(z) = p(z). Inequality (4) shows  $||f - g||_L < \varepsilon$ . Since  $\lim_{n \to \infty} \lambda_n^{(1)} = +\infty$  and p is fixed as mentioned above,  $\lambda_{n_1}^{(1)} > deg(p)$  obviously. It follows that  $T_{\lambda_n^{(1)}}^{(\zeta_0)}(f) = p$ . Therefore, inequality (5) yields

$$\left\|\beta T_{\lambda_{n}^{(1)}}^{(\zeta_{0})}(f)-f_{j_{1}}\right\|_{K_{m_{1}}}=\left\|\beta p-f_{j_{1}}\right\|_{K_{m_{1}}}<\frac{1}{s}.$$

Otherwise, if  $\sigma \geq 2$ , let  $f(z) = p(z) + \sum_{k=2}^{\sigma_0} Q_{n_1}^{(k)}(z-\zeta_0)$  for a suitable choice of  $n_1 \in \mathbb{N}$ . Combining Property (2) with (4),

$$||f-g||_{L} \leq ||p-g||_{L} + \left\| \sum_{k=2}^{\sigma_{0}} Q_{n_{1}}^{(k)}(z-\zeta_{0}) \right\|_{L} < 2||p-g||_{L} < \varepsilon,$$

for  $n_1$  large enough.

Property (1) implies that  $deg(Q_n^{(\sigma)}) \le \lambda_n^{(\sigma)}$ . On the other hand, p is fixed as mentioned above and  $\lim_{n \to \infty} \lambda_n^{(1)} = 0$  $+\infty$ ,  $\lambda_n^{(\sigma+1)}>\lambda_n^{(\sigma)}$ , hence,  $T_{\lambda_n^{(\sigma)}}^{(\zeta_0)}(f)=f$ . Therefore,

$$\left\|\beta T_{\lambda_{n}^{(\zeta_{0})}}^{(\zeta_{0})}(f)-f_{j_{\sigma}}\right\|_{K_{m_{\sigma}}}=\left\|\beta p(z)+\sum_{k=2}^{\sigma}\beta Q_{n_{1}}^{(k)}(z-\zeta_{0})-f_{j_{\sigma}}(z)\right\|_{K}<\frac{1}{s},$$

for  $n_1$  large enough, which used Property (3). This completes the proof of the theorem.

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