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On f-prime radical in ordered semigroups

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Abstract: In this paper, we introduce the concepts of f-prime ideals, f-semiprime ideals and f-prime radicals in ordered semigroups. Furthermore, some results on f-prime radicals and f-primary decomposition of an ideal in an ordered semigroup are obtained.

Keywords: Ordered semigroup, f-prime ideal, f-semiprime ideal, f-prime radical, f-primary decomposition

MSC: 06F05, 20M10

1 Introduction and preliminaries

Prime radical theorem is an important result in commutative ring theory and commutative semigroup theory (see [1,2]). In [3], Hoo and Shum gave a similar result in a residuated negatively ordered semigroup. Wu and Xie extended the result to a commutative ordered semigroup by using m-systems in [4]. In [5], Tang and Xie characterized in detail the radicals of ideals in ordered semigroups. In 1969, Murata, Kurata and Marubayashi introduced the notions of f-prime ideals and f-prime radicals in ring theory (see [6]), which generalized the concepts of prime ideals and prime radicals. In [7], Sardar and Goswami extended these concepts and results of ring theory to semirings. In this paper, we introduce the concepts of f-prime ideals and f-prime radicals in ordered semigroups and extend some results of rings and semirings to ordered semigroups. We also introduce the notion of f-semiprime ideals in ordered semigroups and obtain the result that the f-prime radical of an ideal I in an ordered semigroup is the least f-semiprime ideal containing I.

Next we list some basic concepts and notations on ordered semigroups (see [8]). An *ordered semigroup* is a semigroup $(S, \cdot)$ endowed with an order relation " $\leq$ " such that

$$(\forall a, b, x \in S)\ a \leq b \Rightarrow xa \leq xb \text{ and } ax \leq bx.$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset I of S is called an *ideal* of S if it satisfies the following conditions: (1) $SI \cup IS \subseteq I$; (2) $a \in I$ and $b \in S$, $b \leq a$ implies $b \in I$. An ideal I of S is called *weakly prime* if $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ for any ideals A, B of S; an ideal I of S is called *weakly semiprime* if $A^2 \subseteq I$ implies $A \subseteq I$ for any ideal A of S. For $h \in S$, we denote

$$(h] = \{t \in S \mid t \leq h\};$$

$$[h) = \{s \in S \mid h \leq s\}.$$

A subset M of S is called an *m-system* of S, if for any $a, b \in M$, there exists $x \in S$ such that $(axb] \cap M \neq \varnothing$. A subset N of S is said to be a *n-system* of S if for any $a \in N$, there exists $x \in S$ such that $[axa) \subseteq N$.

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2 f-prime ideals and f-prime radical of an ideal

Definition 2.1. *Let S be an ordered semigroup. Denote by $I(S)$ the set of all ideals of S. Define a mapping $f : S \to I(S)$ which satisfies the following conditions*
(1) $a \in f(a)$;
(2) $x \in f(a) \cup I$ implies that $f(x) \subseteq f(a) \cup I$ for any $I \in I(S)$.
We call such mapping f a good mapping on S.

Example 2.2. *Let S be an ordered semigroup. If $f(a) = I(a)$ for all $a \in S$, where $I(a)$ is the principal ideal generated by a, then it is easy to see that f satisfies the above conditions.*

Example 2.3. *We consider the ordered semigroup $S = \{a, b, c\}$ defined by multiplication and the order below:*

$\cdot$	a	b	c
a	a	a	a
b	a	a	b
c	a	b	b

$$\leq := \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}.$$

The ideals of S are the sets:

$$\{a\},\ \{a, b\} \text{ and } S.$$

If we define $f(a) = \{a\}$, $f(b) = \{a, b\}$ and $f(c) = S$, then it is easy to see that f satisfies the above conditions.

Definition 2.4. *Let S be an ordered semigroup and f a good mapping on S. A subset F of S is called an f-system of S if F contains an m-system F^*, called the kernel of F, such that $f(t) \cap F^* \neq \varnothing$ for any $t \in F$.*

Especially, $\varnothing$ is also defined to be an f-system.

Remark 2.5. *(1) Every m-system is an f-system with kernel itself.*
(2) If F is an f-system with kernel F^*, then $F = \varnothing$ if and only if $F^* = \varnothing$.

Definition 2.6. *Let S be an ordered semigroup and f a good mapping on S. An ideal I of S is called f-prime if its complement $C(I)$ in S is an f-system.*

Remark 2.7. *As we know, the complement of a weakly prime ideal of an ordered semigroup is an m-system. Furthermore, every m-system is an f-system. Hence, every weakly prime ideal of an ordered semigroup is an f-prime ideal. But the converse is not true in general.*

Example 2.8. *We consider the ordered semigroup S in Example 2.3. Let $I = \{a\}$. Then $C(I) = \{b, c\}$ is an f-system with kernel $F^* = \{b\}$. Hence, I is an f-prime ideal. However, I is not weakly prime. Indeed: $\{a, b\}$ is an ideal of S and $\{a, b\}\{a, b\} \subseteq I$, but $\{a, b\}$ is not contained in I.*

Proposition 2.9. *Let P be an f-prime ideal of an ordered semigroup S and $a, b \in S$. If $f(a)f(b) \subseteq P$, then either $a \in P$ or $b \in P$.*

Proof. Suppose that $a, b \in C(P)$. Since P is an f-prime ideal, $C(P)$ is an f-system. Hence, $f(a) \cap [C(P)]^* \neq \varnothing$ and $f(b) \cap [C(P)]^* \neq \varnothing$, where $[C(P)]^*$ is the kernel of $C(P)$. Let $x_1 \in f(a) \cap [C(P)]^*$ and $x_2 \in f(b) \cap [C(P)]^*$. Since $[C(P)]^*$ is an m-system, $(x_1 r x_2] \cap [C(P)]^* \neq \varnothing$ for some $r \in S$. Thus $(x_1 r x_2] \cap C(P) \neq \varnothing$. Also $x_1 r x_2 \in f(a)f(b) \subseteq P$. Thus $(x_1 r x_2] \subseteq P$ which is a contradiction. Therefore, either $a \in P$ or $b \in P$. □

Corollary 2.10. *Let P be an f-prime ideal of an ordered semigroup S and $a_i \in S$ $(i = 1, 2, \cdots, n)$. If $f(a_1)f(a_2)\cdots f(a_n) \subseteq P$, then $a_i \in P$ for some i.*

Let S be an ordered semigroup. Denote by $fPI(S)$ and $fS(S)$ the set of all f-prime ideals of S and the set of all f-systems of S respectively.

Definition 2.11. *Let S be an ordered semigroup and I be an ideal of S. We call the set $\{a \in S \mid (\forall F \in fS(S))\ a \in F \Rightarrow F \cap I \neq \varnothing\}$ the f-prime radical of I, denoted by $r_f(I)$.*

Theorem 2.12. *Let S be an ordered semigroup and I be an ideal of S. Then $r_f(I) = \cap_{P \in \Gamma} P$ where $\Gamma = \{P \in fPI(S) \mid I \subseteq P\}$.*

Proof. Let $J = \cap_{P \in \Gamma} P$ where $\Gamma = \{P \in fPI(S) \mid I \subseteq P\}$. If $x \notin J$, then there exists an f-prime ideal P containing I such that $x \notin P$. Thus $C(P)$ is an f-system containing x but $C(P) \cap I = \varnothing$. Hence $x \notin r_f(I)$. Therefore, $r_f(I) \subseteq J$. On the other hand, if $y \notin r_f(I)$, then there exists an f-system F containing y such that $F \cap I = \varnothing$. Thus $y \notin C(F)$ and $C(F)$ is an f-prime ideal such that $I \subseteq C(F)$. Hence $y \notin J$ and so $J \subseteq r_f(I)$. Therefore, $r_f(I) = J$. □

Corollary 2.13. *Let S be an ordered semigroup and I be an ideal of S. Then $r_f(I)$ is an ideal of S.*

3 f-semiprime ideals

In this section, we introduce the concept of f-semiprime ideals in ordered semigroups and obtain that the f-prime radical of an ideal I is the least f-semiprime ideal containing I.

Definition 3.1. *Let S be an ordered semigroup. A subset A of S is said to be an fn-system of S if $A = \bigcup\limits_{i \in \Gamma} F_i$ where $\{F_i \mid i \in \Gamma\} \subseteq fS(S)$.*

Definition 3.2. *An ideal P of an ordered semigroup S is said to be f-semiprime if its complement $C(P)$ in S is an fn-system.*

Clearly, every f-system is an fn-system. Therefore, every f-prime ideal is an f-semiprime ideal.

Proposition 3.3. *Let S be an ordered semigroup and I a weakly semiprime ideal of S. Then I is an f-semiprime ideal of S.*

Proof. Since I is a weakly semiprime ideal, $C(I)$ is a n-system. Thus $C(I)$ is the union of some m-systems of S. Since every m-system is an f-system, $C(I)$ is the union of some f-systems of S. Hence $C(I)$ is an fn-system. Therefore, I is an f-semiprime ideal. □

Proposition 3.4. *Let P be an f-semiprime ideal of an ordered semigroup S and $a \in S$. If $f(a)f(a) \subseteq P$, then $a \in P$.*

Proof. Suppose that $a \in C(P)$. Since P is an f-semiprime ideal, $C(P)$ is an fn-system. Thus $C(P) = \bigcup\limits_{i \in \Gamma} F_i$, where $\{F_i \mid i \in \Gamma\} \subseteq fS(S)$. Hence, $a \in F_i$ for some $i \in \Gamma$. Therefore, $f(a) \cap F_i^* \neq \varnothing$, where F_i^* is the kernel of F_i. Let $x \in f(a) \cap F_i^*$. Since F_i^* is an m-system, $(xrx] \cap F_i^* \neq \varnothing$ for some $r \in S$. Thus $(xrx] \cap F_i \neq \varnothing$ and so $(xrx] \cap C(P) \neq \varnothing$. Also $xrx \in f(a)f(a) \subseteq P$. Therefore, $(xrx] \subseteq P$ which is a contradiction. Hence $a \in P$. □

Proposition 3.5. *Let S be an ordered semigroup and I be an ideal of S. Then $r_f(I)$ is an f-semiprime ideal of S.*

Proof. By Theorem 2.12, we know that $r_f(I) = \cap_{P \in \Gamma} P$ where $\Gamma = \{P \in fPI(S) \mid I \subseteq P\}$. Thus $C(r_f(I)) = \bigcup_{P \in \Gamma} C(P)$. Since every $C(P)$ is an f-system, $C(r_f(I))$ is an fn-system. Therefore, $r_f(I)$ is an f-semiprime ideal of S. □

Proposition 3.6. *Let S be an ordered semigroup and I be an f-semiprime ideal of S. Then $r_f(I) = I$.*

Proof. By Theorem 2.12, we have $I \subseteq r_f(I)$. Let $x \notin I$. Since $C(I)$ is an fn-system, $C(I) = \bigcup_{i \in \Gamma} F_i$ where $\{F_i \mid i \in \Gamma\} \subseteq fS(S)$. Thus $x \in \bigcup_{i \in \Gamma} F_i$ and so $x \in F_i$ for some $i \in \Gamma$, but $F_i \cap I = \varnothing$. Hence F_i is an f-system containing x but not containing any element of I. By Definition 2.11, $x \notin r_f(I)$. It follows that $r_f(I) \subseteq I$. Therefore $r_f(I) = I$. □

Definition 3.7. *Let S be an ordered semigroup and f a good mapping on S. Let $a \in S$ and $I \in I(S)$. The set $\{x \in S \mid f(a)f(x) \subseteq I\}$, denoted by $I : a$, is called the left f-quotient of I by a. Moreover, for any ideal J of S, the left f-quotient of I by J is defined to be $\bigcap_{a \in J}(I : a)$, denoted by $I : J$.*

Remark 3.8. *Similar to Definition 3.7, we call the set $\{x \in S \mid f(x)f(a) \subseteq I\}$ right f-quotient of I by a, denoted by $a : I$. Moreover, $\bigcap_{a \in J}(a : I)$ is defined to be right f-quotient of I by J. In what follows, unless otherwise mentioned, f-quotient means left f-quotient.*

We note that $I : a$ may be empty. See the following example.

Example 3.9. *We consider the ordered semigroup S of Example 2.3. If we define $f(a) = f(b) = f(c) = S$, then the mapping f is a good mapping. Let $I = \{a\}$. Then $I : a,\ I : b$ and $I : c$ are all empty.*

The following result can be easily obtained from the above definition.

Proposition 3.10. *Let S be an ordered semigroup and f a good mapping on S. If $I, I^{'}, I^{''}, J, J^{'}, J^{''} \in I(S)$ and $a \in S$, then*
(1) $I^{'} \subseteq I^{''} \Rightarrow I^{'} : a \subseteq I^{''} : a$ and $I^{'} : J \subseteq I^{''} : J$;
(2) $J^{'} \subseteq J^{''} \Rightarrow I : J^{'} \supseteq I : J^{''}$;
(3) $(I^{'} \cap I^{''}) : a = (I^{'} : a) \cap (I^{''} : a)$ and $(I^{'} \cap I^{''}) : J = (I^{'} : J) \cap (I^{''} : J)$.

Proposition 3.11. *Let S be an ordered semigroup and f a good mapping on S. If $I \in I(S)$ and $a \in S$, then $I : a$ is either empty or an ideal containing I of S.*

Proof. Suppose that $I : a \neq \varnothing$. Let $x \in I : a$ and $r \in S$. Then $rx,\ xr \in f(x)$. Thus $f(rx) \subseteq f(x)$ and $f(xr) \subseteq f(x)$. Also $f(a)f(x) \subseteq I$. Therefore, $f(a)f(rx) \subseteq I$ and $f(a)f(xr) \subseteq I$. Let $z \leq y \in I : a$. Then $z \in f(y)$ and $f(a)f(y) \subseteq I$. Thus $f(z) \subseteq f(y)$. Therefore, $f(a)f(z) \subseteq I$, which implies that $z \in I : a$. Hence, $I : a$ is an ideal of S. Next we prove that $I \subseteq I : a$.

Let $b \in I$ and $x \in I : a$. Then $b \in f(x) \cup I$. Thus $f(b) \subseteq f(x) \cup I$. Also $f(a)f(x) \subseteq I$. It follows that $f(a)f(b) \subseteq f(a)(f(x) \cup I) = f(a)f(x) \cup f(a)I \subseteq I$. Hence $b \in I : a$ and so $I \subseteq I : a$. □

Let S be an ordered semigroup and f a good mapping on S. Denote the following condition by (α):

$$(\forall F \in fS(S))\ (\forall I \in I(S))\ \ F \cap I \neq \varnothing \Rightarrow F^{*} \cap I \neq \varnothing.$$

If $f(a) = I(a)$ for every $a \in S$, then S satisfies the condition (α). But this is not true for any good mapping f. See the following example.

Example 3.12. *We consider the ordered semigroup $S = \{a, b, c\}$ defined by multiplication and the order below:*

$\cdot$	a	b	c
a	a	a	a
b	a	a	b
c	a	b	c

$$\leq := \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}.$$

It is easy to check that S is an ordered semigroup. The ideals of S are the sets:

$$\{a\},\ \{a, b\} \text{ and } S.$$

If we define $f(a) = \{a\}$, $f(b) = f(c) = S$, then it is easy to see that f is a good mapping. Let $I = \{a, b\}$ and $F = \{b, c\}$. Then F is an f-system with kernel $F^* = \{c\}$ and $F \cap I \neq \varnothing$. However, $F^* \cap I = \varnothing$.

Proposition 3.13. *Let S be an ordered semigroup and f a good mapping on S. If $I, J \in I(S)$, then*
(1) $I \subseteq J \Rightarrow r_f(I) \subseteq r_f(J)$;
(2) $r_f(r_f(I)) = r_f(I)$;
(3) $r_f(I \cup J) = r_f(r_f(I) \cup r_f(J))$;
(4) $r_f(I \cap J) = r_f(I) \cap r_f(J)$, if S satisfies the condition (α).

Proof. (1) and (2) are obvious.

(3) From Theorem 2.12, we have $I \cup J \subseteq r_f(I) \cup r_f(J)$. By condition (1), we have $r_f(I \cup J) \subseteq r_f(r_f(I) \cup r_f(J))$ and $r_f(I) \cup r_f(J) \subseteq r_f(I \cup J)$. Combining conditions (1) and (2), we obtain $r_f(r_f(I) \cup r_f(J)) \subseteq r_f(r_f(I \cup J)) = r_f(I \cup J)$.

(4) It is obvious that $r_f(I \cap J) \subseteq r_f(I) \cap r_f(J)$ from condition (1). Next we prove the other inclusion. Let $x \in r_f(I) \cap r_f(J)$ and F be an f-system containing x. Suppose that $a \in F \cap I$ and $b \in F \cap J$. By assumption (α), there exist $a^* \in F^* \cap I$ and $b^* \in F^* \cap J$. Since F^* is an m-system, $(a^* z b^*] \cap F^* \neq \varnothing$ for some $z \in S$. Thus $(a^* z b^*] \cap F \neq \varnothing$. Moreover, $a^* z b^* \in I \cap J$ and so $(a^* z b^*] \subseteq I \cap J$. Hence $F \cap (I \cap J) \neq \varnothing$. It follows that $x \in r_f(I \cap J)$. Therefore $r_f(I \cap J) = r_f(I) \cap r_f(J)$. □

Combining Proposition 3.5, 3.6 and 3.13 (1), we have the following result.

Theorem 3.14. *Let I be an ideal of an ordered semigroup S. Then $r_f(I)$ is the least f-semiprime ideal containing I.*

4 *f*-primary decomposition of an ideal

In this section, we introduce the concepts of f-primary ideals and f-primary decomposition of an ideal in ordered semigroups, and conclude that the number of f-primary components and the f-prime radicals of f-primary components of a normal decomposition of an ideal I depend only on I under some assumptions.

Definition 4.1. *Let S be an ordered semigroup and f a good mapping on S. An ideal I of S is called left f-primary if $f(a)f(b) \subseteq I$ implies that $a \in r_f(I)$ or $b \in I$.*

Remark 4.2. *By symmetry, we call an ideal I of S right f-primary if $f(a)f(b) \subseteq I$ implies that $a \in I$ or $b \in r_f(I)$. In what follows unless otherwise mentioned, f-primary means left f-primary.*

By Proposition 2.9, we note that f-prime ideals must be f-primary ideals.

From Definition 4.1, we obtain easily the following result.

Proposition 4.3. *Let S be an ordered semigroup satisfying the condition (α). If $I^{'}$ and $I^{''}$ are f-primary ideals of S such that $r_f(I^{'}) = r_f(I^{''})$, then $I = I^{'} \cap I^{''}$ is also an f-primary ideal of S such that $r_f(I) = r_f(I^{'}) = r_f(I^{''})$.*

Let S be an ordered semigroup and f a good mapping on S. Denote the following condition by (β):

$$(\forall I, J \in I(S)) \;\; J \nsubseteq r_f(I) \Rightarrow I : J \neq \varnothing.$$

Theorem 4.4. *Let S be an ordered semigroup satisfying the condition (β). An ideal I of S is f-primary if and only if $I : J = I$ for all ideals $J \nsubseteq r_f(I)$.*

Proof. Suppose that I is an f-primary ideal of S and J is an ideal of S not contained in $r_f(I)$. Since S satisfies the condition (β), $I : J \neq \varnothing$. Thus $I : b \neq \varnothing$ for all $b \in J$. Hence $I \subseteq I : b$ for every $b \in J$ and so $I \subseteq I : J$. Now we choose an element $c \in J \setminus r_f(I)$. By the condition (β), $I : c \neq \varnothing$. Moreover, $f(c)f(a) \subseteq I$ for any $a \in I : c$.

Since I is f-primary and $c \notin r_f(I)$, $a \in I$. Thus $I : c \subseteq I$. Therefore, $I = I : c$ and so $I : J \subseteq I : c = I$. Consequently, $I = I : J$.

Conversely, suppose that $I : J = I$ for all ideals J not contained in $r_f(I)$. Let $f(a)f(b) \subseteq I$ and $a \notin r_f(I)$. Since $a \in f(a)$, $f(a)$ is not a subset of $r_f(I)$ and so $I : f(a) = I$. For any $a^{'} \in f(a)$, $f(a^{'}) \subseteq f(a)$. Thus $f(a^{'})f(b) \subseteq f(a)f(b) \subseteq I$ and thus $b \in I : f(a) = I$. It follows that I is f-primary. □

Definition 4.5. *If an ideal I of an ordered semigroup S can be written as $I = I_1 \cap I_2 \cap \cdots \cap I_n$ where each I_i is an f-primary ideal, then this is called an f-primary decomposition of I and each I_i is called the f-primary component of the decomposition.*

A f-primary decomposition in which no I_i contains the intersection of the remaining I_j is called irredundant. Moreover, an irredundant f-primary decomposition in which the radicals of the various f-primary components are all different is called a normal decomposition.

From Proposition 4.3, we note that each f-primary decomposition can be refined into one which is normal.

Let S be an ordered semigroup and f a good mapping on S. Denote the following condition by (γ): if for any f-primary ideal I of S, we have $I : I = S$. If $f(a) = I(a)$ for all $a \in S$, then S satisfies the condition (γ). But the condition (γ) need not be satisfied for any good mapping f.

Example 4.6. *From Example 2.8, $P = \{a\}$ is an f-prime ideal of S and so P is f-primary. However $P : P = \{a\} \neq S$.*

Theorem 4.7. *Let S be an ordered semigroup satisfying the conditions (α), (β) and (γ). If an ideal A of S has two normal f-primary decompositions $A = \bigcap_{i=1}^{n} I_i = \bigcap_{i=1}^{m} I'_i$, then $n = m$ and $r_f(I_i) = r_f(I'_i)$ for $1 \le i \le n = m$ by a suitable ordering.*

Proof. It is easy to see that the result holds in the case $A = S$. Next we prove the case that $A \neq S$, where all f-primary components $I_1, \cdots, I_n, I'_1, \cdots, I'_m$ are proper ideals. We may assume that $r_f(I_1)$ is maximal in the set $\{r_f(I_1), \cdots, r_f(I_n), r_f(I'_1),$
$\cdots, r_f(I'_m)\}$. Now we prove that $r_f(I_1) = r_f(I'_i)$ for some i. It is enough to show that $I_1 \subseteq r_f(I'_i)$. Suppose that $I_1 \nsubseteq r_f(I'_i)$ for all $1 \le i \le m$. Then we have, by Theorem 4.4, $I'_i : I_1 = I'_i$ for all $1 \le i \le m$, and so

$$\begin{aligned} A : I_1 &= (I'_1 \cap I'_2 \cap \cdots \cap I'_n) : I_1 \\ &= (I'_1 : I_1) \cap (I'_2 : I_1) \cap \cdots \cap (I'_n : I_1) \\ &= I'_1 \cap I'_2 \cap \cdots \cap I'_n \\ &= A. \end{aligned}$$

If $n = 1$, then, by the condition (γ), $S = I_1 : I_1 = A : I_1 = A$ which is a contradiction. If $n > 1$, then, by the condition (γ) and the fact that $I_1 \nsubseteq r_f(I_i)$ for all $2 \le i \le n$, we have

$$\begin{aligned} A &= A : I_1 \\ &= (I_1 \cap I_2 \cap \cdots \cap I_n) : I_1 \\ &= (I_1 : I_1) \cap (I_2 : I_1) \cap \cdots \cap (I_n : I_1) \\ &= I_2 \cap \cdots \cap I_n. \end{aligned}$$

This is also a contradiction. By a suitable ordering, we may have $r_f(I_1) = r_f(I'_1)$.

We use an induction on the number n of f-primary components. If $n = 1$, then $A = I_1 = \bigcap_{i=1}^{m} I'_i$. Suppose that $m > 1$. Then $I_1 \nsubseteq r_f(I_i)$ for all $2 \le i \le m$. Since $S = I_1 : I_1 = \bigcap_{i=1}^{m} (I'_i : I_1)$, we have, by Theorem 4.4, $S = I'_2 = \cdots = I'_m$ which is a contradiction. Thus $m = 1 = n$. Now let us suppose that the conclusions hold for the ideals which are represented by fewer than n f-primary components. Let $I = I_1 \cap I'_1$. Then I is an f-primary ideal such that $r_f(I) = r_f(I_1) = r_f(I'_1)$ from Proposition 4.3. By the condition (γ), we have $S = I_1 : I_1 \subsetneqq I_1 : I$

and thus $I_1 : I = S$. From the fact that $I \nsubseteq r_f(I_i)$ for all $2 \leq i \leq n$, we obtain $I_i : I = I_i$. Hence $A : I = \bigcap_{i=2}^{n} I_i$. Similarly, we can show that $A : I = \bigcap_{i=2}^{m} I'_i$. Consequently, $A : I = \bigcap_{i=2}^{n} I_i = \bigcap_{i=2}^{m} I'_i$ and the decompositions are both normal. By the induction hypothesis, we have $n - 1 = m - 1$ and so $n = m$. Moreover, by a suitable ordering, we have $r_f(I_i) = r_f(I'_i)$ for all $2 \leq i \leq n = m$. □

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