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Research Article

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One kind power mean of the hybrid Gauss sums

Abstract: In this paper, we use the analysis method and the properties of trigonometric sums to study the computational problem of one kind power mean of the hybrid Gauss sums. After establishing some relevant lemmas, we give an exact computational formula for it. As an application of our result, we give an exact formula for the number of solutions of one kind diagonal congruence equation mod p, where p be an odd prime.

Keywords: Gauss sums, Power mean, Analysis method, Computational formula

MSC: 11L03, 11L05

1 Introduction

As usual, let $q \geq 3$ be a positive integer. For any positive integer $n \geq 2$, the classical n-th Gauss sums $G(m, n; q)$ is defined by

$$G(m, n; q) = \sum_{a=0}^{q-1} e\left(\frac{ma^n}{q}\right),$$

where $e(y) = e^{2\pi i y}$.

Many mathematical scholars have studied the arithmetical properties concerning $G(m, n; q)$ and have obtained various interesting results, see references [1]-[9] and [11]. For example, Shimeng Shen and Wenpeng Zhang [2] studied the computational problem of the number $M_n(p)$ of solutions of the congruence equation

$$x_1^4 + x_2^4 + \cdots + x_n^4 \equiv 0 \bmod p, \quad 0 \leq x_i \leq p - 1, \ i = 1, \ 2, \cdots, n,$$

and proved the following conclusions:

Let p be a prime with $p = 8k + 5$, $U_n(p) = M_n(p) - p^{n-1}$. Then for any positive integer $n \geq 5$, one has the fourth-order linear recurrence formula

$$U_n(p) = -2pU_{n-2}(p) + 4p\alpha(p)U_{n-3}(p) - \left(9p^2 - p\alpha^2(p)\right)U_{n-4}(p),$$

where the first four terms are $U_1(p) = 0$, $U_2(p) = -(p - 1)$, $U_3(p) = 3(p - 1)\alpha(p)$ and $U_4(p) = -7p(p - 1) + (p - 1)\alpha^2(p)$.

If $p = 8k + 1$, then for any positive integer $n \geq 5$, one has the fourth-order linear recurrence formula

$$U_n(p) = 6pU_{n-2}(p) + 4p\alpha(p)U_{n-3}(p) - \left(p^2 - p\alpha^2(p)\right)U_{n-4}(p),$$

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where the first four terms are $U_1(p) = 0$, $U_2(p) = 3(p - 1)$, $U_3(p) = 3(p - 1)\alpha(p)$, $U_4(p) = 17p(p - 1) + (p - 1)\alpha^2(p)$, and $\alpha(p) = \sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + \overline{a}}{p}\right)$, $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol mod p, and $\overline{a}$ denotes the multiplicative inverse of a mod p.

Xiaoxue Li and Jiayuan Hu [3] obtained the identity

$$\sum_{b=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right)\right|^2 \cdot \left|\sum_{c=1}^{p-1} e\left(\frac{bc + \overline{c}}{p}\right)\right|^2$$

$$= \begin{cases} 3p^3 - 3p^2 - 3p + p\left(\tau^2\left(\overline{\chi}_4\right) + \tau^2(\chi_4)\right), & \text{if } p \equiv 5 \bmod 8; \\ 3p^3 - 3p^2 - 3p - p\tau^2\left(\overline{\chi}_4\right) - p\tau^2(\chi_4) + 2\tau^5\left(\overline{\chi}_4\right) + 2\tau^5(\chi_4), & \text{if } p \equiv 1 \bmod 8, \end{cases}$$

where χ_4 denotes any fourth-order character mod p, $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a)e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums.

At the same time, Xiaoxue Li and Jiayuan Hu [3] also pointed out that how to compute the exact value of $\tau^2\left(\overline{\chi}_4\right) + \tau^2(\chi_4)$ and $\tau^5\left(\overline{\chi}_4\right) + \tau^5(\chi_4)$ are two meaningful problems.

Let $A(k, p) = \tau^k(\chi_4) + \tau^k\left(\overline{\chi_4}\right)$. Zhuoyu Chen and Wenpeng Zhang [9] studied the computational problem of $A(k, p)$, and obtained two interesting linear recurrence formulas. That is, let p be an odd prime with $p \equiv 1 \bmod 4$. Then for any positive integer k, one has the linear recurrence formulas

$$A(2k + 2, p) = 2\sqrt{p} \cdot \alpha(p) \cdot A(2k, p) - p^2 \cdot A(2k - 2, p)$$

and

$$A(2k + 3, p) = 2\sqrt{p} \cdot \alpha(p) \cdot A(2k + 1, p) - p^2 \cdot A(2k - 1, p),$$

where $A(0, p) = 2$, $A(1, p) = G(1) - \sqrt{p}$, $G(1) = G(1, 4; p)$, $A(2, p) = 2\sqrt{p}\alpha(p)$, $A(3, p) = \sqrt{p} \cdot \left(2\alpha - (-1)^{\frac{p-1}{4}} \sqrt{p}\right) \cdot (G(1) - \sqrt{p})$.

In this paper, as a note of [2] and [9], we shall consider the computational problem of one kind hybrid power mean of two different Gauss sums

$$\sum_{m=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^{2h} \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right)\right|^{2k}, \tag{1}$$

where $p = 12r + 1$ is an odd prime, k and h are two non-negative integers.

What we are interested in is whether there exits an exact computational formula for (1). Through researches mentioned above we found that for some special prime p we can give an an efficient method to compute the value of (1). The main purpose of this paper is to illustrate this point. That is, we shall prove the following main results:

Theorem 1.1. *Let p be a prime with $p = 24r + 13$. Then for any positive integers h and k, we have the identity*

$$\sum_{m=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^{2h} \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right)\right|^{2k} = \frac{1}{2} \cdot p^{\frac{k}{2}} \cdot \left[\left(3\sqrt{p} + 2\alpha(p)\right)^k + \left(3\sqrt{p} - 2\alpha(p)\right)^k\right] \cdot \sum_{m=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^{2h},$$

where $\alpha(p) = \sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + \overline{a}}{p}\right)$, $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol mod p, $\overline{a}$ denotes the multiplicative inverse of a mod p.

From Theorem 1.1 we may immediately deduce the corollaries as follows.

Corollary 1.2. *If p is a prime with $p = 24r + 13$, then for any positive integer k, we have*

$$\sum_{m=1}^{p-1} \left|\sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right)\right|^{2k} = \frac{1}{2}(p - 1) \cdot p^{\frac{k}{2}} \cdot \left[\left(3\sqrt{p} + 2\alpha(p)\right)^k + \left(3\sqrt{p} - 2\alpha(p)\right)^k\right].$$

Corollary 1.3. *If p is a prime with $p = 24r + 13$, then for any positive integer k, we have*

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right|^2 \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right) \right|^{2k} = (p-1) \cdot p^{\frac{k+2}{2}} \cdot \left[\left(3\sqrt{p} - 2\alpha(p)\right)^k + \left(3\sqrt{p} + 2\alpha(p)\right)^k \right].$$

Corollary 1.4. *If $p = 24r + 13$ is an odd prime, then for any positive integer k, we have*

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right|^4 \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right) \right|^{2k} = 3(p-1) \cdot p^{\frac{k+4}{2}} \cdot \left[\left(3\sqrt{p} + 2\alpha(p)\right)^k + \left(3\sqrt{p} - 2\alpha(p)\right)^k \right].$$

Corollary 1.5. *If $p = 24r + 13$ is an odd prime, then for any positive integer k, we have*

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right|^6 \cdot \left| \sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right) \right|^{2k} = \frac{1}{2}(p-1)(18p + d^2) \cdot p^{\frac{k+4}{2}} \cdot \left[\left(3\sqrt{p} - 2\alpha(p)\right)^k + \left(3\sqrt{p} + 2\alpha(p)\right)^k \right],$$

where d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

Let k and h be two positive integers, p is a prime with $p = 24r + 13$, and $M(h, k; p)$ denotes the number of solutions of the congruence equation

$$x_1^3 + \cdots + x_h^3 + y_1^4 + \cdots + y_k^4 \equiv z_1^3 + \cdots + z_h^3 + w_1^4 + \cdots + w_k^4 \bmod p,$$

where $0 \le x_i,\ z_i\ y_j,\ w_j \le p - 1$, $i = 1,\ 2, \cdots,\ h, j = 1,\ 2, \cdots,\ k$.

Then from Theorem 1.1 we can give an exact computational method for $M(h, k; p)$. In particular, we have the following:

Corollary 1.6. *If $p = 24r + 13$ is an odd prime, then for any positive integer k, we have*

$$M(2, k; p) = p^{2k+3} + 3(p-1) \cdot p^{\frac{k+2}{2}} \cdot \left[\left(3\sqrt{p} - 2\alpha(p)\right)^k + \left(3\sqrt{p} + 2\alpha(p)\right)^k \right].$$

If prime $p = 24r + 1$, then the situation is more complex, we can only give an effective calculation method one by one. Theorem 1.7 indicates some examples of it.

Theorem 1.7. *If p is an odd prime with $p = 24r + 1$, then we have the identities*

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right|^2 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^2 = 6(p-1) \cdot p^2.$$

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right|^4 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^4 = 6p^3(p-1)\left(17p + 4\alpha^2(p)\right).$$

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right|^6 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^6 = 3p^4(p-1)\left(33p + 28\alpha^2(p)\right) \cdot \left(18p + d^2\right).$$

Some notes: If $3 \nmid (p - 1)$, then for any integer m with $(m, p) = 1$, we have

$$\left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right| = \left| \sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) \right| = 0.$$

If prime $p = 4r + 3$, then we have

$$\left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right| = \left| 1 + \sum_{a=1}^{p-1} e\left(\frac{ma^2}{p}\right) + \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) e\left(\frac{ma^2}{p}\right) \right| = \sqrt{p}.$$

So in these cases, the problem we are studying is trivial.

2 Some simple lemmas

To prove our main results, we first propose several simple lemmas. During the proof process, we will apply some analytic number theory knowledge and the properties of character and trigonometric sums, all of which can be found in [1].

Lemma 2.1. *If p is an odd prime with $3|(p-1)$, ψ is any third-order character* mod p, *then we have*

$$\tau^3(\psi) + \tau^3(\overline{\psi}) = dp,$$

where $\tau(\psi)$ denotes the classical Gauss sums, d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1$ mod 3.

Proof. See [7] or [11]. □

Lemma 2.2. *If p is an odd prime with $p \equiv 1$ mod 4, ψ is any fourth-order character* mod p, *then we have*

$$\tau^2(\psi) + \tau^2(\overline{\psi}) = \sqrt{p} \cdot \sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right) = 2\sqrt{p} \cdot \alpha(p),$$

where $\left(\frac{*}{p}\right)$ is the Legendre's symbol mod p, $\alpha(p) = \sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)$ *is an integer and it satisfies the identity (see Theorem 4-11 in [10])*

$$p = \alpha^2 + \beta^2 \equiv \left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)\right)^2 + \left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+r\overline{a}}{p}\right)\right)^2,$$

and r is any quadratic non-residue mod p.

Proof. In fact this is Lemma 2 of [9], so its proof is omitted. □

Lemma 2.3. *If p is a prime with $p \equiv 5$ mod 8, then for any positive integer k, we have the identity*

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4}{p}\right)\right|^{2k} = \frac{1}{2}(p-1)\cdot p^{\frac{k}{2}}\cdot\left[\left(3\sqrt{p}-2\alpha(p)\right)^k + \left(3\sqrt{p}+2\alpha(p)\right)^k\right].$$

Proof. If $p = 8h+5$, then for any fourth-order character ψ mod p and any integer m with $(m,p) = 1$, applying the properties of the classic Gauss sums we have $\psi(-1) = -1$ and

$$\begin{aligned}
B(m) &= \sum_{a=0}^{p-1}e\left(\frac{ma^4}{p}\right) = 1 + \sum_{a=1}^{p-1}\left(1 + \psi(a) + \psi^2(a) + \overline{\psi}(a)\right)e\left(\frac{ma}{p}\right) \\
&= \sum_{a=0}^{p-1}e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1}\left(\psi(m) + \overline{\psi}(m)\right)e\left(\frac{ma}{p}\right) \\
&= \chi_2(m)\sqrt{p} + \overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi}),
\end{aligned} \tag{2}$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre's symbol mod p.

Note that $\overline{\overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})} = -\left(\overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})\right)$ (that is, it is a pure imaginary number) and $\psi^2 = \chi_2$, from (2) and Lemma 2.2 we have

$$\begin{aligned}
|B(m)|^2 &= \left|\chi_2(m)\sqrt{p} + \overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})\right|^2 \\
&= p + \left|\overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})\right|^2 = 3p - \chi_2(m)\left(\tau^2(\psi) + \tau^2(\overline{\psi})\right) \\
&= 3p - 2\chi_2(m)\sqrt{p}\alpha(p).
\end{aligned} \tag{3}$$

So for any positive integer k, from (3) and binomial theorem we have

$$|B(m)|^{2k} = \left(3p - 2\chi_2(m)\sqrt{p}\alpha(p)\right)^k = \sum_{i=0}^{k} \binom{k}{i}(3p)^{k-i}\left(-2\chi_2(m)\sqrt{p}\alpha(p)\right)^i$$

$$= \sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{k}{2i}3^{k-2i} \cdot p^{k-i} \cdot (2\alpha(p))^{2i} - \chi_2(m) \sum_{i=0}^{\left[\frac{k-1}{2}\right]} \binom{k}{2i+1}(3p)^{k-2i-1} \cdot \left(2\sqrt{p}\alpha(p)\right)^{2i+1}. \tag{4}$$

Note that $\sum_{m=1}^{p-1} \chi_2(m) = 0$, from (4) we may immediately deduce that

$$\sum_{m=1}^{p-1} |B(m)|^{2k} = \sum_{m=1}^{p-1}\sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{k}{2i}3^{k-2i} \cdot p^{k-i} \cdot (2\alpha(p))^{2i} - \sum_{m=1}^{p-1}\chi_2(m) \sum_{i=0}^{\left[\frac{k-1}{2}\right]} \binom{k}{2i+1}(3p)^{k-2i-1} \cdot \left(2\sqrt{p}\alpha(p)\right)^{2i+1}$$

$$= (p-1) \cdot \sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{k}{2i}3^{k-2i} \cdot p^{k-i} \cdot (2\alpha(p))^{2i}$$

$$= \frac{1}{2}(p-1)p^{\frac{k}{2}} \cdot \left[\left(3\sqrt{p} + 2\alpha(p)\right)^k + \left(3\sqrt{p} - 2\alpha(p)\right)^k\right].$$

This proves Lemma 2.3. □

Lemma 2.4. *If p is a prime with $p \equiv 1 \bmod 8$, then for any positive integer k, we have $S_1(p) = 0$, $S_2(p) = 3p(p-1)$, $S_3(p) = 6p(p-1)\alpha(p)$, and for $k \geq 4$, $S_k(p)$ satisfy the fourth-order linear recurrence formula*

$$S_k(p) = 6pS_{k-2}(p) + 8p\alpha(p)S_{k-3}(p) + p\left(4\alpha^2(p) - p\right)S_{k-4}(p),$$

where $S_k(p) = \sum_{m=1}^{p-1} B^k(m) = \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^k$.

Proof. If $p = 8r + 1$, then $\psi(-1) = 1$, so $\overline{\psi}(m)\tau(\psi) + \psi(m)\tau\left(\overline{\psi}\right)$ is a real number. From (2) and Lemma 2.2 we have

$$|B(m)|^2 = B^2(m) = 3p + 2\chi_2(m)\sqrt{p}\alpha(p) + 2\psi(m)\sqrt{p}\tau(\psi) + 2\overline{\psi}(m)\sqrt{p}\tau\left(\overline{\psi}\right). \tag{5}$$

From (2) and (5) we can deduce that

$$B^4(m) - 6pB^2(m) - 8p\alpha(p)B(m) + p^2 - 4p\alpha^2(p) = 0. \tag{6}$$

Note that the identities

$$\sum_{m=1}^{p-1} B(m) = 0, \quad \sum_{m=1}^{p-1} B^2(m) = 3p(p-1), \quad \sum_{m=1}^{p-1} B^3(m) = 6p(p-1)\alpha(p). \tag{7}$$

If $n \geq 4$, then from (6) we have

$$\sum_{m=1}^{p-1} B^n(m) = \sum_{m=1}^{p-1} B^{n-4}(m)B^4(m)$$

$$= \sum_{m=1}^{p-1} B^{n-4}(m)\left(6pB^2(m) + 8p\alpha(p)B(m) - p^2 + 4p\alpha^2(p)\right)$$

$$= 6p\sum_{m=1}^{p-1} B^{n-2}(m) + 8p\alpha(p)\sum_{m=1}^{p-1} B^{n-3}(m) + \left(4p\alpha^2(p) - p^2\right)\sum_{m=1}^{p-1} B^{n-4}(m). \tag{8}$$

So for any integer $k \geq 4$, from (7), (8) we know that $S_k(p)$ satisfy the fourth-order linear recurrence formula

$$S_k(p) = 6pS_{k-2}(p) + 8p\alpha(p)S_{k-3}(p) + p\left(4\alpha^2(p) - p\right)S_{k-4}(p).$$

This proves Lemma 2.4. □

Lemma 2.5. *If p is an odd prime with $p \equiv 1 \bmod 3$, then for any integer m with $(m, p) = 1$, we have $M_1(p) = 0$, $M_2(p) = 2p(p-1)$, $M_3(p) = dp(p-1)$, and for all $h \geq 4$, $M_h(p)$ satisfy the linear recurrence formula*

$$M_h(p) = 3pM_{h-2}(p) + dpM_{h-3}(p), \tag{9}$$

where $M_h(p) = \sum_{m=1}^{p-1} G^h(m) = \sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right)^h$, where d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$

Proof. It can be found in reference [4]. Here we give a simple proof. Let λ be any third-order character mod p. Then for any integer m with $p \nmid m$, from the definition and properties of the classical Gauss sums we have

$$G(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) = \sum_{a=0}^{p-1} \left(1 + \lambda(a) + \overline{\lambda}(a)\right) e\left(\frac{ma}{p}\right)$$

$$= \sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) + \sum_{a=0}^{p-1} \lambda(a) e\left(\frac{ma}{p}\right) + \sum_{a=0}^{p-1} \overline{\lambda}(a) e\left(\frac{ma}{p}\right)$$

$$= \overline{\lambda}(m)\tau(\lambda) + \lambda(m)\tau\left(\overline{\lambda}\right). \tag{10}$$

Note that $\tau(\lambda)\tau\left(\overline{\lambda}\right) = p$, $\lambda^3 = \chi_0$, the principal character mod p, from (10) and Lemma 2.1 we may deduce that

$$G^3(m) = \tau^3(\lambda) + \tau^3\left(\overline{\lambda}\right) + 3p\left(\overline{\lambda}(m)\tau(\lambda) + \lambda(m)\tau\left(\overline{\lambda}\right)\right) = dp + 3pG(m). \tag{11}$$

From (11) and the definition of $M_h(p)$ we can deduce Lemma 2.5. $\qquad\square$

3 Proofs of the main results

Now we will use the lemmas in section 2 to prove our main theorems. If $p = 12r + 1$, let ψ be any fourth-order character mod p, and λ be any third-order character mod p. Then note that

$$\sum_{m=1}^{p-1} \lambda^i(m)\psi^j(m) = 0, \ i = 0, 1, 2, j = 0, 1, 2, 3, (i, j) \neq (0, 0).$$

So the value of the power mean in (1) only depend on the constant terms in $|G(m)|^{2h}$ and $|B(m)|^{2k}$, those terms are independent of m. So we have

$$\sum_{m=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^{2h} \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right)\right|^{2k}$$

$$= \frac{1}{p-1} \left(\sum_{m=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^{2h}\right) \cdot \left(\sum_{m=1}^{p-1} \left|\sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right)\right|^{2k}\right). \tag{12}$$

If $p = 24r + 13$, then $p \equiv 5 \bmod 8$ and $p \equiv 1 \bmod 3$, from Lemma 2.3 we have

$$\sum_{m=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^{2h} \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^4}{p}\right)\right|^{2k}$$

$$= \left(\sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{k}{2i} 3^{k-2i} \cdot p^{k-i} \cdot (2\alpha(p))^{2i}\right) \cdot \sum_{m=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^{2h}$$

$$= \frac{1}{2} \cdot p^{\frac{k}{2}} \cdot \left[\left(3\sqrt{p} + 2\alpha(p)\right)^k + \left(3\sqrt{p} - 2\alpha(p)\right)^k\right] \cdot \sum_{m=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^{2h}.$$

This proves Theorem 1.1.

If $p = 24r + 1$, then we can calculate the value of $S_k(p)$ by Lemma 2.4 for any even number $k \geq 2$. We can also calculate the value of $M_h(p)$ by Lemma 2.5 for any even number $h \geq 2$. In fact this time note that $G(m)$ and $B(m)$ are both real numbers. So from (11) we have

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^2 = M_2(p) = 2p(p-1). \tag{13}$$

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^4 = M_4(p) = 6p^2(p-1). \tag{14}$$

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^6 = M_6(p) = (p-1)p^2\left(18p + d^2\right). \tag{15}$$

From Lemma 2.4 we have

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 = S_2(p) = 3p(p-1). \tag{16}$$

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^4 = S_4(p) = (p-1)p\left(17p + 4\alpha^2(p)\right). \tag{17}$$

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^6 = S_6(p) = 3(p-1)p^2\left(33p + 28\alpha^2(p)\right). \tag{18}$$

Then from (12)-(18) we may immediately deduce the identities

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^2 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 = 6(p-1)\cdot p^2.$$

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^4 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^4 = 6p^3(p-1)\left(17p + 4\alpha^2(p)\right).$$

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right|^6 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^6 = 3p^4(p-1)\left(33p + 28\alpha^2(p)\right)\cdot\left(18p + d^2\right).$$

This completes the proof of Theorem 1.7.

Competing interests
The authors declare that there are no conflicts of interest regarding the publication of this paper.

Author's contributions
All authors read and approved the final manuscript.

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