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Some recurrence formulas for the Hermite polynomials and their squares

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Abstract: In this paper, by making use of the generating function methods and Padé approximation techniques, we establish some new recurrence formulas for the Hermite polynomials and their squares. These results presented here are the corresponding extensions of some known formulas.

Keywords: Hermite polynomials, Padé approximants, Summation formulas, Recurrence formulas

MSC: 11B83, 05A19

1 Introduction

In the Sturm-Liouville boundary value problem, there is a special case called Hermite's differential equation which arises in the treatment of the harmonic oscillator in quantum mechanics. It is well known that Hermite's differential equation is defined as

$$y'' - 2xy' + 2ny = 0, \quad (1)$$

where n is a real number. In particular, for non-negative integer n , the solutions of Hermite's differential equation are usually referred to as the Hermite polynomials $H_n(x)$, which are defined by means of the exponential generating function

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \quad (|t| < \infty). \quad (2)$$

It is easily seen from (2) that the Hermite polynomials can be determined by

$$H_n(x) = \left. \frac{\partial^n}{\partial t^n} \exp(2xt - t^2) \right|_{t=0} \quad (n \geq 0). \quad (3)$$

The first several Hermite polynomials are

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x. \quad (4)$$

These polynomials have played important roles in various fields of mathematics, physics and engineering, such as quantum mechanics, mathematical physics, ucleon physics and quantum optics.

It is clear that the Poisson kernel for the Hermite polynomials is (see, e.g., [8])

$$\frac{1}{\sqrt{1-t^2}} \exp\left(\frac{2xyt - (x^2 + y^2)t^2}{1-t^2}\right) = \sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n} \cdot \frac{t^n}{n!} \quad (|t| < 1). \quad (5)$$

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In particular, the case $x = y$ in (5) yields the squares $H_n^2(x)$ of the Hermite polynomials given by

$$\frac{1}{\sqrt{1-t^2}} \exp\left(\frac{2x^2 t}{1+t}\right) = \sum_{n=0}^{\infty} \frac{H_n^2(x)}{2^n} \cdot \frac{t^n}{n!} \quad (|t| < 1). \quad (6)$$

It is easily seen that (6) can be reformulated as

$$\frac{1}{\sqrt{1-t^2}} \exp\left(\frac{xt}{1+t}\right) = \sum_{n=0}^{\infty} \frac{H_n^2 \sqrt{x/2}}{2^n} \cdot \frac{t^n}{n!} \quad (|t| < 1). \quad (7)$$

Recently, Kim et al. [9–12], Qi and Guo [17] studied the generating functions of the Hermite polynomials and their squares, and presented some explicit formulas for the Hermite polynomials and their squares. Further, Qi and Guo [17] used the properties of the Bell polynomials of the second kind stated in [16] to obtain some explicit formulas and recurrence relations for the Hermite polynomials and their squares, for example, they showed that for non-negative integer n , the Hermite polynomials and their squares can be computed by

$$H_n(x) = (-1)^n \frac{n!}{2^n} \sum_{k=0}^n (-1)^k \frac{2^{2k}}{k!} \binom{k}{n-k} x^{2k-n}, \quad (8)$$

and

$$H_n^2(x) = (-1)^n 2^n \cdot n! \sum_{k=0}^n (-1)^k \frac{2^k}{k!} \sum_{l=0}^{n-k} \frac{1 + (-1)^k}{2} \frac{(l-1)!!}{l!!} \binom{n-l-1}{n-l-k} x^{2k}, \quad (9)$$

and there exist the following recurrence formulas for the Hermite polynomials and their squares, as follows,

$$\sum_{k=0}^n \frac{1 + (-1)^{n-k}}{2} \frac{2^{\frac{n-k}{2}}}{k! \cdot (n-k)!!} H_k(x) = \frac{(2x)^n}{n!} \quad (n \geq 0), \quad (10)$$

and

$$\sum_{k=0}^n \frac{(-1)^k}{2^k \cdot k!} \left[\sum_{l=0}^{n-k} \frac{2^l}{l!} \binom{n-k-1}{n-k-l} x^{2l} \right] H_k^2(x) = \frac{1 + (-1)^n}{2} \frac{(n-1)!!}{n!!} \quad (n \geq 0), \quad (11)$$

where, and in what follows, $\binom{a}{k}$ is the binomial coefficient defined for complex number a and non-negative integer k by

$$\binom{a}{0} = 1, \quad \binom{a}{k} = \frac{a(a-1)(a-2)\cdots(a-k+1)}{k!} \quad (k \geq 1). \quad (12)$$

Motivated and inspired by the work of Kim et al. [9], Qi and Guo [17], in this paper we establish some new recurrence formulas for the Hermite polynomials and their squares by making use of the generating function methods and Padé approximation techniques. It turns out that the formulas (8), (9) and (11) and an analogous formula to (10) described in [9] are derived as special cases.

2 Padé approximants

We here recall the definition of Padé approximation to general series and their expression in the case of the exponential function, which have been widely used in various fields of mathematics, physics and engineering; see, for example [3, 13].

Let m, n be non-negative integers and let $\mathcal{P}_k$ be the set of all polynomials of degree $\leq k$. Assume that f is a function given by a Taylor expansion

$$f(t) = \sum_{k=0}^{\infty} c_k t^k \quad (13)$$

in a neighborhood of the origin, a Padé form of type (m, n) is the following pair (P, Q) such that

$$P = \sum_{k=0}^m p_k t^k \in \mathcal{P}_m, \quad Q = \sum_{k=0}^n q_k t^k \in \mathcal{P}_n \quad (Q \neq 0), \quad (14)$$

and

$$Qf - P = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \rightarrow 0. \quad (15)$$

It is clear that every Padé form of type (m, n) for $f(t)$ always exists and satisfies the same rational function, and the uniquely determined rational function P/Q is usually called the Padé approximant of type (m, n) for $f(t)$ (see, e.g., [1, 4]). For non-negative integers m, n , the Padé approximant of type (m, n) for the exponential function $\exp(t)$ is the unique rational function (see, e.g., [7, 14])

$$R_{m,n}(t) = \frac{P_m(t)}{Q_n(t)} \quad (P_m \in \mathcal{P}_m, Q_n \in \mathcal{P}_n, Q_n(0) = 1), \quad (16)$$

which obeys the property

$$\exp(t) - R_{m,n}(t) = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \rightarrow 0. \quad (17)$$

In fact, the explicit formulas for P_m and Q_n can be expressed in the following way (see, e.g., [2, 15]):

$$P_m(t) = \sum_{k=0}^m \frac{(m+n-k)! \cdot m!}{(m+n)! \cdot (m-k)!} \cdot \frac{t^k}{k!}, \quad (18)$$

$$Q_n(t) = \sum_{k=0}^n \frac{(m+n-k)! \cdot n!}{(m+n)! \cdot (n-k)!} \cdot \frac{(-t)^k}{k!}, \quad (19)$$

and

$$Q_n(t) \exp(t) - P_m(t) = (-1)^n \frac{t^{m+n+1}}{(m+n)!} \int_0^1 x^n (1-x)^m \exp(xt) dx, \quad (20)$$

where $P_m(t)$ and $Q_n(t)$ is called the Padé numerator and denominator of type (m, n) for the exponential function $\exp(t)$, respectively.

We shall use the above properties of Padé approximants to the exponential function to establish some new recurrence formulas for the Hermite polynomials and their squares in next section.

3 The statement of results

Theorem 3.1. *Let m, n be non-negative integers. Then, for non-negative integer l with $0 \leq l \leq 2(m+n)+1$,*

$$\sum_{\substack{2i+k=l \\ i,k \geq 0}} \binom{m}{i} (m+n-i)! \frac{H_k(x)}{k!} = \sum_{\substack{2i+k=l \\ i,k \geq 0}} \binom{n}{i} (-1)^i (m+n-i)! \frac{(2x)^k}{k!}. \quad (21)$$

Proof. Let m, n be non-negative integers. If we denote the right hand side of (20) by $S_{m,n}(t)$ then we have

$$\exp(t) = \frac{P_m(t) + S_{m,n}(t)}{Q_n(t)}. \quad (22)$$

It is easily seen from (2) that

$$\exp(2xt) = \left(\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \right) \exp(t^2). \quad (23)$$

By applying (22) to (23), we discover

$$\left(\frac{P_m(t^2) + S_{m,n}(t^2)}{Q_n(t^2)} \right) \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \exp(2xt), \quad (24)$$

which can be rewritten as

$$P_m(t^2) \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} + S_{m,n}(t^2) \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = Q_n(t^2) \exp(2xt). \quad (25)$$

We now apply the exponential series $\exp(xt) = \sum_{k=0}^{\infty} x^k t^k / k!$ in the right hand side of (20). With the help of the beta function, we get

$$S_{m,n}(t) = (-1)^n \frac{t^{m+n+1}}{(m+n)!} \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^1 x^{n+k} (1-x)^m dx = \sum_{k=0}^{\infty} \frac{(-1)^n \cdot m! \cdot (n+k)!}{(m+n)! \cdot (m+n+k+1)!} \cdot \frac{t^{m+n+k+1}}{k!}. \quad (26)$$

For convenience, let $p_{m,n;k}$, $q_{m,n;k}$ and $s_{m,n;k}$ be the coefficients of the polynomials $P_m(t)$, $Q_n(t)$ and $S_{m,n}(t)$ given by

$$P_m(t) = \sum_{k=0}^m p_{m,n;k} t^k, \quad Q_n(t) = \sum_{k=0}^n q_{m,n;k} t^k, \quad (27)$$

and

$$S_{m,n}(t) = \sum_{k=0}^{\infty} s_{m,n;k} t^{m+n+k+1}. \quad (28)$$

It follows from (18), (19) and (26) that

$$p_{m,n;k} = \frac{m! \cdot (m+n-k)!}{k! \cdot (m+n)! \cdot (m-k)!}, \quad q_{m,n;k} = \frac{(-1)^k \cdot n! \cdot (m+n-k)!}{k! \cdot (m+n)! \cdot (n-k)!}, \quad (29)$$

and

$$s_{m,n;k} = \frac{(-1)^n \cdot m! \cdot (n+k)!}{k! \cdot (m+n)! \cdot (m+n+k+1)!}. \quad (30)$$

If we apply (27) and (28) to (25) then we have

$$\sum_{i=0}^m p_{m,n;i} t^{2i} \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!} + \sum_{i=0}^{\infty} s_{m,n;i} t^{2(m+n+i+1)} \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!} = \sum_{i=0}^n q_{m,n;i} t^{2i} \sum_{k=0}^{\infty} (2x)^k \frac{t^k}{k!}, \quad (31)$$

which together with the Cauchy product yields

$$\sum_{l=0}^{\infty} \sum_{\substack{2i+k=l \\ i,k \geq 0}} p_{m,n;i} \frac{H_k(x)}{k!} t^l + \sum_{l=0}^{\infty} \sum_{\substack{2i+k=l-2(m+n+1) \\ i,k \geq 0}} s_{m,n;i} \frac{H_k(x)}{k!} t^l = \sum_{l=0}^{\infty} \sum_{\substack{2i+k=l \\ i,k \geq 0}} q_{m,n;i} \frac{(2x)^k}{k!} t^l. \quad (32)$$

By comparing the coefficients of t^l in (32), we obtain that for non-negative integer l with $0 \leq l \leq 2(m+n)+1$,

$$\sum_{\substack{2i+k=l \\ i,k \geq 0}} p_{m,n;i} \frac{H_k(x)}{k!} = \sum_{\substack{2i+k=l \\ i,k \geq 0}} q_{m,n;i} \frac{(2x)^k}{k!}. \quad (33)$$

Thus, applying (29) to (33) gives the desired result. $\square$

We next discuss some special cases of Theorem 3.1. By taking $m = 0$ in Theorem 3.1, we obtain that for non-negative integer l with $0 \leq l \leq 2n+1$,

$$\frac{n!}{l!} H_l(x) = \sum_{\substack{2i+k=l \\ i,k \geq 0}} \binom{n}{i} (-1)^i (n-i)! \frac{(2x)^k}{k!}, \quad (34)$$

which means

$$H_l(x) = l! \cdot \sum_{i=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^i \frac{(2x)^{l-2i}}{i! \cdot (l-2i)!} \quad (l \geq 0). \quad (35)$$

If we take $n = 0$ in Theorem 3.1, we obtain that for non-negative integer l with $0 \leq l \leq 2m+1$,

$$\sum_{\substack{2i+k=l \\ i,k \geq 0}} \binom{m}{i} (m-i)! \frac{H_k(x)}{k!} = \frac{m!}{l!} (2x)^l, \quad (36)$$

which implies

$$(2x)^l = l! \cdot \sum_{i=0}^{\lfloor \frac{l}{2} \rfloor} \frac{H_{l-2i}(x)}{i! \cdot (l-2i)!} \quad (l \geq 0). \quad (37)$$

Remark 3.2. It becomes obvious that (35) is an equivalent version of the formula (8), and (37) can be regarded as an analogous version of the formula (10). In fact, (35) and (37) were rediscovered by Kim et al. [9] where some interesting identities between the Hermite polynomials and the Bernoulli and Euler polynomials can be also found. We here refer to [5] for some analogous formulas for the generalized Hermite polynomials to (35) and (37).

Theorem 3.3. Let m, n be non-negative integers. Then, for non-negative integer l with $l \geq 2(m + n + 1)$,

$$\begin{aligned} & \sum_{\substack{2i+k=l \\ i,k \geq 0}} \binom{m}{i} (m+n-i)! \frac{H_k(x)}{k!} + (-1)^n m! \cdot n! \sum_{\substack{2i+k=l-2(m+n+1) \\ i,k \geq 0}} \binom{n+i}{n} \frac{H_k(x)}{(m+n+i+1)! \cdot k!} \\ &= \sum_{\substack{2i+k=l \\ i,k \geq 0}} \binom{n}{i} (-1)^i (m+n-i)! \frac{(2x)^k}{k!}. \end{aligned} \quad (38)$$

Proof. It is easily seen that comparing the coefficients of t^l in (32) gives that for $l \geq 2(m + n + 1)$,

$$\sum_{\substack{2i+k=l \\ i,k \geq 0}} p_{m,n;i} \frac{H_k(x)}{k!} + \sum_{\substack{2i+k=l-2(m+n+1) \\ i,k \geq 0}} s_{m,n;i} \frac{H_k(x)}{k!} = \sum_{\substack{2i+k=l \\ i,k \geq 0}} q_{m,n;i} \frac{(2x)^k}{k!}. \quad (39)$$

Thus, by applying (29) and (30) to (39), we obtain the desired result. $\square$

In particular, the case $l = 2(m + n + 1)$ in Theorem 3.3 gives that for non-negative integers m, n ,

$$\begin{aligned} & \sum_{\substack{2i+k=2(m+n+1) \\ i,k \geq 0}} \binom{m}{i} (m+n-i)! \frac{H_k(x)}{k!} + (-1)^n \frac{m! \cdot n!}{(m+n+1)!} \\ &= \sum_{\substack{2i+k=2(m+n+1) \\ i,k \geq 0}} \binom{n}{i} (-1)^i (m+n-i)! \frac{(2x)^k}{k!}. \end{aligned} \quad (40)$$

If we take $m = 0$ in (40), we get that for non-negative integer n ,

$$\frac{n!}{(2n+2)!} H_{2n+2}(x) + (-1)^n \frac{n!}{(n+1)!} = \sum_{\substack{2i+k=2n+2 \\ i,k \geq 0}} \binom{n}{i} (-1)^i (n-i)! \frac{(2x)^k}{k!}. \quad (41)$$

It is clear that (41) is the case $l = 2n + 2$ in (35).

Theorem 3.4. Let m, n be non-negative integers. Then, for non-negative integer l with $0 \leq l \leq m + n$,

$$\sum_{\substack{i+j+k=l \\ i,j,k \geq 0}} \binom{m}{i} (m+n-i)! (-2x^2)^i \binom{-i}{j} \frac{H_k^2(x)}{2^k \cdot k!} = \sum_{\substack{i+j+2k=l \\ i,j,k \geq 0}} \binom{n}{i} (m+n-i)! (2x^2)^i \binom{-i}{j} \binom{-\frac{1}{2}}{k} (-1)^k. \quad (42)$$

Proof. We rewrite (6) as

$$\frac{1}{\sqrt{1-t^2}} = \exp\left(-\frac{2x^2 t}{1+t}\right) \sum_{n=0}^{\infty} \frac{H_n^2(x)}{2^n} \cdot \frac{t^n}{n!}, \quad (43)$$

which together with (22) gives

$$\left\{ P_m\left(-\frac{2x^2 t}{1+t}\right) + S_{m,n}\left(-\frac{2x^2 t}{1+t}\right) \right\} \sum_{n=0}^{\infty} \frac{H_n^2(x)}{2^n} \cdot \frac{t^n}{n!} = \frac{1}{\sqrt{1-t^2}} Q_n\left(-\frac{2x^2 t}{1+t}\right). \quad (44)$$

If we apply (27) and (28) to (44) we have

$$\sum_{i=0}^m p_{m,n;i} (-2x^2)^i \left(\frac{t}{1+t}\right)^i \sum_{k=0}^{\infty} \frac{H_k^2(x)}{2^k} \cdot \frac{t^k}{k!}$$

$$\begin{aligned}
& + \sum_{i=0}^{\infty} s_{m,n;i} (-2x^2)^{m+n+i+1} \left(\frac{t}{1+t} \right)^{m+n+i+1} \sum_{k=0}^{\infty} \frac{H_k^2(x)}{2^k} \cdot \frac{t^k}{k!} \\
& = \frac{1}{\sqrt{1-t^2}} \sum_{i=0}^n q_{m,n;i} (-2x^2)^i \left(\frac{t}{1+t} \right)^i.
\end{aligned} \tag{45}$$

Notice that for complex number α ,

$$(1+t)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} t^n. \tag{46}$$

It follows from (46) that

$$\frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n t^{2n}, \tag{47}$$

and for non-negative integer i ,

$$\frac{1}{(1+t)^i} = \sum_{n=0}^{\infty} \binom{-i}{n} t^n. \tag{48}$$

Hence, by applying (47) and (48) to (45), with the help of the Cauchy products, we get

$$\begin{aligned}
& \sum_{l=0}^{\infty} \sum_{\substack{i+j+k=l \\ i,j,k \geq 0}} p_{m,n;i} (-2x^2)^i \binom{-i}{j} \frac{H_k^2(x)}{2^k \cdot k!} t^l + (-2x^2)^{m+n+1} \\
& \times \sum_{l=0}^{\infty} \sum_{\substack{i+j+k=l-(m+n+1) \\ i,j,k \geq 0}} s_{m,n;i} (-2x^2)^i \binom{-(m+n+i+1)}{j} \frac{H_k^2(x)}{2^k \cdot k!} t^l \\
& = \sum_{l=0}^{\infty} \sum_{\substack{i+j+2k=l \\ i,j,k \geq 0}} q_{m,n;i} (-2x^2)^i \binom{-i}{j} \binom{-\frac{1}{2}}{k} (-1)^k t^l.
\end{aligned} \tag{49}$$

If we compare the coefficients of t^l in (49), we obtain that for non-negative integer l with $0 \leq l \leq m+n$,

$$\sum_{\substack{i+j+k=l \\ i,j,k \geq 0}} p_{m,n;i} (-2x^2)^i \binom{-i}{j} \frac{H_k^2(x)}{2^k \cdot k!} = \sum_{\substack{i+j+2k=l \\ i,j,k \geq 0}} q_{m,n;i} (-2x^2)^i \binom{-i}{j} \binom{-\frac{1}{2}}{k} (-1)^k. \tag{50}$$

Thus, applying (29) to (50) gives the desired result. $\square$

It follows that we show some special cases of Theorem 3.4. It is obvious that the case $m = 0$ in Theorem 3.4 gives that for non-negative integer l with $0 \leq l \leq n$,

$$\frac{n!}{2^l \cdot l!} H_l^2(x) = \sum_{\substack{i+j+2k=l \\ i,j,k \geq 0}} \binom{n}{i} (n-i)! (2x^2)^i \binom{-i}{j} \binom{-\frac{1}{2}}{k} (-1)^k, \tag{51}$$

which implies that for non-negative integer l ,

$$H_l^2(x) = 2^l \cdot l! \sum_{\substack{i+j+2k=l \\ i,j,k \geq 0}} \frac{(2x^2)^i}{i!} \binom{-i}{j} \binom{-\frac{1}{2}}{k} (-1)^k. \tag{52}$$

Observe that for non-negative integer k ,

$$(-1)^k \binom{-\frac{1}{2}}{k} = \frac{1 \cdot 3 \cdots (2k-1)}{2^k \cdot k!} = \frac{(2k-1)!!}{(2k)!!}, \tag{53}$$

and for non-negative integers i, k, l with $k \leq l-i$,

$$\binom{-i}{l-i-k} = (-1)^{l-i-k} \frac{i(i+1) \cdots (l-k-1)}{(l-i-k)!} = (-1)^{l-i-k} \binom{l-k-1}{l-k-i}. \tag{54}$$

It follows from (52)-(54) that for non-negative integer l ,

$$\begin{aligned} H_l^2(x) &= 2^l \cdot l! \sum_{i=0}^l \frac{(2x^2)^i}{i!} \sum_{\substack{j+2k=l-i \\ j,k \geq 0}} \binom{-i}{j} \frac{(2k-1)!!}{(2k)!!} \\ &= 2^l \cdot l! \sum_{i=0}^l \frac{(2x^2)^i}{i!} \sum_{\substack{k=0 \\ 2|k}}^{l-i} \binom{-i}{l-i-k} \frac{(k-1)!!}{k!!} \\ &= (-1)^l 2^l \cdot l! \sum_{i=0}^l (-1)^i \frac{(2x^2)^i}{i!} \sum_{\substack{k=0 \\ 2|k}}^{l-i} \binom{l-k-1}{l-k-i} \frac{(k-1)!!}{k!!}, \end{aligned} \quad (55)$$

which gives the formula (9). For some interesting formulas for the product of two Hermite polynomials, one is referred to [10]. If we take $n = 0$ in Theorem 3.4, in light of (53), we get that for non-negative integer l with $0 \leq l \leq m$ and $2 \mid l$,

$$\sum_{\substack{i+j+k=l \\ i,j,k \geq 0}} \binom{m}{i} (m-i)! (-2x^2)^i \binom{-i}{j} \frac{H_k^2(x)}{2^k \cdot k!} = m! \frac{(l-1)!!}{l!!}, \quad (56)$$

which is equivalent to

$$\sum_{k=0}^l \frac{H_k^2(x)}{2^k \cdot k!} \sum_{i=0}^{l-k} \binom{-i}{l-k-i} \frac{(-2x^2)^i}{i!} = \frac{(l-1)!!}{l!!}. \quad (57)$$

So from (54) and (57), we obtain that for non-negative integer l with $2 \mid l$,

$$\sum_{k=0}^l (-1)^k \frac{H_k^2(x)}{2^k \cdot k!} \sum_{i=0}^{l-k} \binom{l-k-1}{l-k-i} \frac{(2x^2)^i}{i!} = \frac{(l-1)!!}{l!!}, \quad (58)$$

which gives the formula (11).

Theorem 3.5. Let m, n be non-negative integers. Then, for non-negative integer l with $l \geq m + n + 1$,

$$\begin{aligned} &\sum_{\substack{i+j+k=l \\ i,j,k \geq 0}} \binom{m}{i} (m+n-i)! (-2x^2)^i \binom{-i}{j} \frac{H_k^2(x)}{2^k \cdot k!} + (-1)^{m+1} (2x^2)^{m+n+1} m! \cdot n! \\ &\times \sum_{\substack{i+j+k=l-(m+n+1) \\ i,j,k \geq 0}} \binom{n+i}{n} \frac{(-2x^2)^i}{(m+n+i+1)!} \binom{-(m+n+i+1)}{j} \frac{H_k^2(x)}{2^k \cdot k!} \\ &= \sum_{\substack{i+j+2k=l \\ i,j,k \geq 0}} \binom{n}{i} (m+n-i)! (2x^2)^i \binom{-i}{j} \binom{-\frac{1}{2}}{k} (-1)^k. \end{aligned} \quad (59)$$

Proof. By comparing the coefficients of t^l in (49), we obtain that for non-negative integer l with $l \geq m + n + 1$,

$$\begin{aligned} &\sum_{\substack{i+j+k=l \\ i,j,k \geq 0}} p_{m,n;i} (-2x^2)^i \binom{-i}{j} \frac{H_k^2(x)}{2^k \cdot k!} + (-2x^2)^{m+n+1} \\ &\times \sum_{\substack{i+j+k=l-(m+n+1) \\ i,j,k \geq 0}} s_{m,n;i} (-2x^2)^i \binom{-(m+n+i+1)}{j} \frac{H_k^2(x)}{2^k \cdot k!} \\ &= \sum_{\substack{i+j+2k=l \\ i,j,k \geq 0}} q_{m,n;i} (-2x^2)^i \binom{-i}{j} \binom{-\frac{1}{2}}{k} (-1)^k. \end{aligned} \quad (60)$$

Thus, applying (29) and (30) to (60) gives the desired result. $\square$

It is clear that the case $l = m + n + 1$ in Theorem 3.5 gives that for non-negative integers m, n ,

$$\begin{aligned} & \sum_{\substack{i+j+k=m+n+1 \\ i,j,k \geq 0}} \binom{m}{i} (m+n-i)! (-2x^2)^i \binom{-i}{j} \frac{H_k^2(x)}{2^k \cdot k!} \\ &= \sum_{\substack{i+j+2k=m+n+1 \\ i,j,k \geq 0}} \binom{n}{i} (m+n-i)! (2x^2)^i \binom{-i}{j} \binom{-\frac{1}{2}}{k} (-1)^k + (-1)^m (2x^2)^{m+n+1} \frac{m! \cdot n!}{(m+n+1)!}. \end{aligned} \quad (61)$$

If we take $m = 0$ in (61) we have

$$\begin{aligned} \frac{n!}{2^{n+1} \cdot (n+1)!} H_{n+1}^2(x) &= \sum_{\substack{i+j+2k=n+1 \\ i,j,k \geq 0}} \binom{n}{i} (n-i)! (2x^2)^i \binom{-i}{j} \binom{-\frac{1}{2}}{k} (-1)^k \\ &\quad + (2x^2)^{n+1} \frac{n!}{(n+1)!} \quad (n \geq 0), \end{aligned} \quad (62)$$

which together with (53) and (54) yields (55). If we take $n = 0$ in (61) we get that for non-negative m with $2 \nmid m$,

$$\sum_{\substack{i+j+k=m+1 \\ i,j,k \geq 0}} \binom{m}{i} (m-i)! (-2x^2)^i \binom{-i}{j} \frac{H_k^2(x)}{2^k \cdot k!} = m! \frac{m!!}{(m+1)!!} + (-1)^m (2x^2)^{m+1} \frac{m!}{(m+1)!}, \quad (63)$$

which is an equivalent version of (58). Accordingly, (62) and (63) gives the formula (9) and (11), respectively.

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