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Research Article

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On $\mathcal{Q}$-regular semigroups

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Abstract: In this paper, we give some characterizations of $\mathcal{Q}$-regular semigroups and show that the class of $\mathcal{Q}$-regular semigroups is closed under the direct product and homomorphic images. Furthermore, we characterize the $\mathcal{Q}$-subdirect products of this class of semigroups and study the E-unitary $\mathcal{Q}$-regular covers for $\mathcal{Q}$-regular semigroups, in particular for those whose maximum group homomorphic image is a given group. As an application of these results, we claim that the similar results on V-regular semigroups also hold.

Keywords: $\mathcal{Q}$-regular semigroup, $\mathcal{Q}$-subdirect product, Surjective $\mathcal{Q}$-subhomomorphism, E-unitary $\mathcal{Q}$-regular cover

MSC: 20M10

1 Introduction and Preliminaries

Let S be a regular semigroup. Let $V(a)$ be the set of all inverses of a for each $a \in S$. We also use $E(S)$ to denote the set of all idempotents in S. It is well known that a regular semigroup S is orthodox if and only if for all $a, b \in S$,

$$V(b)V(a) \subseteq V(ab).$$

The concept of $\mathcal{P}$-regular semigroups was first introduced by Yamada and Sen [1]. In [2], Zhang and He characterized the structure of $\mathcal{P}$-regular semigroups. In fact, the class of $\mathcal{P}$-regular semigroups is a generalization of orthodox semigroups and regular $*$-semigroups.

On the other hand, by Onstad [3], a regular semigroup is said to be a V-regular semigroup if for all $a, b \in S$,

$$V(ab) \subseteq V(b)V(a).$$

According to the definition of orthodox semigroups, V-regular semigroup is a dual form of orthodox semigorup. Evidently, a regular semigroup S is both orthodox and V-regular if and only if for all $a, b \in S$,

$$V(ab) = V(b)V(a).$$

The class of inverse semigroups forms the most important class of regular semigroups which satisfy the above condition. As a generalization of inverse semigroups and orthodox semigroups, Gu and Tang [4] investigated V^n-semigroups and showed that the class of V^n-semigroups is closed under direct products and homomorphic images.

In V-regular semigroups case, Nambooripad and Pastijn [5] gave a characterization of this class of semigroups and Zheng and Ren [6] described congruences on V-regular semigroups in terms of certain

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congruence pairs. Then, Li [7] generalized the concept of V-regular semigroups and investigated $\mathcal{Q}$-regular semigroups which is a dual form of $\mathcal{P}$-regular semigroups.

A regular semigroup S is called $\mathcal{Q}$-regular if for any $a \in S$ there exists a non-empty set $V_Q \subseteq V(a)$ satisfying the following conditions:
(1) $aa^+ \in V_Q(aa^+)$ and $a^+a \in V_Q(a^+a)$ for any $a \in S$ and $a^+ \in V_Q(a)$,
(2) $V_Q(ab) \subseteq V_Q(b)V_Q(a)$ for all $a, b \in S$, where a^+ satisfying the above conditions is called a $\mathcal{Q}$-inverse of a and $V_Q(a)$ denotes the set of all $\mathcal{Q}$-inverses of a.

In [7], Li also gave an example of $\mathcal{Q}$-regular semigroup and showed that this class of semigroups properly contains the class of V-regular semigroups. At the same time, an equivalent characterization of $\mathcal{Q}$-regular semigroups was obtained.

A regular semigroup S is $\mathcal{Q}$-regular if and only if there exists a set $Q \subseteq E(S)$ satisfying the following conditions:
(1) $Q \cap L_a \neq \varnothing$ and $Q \cap R_a \neq \varnothing$ for any $a \in S$;
(2) $\omega^l|_Q = \omega|_Q \circ \mathcal{L}|_Q$ and $\omega^r|_Q = \omega|_Q \circ \mathcal{R}|_Q$;
(3) For any $a, b \in S$, if $(ab)^+ \in V(ab)$ such that $(ab)^+(ab), (ab)(ab)^+ \in Q$, then there exists $e_1 \in Q \cap L_a$ and $f_2 \in Q \cap R_a$ such that $b(ab)^+a = f_2e_1$.

In this paper, a $\mathcal{Q}$-regular semigroup S will be denoted by $S(Q)$. A subset Q of $E(S)$ satisfying $(1)-(3)$ is called a charcateristic set (for simple a C-set) of S.

Some results on subdirect products of inverse semigroups were characterized by McAlister and Reilly [8]. In [9], Nambooripad and Veeramony discussed the subdirect products of regular semigroups. Mitsch [10] studied the subdirect products of E-inversive semigroups. In [11], Zheng characterized the subdirect products of $\mathcal{P}$-regular semigroups. Throughout this paper, we investigate some properties of $\mathcal{Q}$-regular semigroups at first and characterize the $\mathcal{Q}$-subdirect products of this class of semigroups. Furthermore, we introduce the concept of the E-unitary $\mathcal{Q}$-regular covers for $\mathcal{Q}$-regular semigroups, in particular for those whose maximum group homomorphic image is a given group. Finally, we deduce the similar results on V-regular semigroups also hold up.

For notations and definitions given in this paper, the reader is referred to Howie [12].

2 On $\mathcal{Q}$-regular semigroups

As a dual form of $\mathcal{P}$-regular semigroups, Li [7] introduced the concept of $\mathcal{Q}$-regular semigroups. In this section, we give some characterizations of $\mathcal{Q}$-regular semigroups. In particular, we show that the class of $\mathcal{Q}$-regular semigroups is closed under the homomorphic images.

Proposition 2.1. *Let $S(Q)$ be a $\mathcal{Q}$-regular semigroup. For any $p \in Q$,*

$$V_Q(p) = (Q \cap L_p)(Q \cap R_p).$$

Proof. Since $p \in Q \subseteq E(S)$, $Q \cap L_p \neq \varnothing$, $Q \cap R_p \neq \varnothing$. For any $h \in V_Q(p)$, we have $h = hph = (hp)(ph) \in (Q \cap L_p)(Q \cap R_p)$, and so $V_Q(p) \subseteq (Q \cap L_p)(Q \cap R_p)$.

Conversely, let $f \in Q \cap L_p$, $g \in Q \cap R_p$. Then

$$p(fg)p = (pf)(gp) = p \cdot p = p,$$
$$(fg)p(fg) = f(gp)fg = fpfg = fg.$$

So $fg \in V(p)$. Additionally,

$$(fg)p = f(gp) = fp = f \in Q,$$
$$p(fg) = (pf)g = pg = g \in Q.$$

Thus $fg \in V_Q(p)$, namely, $(Q \cap L_p)(Q \cap R_p) \subseteq V_Q(p)$. Therefore, $V_Q(p) = (Q \cap L_p)(Q \cap R_p)$. □

Proposition 2.2 ([13]). *Let $S(Q)$ be a $\mathcal{Q}$-regular semigroup. If $a \in S(Q)$, $a^+ \in V_Q(a)$, then*

$$V_Q(a) = V_Q(a^+a)a^+V_Q(aa^+).$$

Theorem 2.3. *Let $S(Q)$ be a $\mathcal{Q}$-regular semigroup. If $e \in Q \cap L_a$, $f \in Q \cap R_a$, for any $a \in S$, then there exists a unique $a^+ \in V_Q(a)$ such that $a^+a = e$, $aa^+ = f$.*

Proof. For any $a \in S$, let $e \in Q \cap L_a$ and $f \in Q \cap R_a$. And so there is an inverse a' of a in $R_e \cap L_f$ such that $a'a = e \in Q$, $aa' = f \in Q$. Thus $a' \in V_Q(a)$. If there exists a $a^+ \in V_Q(a)$ such that $a^+a = e$, $aa^+ = f$, then

$a' = a'aa' = a'f = a'aa^+ = ea^+ = a^+aa^+ = a^+$. □

Corollary 2.4. *Let $S(Q)$ be a $\mathcal{Q}$-regular semigroup. For any $a, b \in S$,*
(1) $(a, b) \in \mathcal{L}$ if and only if there exists $a^+ \in V_Q(a)$, $b^+ \in V_Q(b)$ such that $a^+a = b^+b$;
(2) $(a, b) \in \mathcal{R}$ if and only if there exists $a^+ \in V_Q(a)$, $b^+ \in V_Q(b)$ such that $aa^+ = bb^+$;
(3) $(a, b) \in \mathcal{H}$ if and only if there exists $a^+ \in V_Q(a)$, $b^+ \in V_Q(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$.

Proof. (1) For any $a^+ \in V_Q(a)$, we have $a^+a \in Q \cap L_a$. If $(a, b) \in \mathcal{L}$ and there exists a $e \in Q \cap R_b$, then $e \mathcal{R} b \mathcal{L} a \mathcal{L} a^+a$. By Theorem 2.3, there exists a unique $b^+ \in V_Q(b)$ such that $b^+b = a^+a$.

On the other hand, if there exist $a^+ \in V_Q(a)$, $b^+ \in V_Q(b)$ such that $b^+b = a^+a$, then $a \mathcal{L} a^+a = b^+b \mathcal{L} b$, that is $a \mathcal{L} b$.

By using similar arguments as the above, we can proof (2) and (3). □

Corollary 2.5. *Let $S(Q)$ be a $\mathcal{Q}$-regular semigroup. If $e, f \in Q$, then $(e, f) \in \mathcal{D}$ if and only if there exist $a \in S$, $a^+ \in V_Q(a)$ such that $a^+a = f$, $aa^+ = e$.*

Proof. If $(e, f) \in \mathcal{D}$, there exists $a \in R_e \cap L_f$. By Theorem 2.3, there exists $a^+ \in V_Q(a)$ such that $a^+a = f$, $aa^+ = e$.

Conversely, if there exist $a \in S$, $a^+ \in V_Q(a)$ such that $a^+a = f$, $aa^+ = e$, then $e \mathcal{R} a$, $a \mathcal{L} f$. Hence $e \mathcal{D} f$. □

Proposition 2.6. *Let $S(Q)$ be $\mathcal{Q}$-regular semigroup. If $q \mathcal{R} p$ ($q \mathcal{L} p$) for any $p, q \in Q$, then $q \in V_Q(p)$.*

Proof. Let $q \in R_p$. Since $p, q \in Q \subseteq E(S)$, so $pq = q$, $qp = p$. Hence $pqp = qp = p$, $qpq = pq = q$, namely $q \in V(p)$. For another, $pq = q \in Q$, $qp = p \in Q$. Hence $q \in V_Q(p)$. In the case of $p \mathcal{L} q$, the proof is similar. □

Corollary 2.7. *Let $S(Q)$ be a $\mathcal{Q}$-regular semigroup. Then the following statements are equivalent:*
(1) for any $q, p \in Q$, if $V_Q(q) \cap V_Q(p) \neq \varnothing$, then $V_Q(q) = V_Q(p)$;
(2) for any $e, f \in E(S)$, if $V_Q(e) \cap V_Q(f) \neq \varnothing$, then $V_Q(e) = V_Q(f)$;
(3) for any $a, b \in S(Q)$, if $V_Q(a) \cap V_Q(b) \neq \varnothing$, then $V_Q(a) = V_Q(b)$.

Proof. We only need to proof (1) ⇒ (3). Let $c \in V_Q(a) \cap V_Q(b)$. By Proposition 2.2,

$$V_Q(a) = V_Q(ca)\, c\, V_Q(ac),$$
$$V_Q(b) = V_Q(cb)\, c\, V_Q(bc).$$

Since $ca, ac, cb, bc \in Q$ and $ca \mathcal{R} c \mathcal{R} cb$, $ac \mathcal{L} c \mathcal{L} bc$, by Proposition 2.6,

$$ca \in V_Q(ca) \cap V_Q(cb), \qquad ac \in V_Q(ac) \cap V_Q(bc).$$

By (1),

$$V_Q(ca) = V_Q(cb),\ V_Q(ac) = V_Q(bc).$$

Thus, $V_Q(a) = V_Q(b)$. □

Proposition 2.8. *Let $S(Q)$ be a $\mathcal{Q}$-regular semigroup and ρ an idempotent-separating congruence on S. For any $a, b \in S$, if $a \rho b$, then $a^+ \rho b^+$ for some $a^+ \in V_Q(a), b^+ \in V_Q(b)$.*

Proof. Since ρ is an idempotent-separating congruence on S, $a \rho b$ implies that $a \mathcal{H} b$. By Corollary 2.4, there exist $a^+ \in V_Q(a), b^+ \in V_Q(b)$ such that $aa^+ = bb^+, a^+a = b^+b$. Thus

$$a^+ = a^+aa^+ = a^+bb^+,$$
$$b^+ = b^+bb^+ = a^+ab^+.$$

Since ρ is a congruence, $a^+ab^+ \rho a^+bb^+$, that is $b^+ \rho a^+$. □

Theorem 2.9. *Let $S(Q)$ be a $\mathcal{Q}$-regular semigroup and T a regular semigroup. If $\psi : S(Q) \to T$ is a semigroup homomorphism and $\bar{Q} = \{q\psi : q \in Q\}$, then $T(\bar{Q})$ is $\mathcal{Q}$-regular.*

Proof. For any $a \in S$, there exists a $a^+ \in V_Q(a)$ such that $aa^+, a^+a \in Q$. And so

$$(a\psi)(a^+\psi)(a\psi) = (aa^+a)\psi = a\psi, \qquad (a^+\psi)(a\psi)(a^+\psi) = (a^+aa^+)\psi = a^+\psi.$$

That is $a^+\psi \in V(a\psi)$. Since $(a\psi)(a^+\psi) = (aa^+)\psi \in \bar{Q}$, $a^+\psi \in V_{\bar{Q}}(a\psi) \neq \varnothing$. It is easy to see that

$$(a\psi)(a^+\psi) \in V_Q((a\psi)(a^+\psi)) \text{ and } (a^+\psi)(a\psi) \in V_Q((a^+\psi)(a\psi)).$$

On the other hand, for any $(ab)^+ \in V_Q(ab)$, since S is $\mathcal{Q}$-regular, there exist $a^+ \in V_Q(a),\ b^+ \in V_Q(b)$ such that

$$(ab)^+\psi = (b^+a^+)\psi = (b^+\psi)(a^+\psi).$$

Thus $V_{\bar{Q}}((ab)\psi) \subseteq V_{\bar{Q}}(b\psi)V_{\bar{Q}}(a\psi)$ and $T(\bar{Q})$ is $\mathcal{Q}$-regular. □

3 $\mathcal{Q}$-subdirect products of $\mathcal{Q}$-regular semigroups

Some results on subdirect products of regular semigroups and E-inversive semigroups were characterized by Nambooripad [9] and Mitsch [10]. In this section, we show that the class of $\mathcal{Q}$-regular semigroups is closed under the direct product and characterize the $\mathcal{Q}$-subdirect products of $\mathcal{Q}$-regular semigroups.

For arbitrary semigroup S, Petrich [14] introduced the concept of filters of S, that is, a subsemigroup F of S such that $ab \in F$ implies $a \in F$ and $b \in F$.

Example 3.1. *Let $S = \{a, b, c, d, e\}$ be the semigroup with operation defined by*

·	a	b	c	d	e
a	a	b	c	c	c
b	a	b	c	c	c
c	a	b	c	c	c
d	a	b	c	e	d
e	a	b	c	d	e

Then $F = \{d, e\}$ is a filter of S.

In what follows, we introduce the concept of $\mathcal{Q}$-inverse filters of a $\mathcal{Q}$-regular semigroup.

Definition 3.2. *A regular subsemigroup T of $\mathcal{Q}$-regular semigroup $S(Q)$ is called a $\mathcal{Q}$-inverse filter, if*
(1) *$T(Q \cap E_T)$ is a $\mathcal{Q}$-regular semigroup;*
(2) *For any $t_1, t_2 \in T$ and $t_1^+ \in V_Q^S(t_1), t_2^+ \in V_Q^S(t_2)$, if $t_1^+t_2^+ \in T$, then $t_1^+,\ t_2^+ \in T$.*

Definition 3.3. *Let $S_1(Q_1)$ and $S_2(Q_2)$ be two $\mathcal{Q}$-regular semigroups. A homomorphism f of $S_1(Q_1)$ into $S_2(Q_2)$ is called a $\mathcal{Q}$-homomorphism if $Q_1 f = Q_2 \cap S_1(Q_1)f$. A $\mathcal{Q}$-homomorphism $f : S_1(Q_1) \to S_2(Q_2)$ is called a $\mathcal{Q}$-isomorphism if f is bijective, in such the case, $S_1(Q_1)$ is called $\mathcal{Q}$-isomorphic to $S_2(Q_2)$, and denoted by $S_1(Q_1) \overset{\mathcal{Q}}{\cong} S_2(Q_2)$.*

Proposition 3.4. *Let $S_1(Q_1)$ and $S_2(Q_2)$ be two $\mathcal{Q}$-regular semigroups. If $S(Q) = S_1(Q_1) \times S_2(Q_2)$, where $Q = \{(p_1, p_2) : p_1 \in Q_1, p_2 \in Q_2\}$, then $S(Q)$ is a $\mathcal{Q}$-regular semigroup.*

Proof. Obviously, $S(Q)$ is a semigroup. If $(s, t) \in S$, where $s \in S_1, t \in S_2$, since S_1, S_2 are all $\mathcal{Q}$-regular, there exists $s' \in V_{S_1}(s), t' \in V_{S_2}(t)$ such that $(s', t') \in S$. It is easy to see that $(s', t') \in V(s, t)$, and so S is regular. Let

$$V_{Q_1}(s) \times V_{Q_2}(t) = \{(s^+, t^+) : s^+ \in V_{Q_1}(s), t^+ \in V_{Q_2}(t)\}.$$

Clearly, it is non-empty and contained in $V(s, t)$. For any $(s^+, t^+) \in V_{Q_1}(s) \times V_{Q_2}(t)$, we have

$$(s, t)(s^+, t^+) \in V_{Q_1}(ss^+) \times V_{Q_2}(tt^+),$$
$$(s^+, t^+)(s, t) \in V_{Q_1}(s^+ s) \times V_{Q_2}(t^+ t).$$

For another, let $(s, t),\ (x, y) \in S$. For any $((sx)^+, (ty)^+) \in V_{Q_1}(sx) \times V_{Q_2}(ty)$, since S_1, S_2 are all $\mathcal{Q}$-regular, there exists $s^+ \in V_{Q_1}(s), x^+ \in V_{Q_1}(x), t^+ \in V_{Q_2}(t), y^+ \in V_{Q_2}(y)$ such that

$$\begin{aligned}((sx)^+, (ty)^+) &= (x^+ s^+, y^+ t^+)\\ &= (x^+, y^+)(s^+, t^+)\\ &\in (V_{Q_1}(x) \times V_{Q_2}(y))(V_{Q_1}(s) \times V_{Q_2}(t)).\end{aligned}$$

Hence, $S(Q)$ is a $\mathcal{Q}$-regular semigroup. □

Definition 3.5. *Let $S_1(Q_1), S_2(Q_2)$ be $\mathcal{Q}$-regular semigroups and $S(Q)$ be the direct product of them, where $Q = \{(p_1, p_2) : p_1 \in Q_1, p_2 \in Q_2\}$. If $T(Q')$ is a $\mathcal{Q}$-inverse filter of $S(Q)$ and the projections*

$$f_1 : \ T(Q') \to S_1(Q_1), (s_1, s_2) \mapsto s_1,$$
$$f_2 : \ T(Q') \to S_2(Q_2), (s_1, s_2) \mapsto s_2$$

are all surjective $\mathcal{Q}$-homomorphisms, then $T(Q')$ is called a $\mathcal{Q}$-subdirect product of $S_1(Q_1)$ and $S_2(Q_2)$.

Definition 3.6. *Let $S(Q_1)$ and $T(Q_2)$ be two $\mathcal{Q}$-regular semigroups. A mapping $\varphi : S \to 2^T$ (the power set of T) is called a surjective $\mathcal{Q}$-subhomomorphism of $S(Q_1)$ onto $T(Q_2)$, if it satisfies the following:*
(1) for any $s \in S$, $s\varphi \neq \varnothing$;
(2) for any $s_1, s_2 \in S$, $(s_1\varphi)(s_2\varphi) \subseteq (s_1 s_2)\varphi$;
(3) $\bigcup_{s \in S} s\varphi = T$;
(4) for any $t \in s\varphi$ $(s \in S, t \in T)$, there exist $s^+ \in V_{Q_1}(s), t^+ \in V_{Q_2}(t)$ such that $t^+ \in s^+\varphi$;
(5) for any $p_1 \in Q_1$, there exists $p_2 \in Q_2$ such that $p_2 \in p_1\varphi$; and for any $p_2 \in Q_2$, there exists $p_1 \in Q_1$ such that $p_2 \in p_1\varphi$;
(6) for any $t_1 \in s_1\varphi, t_2 \in s_2\varphi$, if $t_1^+ t_2^+ \in (s_1^+ s_2^+)\varphi$, then $t_1^+ \in s_1^+\varphi, t_2^+ \in s_2^+\varphi$, where $t_i^+ \in V_{Q_2}(t_i), s_i^+ \in V_{Q_1}(s_i)$, $i = 1, 2$.

Theorem 3.7. *Let $S(Q_1)$ and $T(Q_2)$ be $\mathcal{Q}$-regular semigroups, φ a surjective $\mathcal{Q}$-subhomomorphism of $S(Q_1)$ onto $T(Q_2)$. If*

$$\pi = \pi(S, T, \varphi) = \{(s, t) \in S \times T : \ t \in s\varphi\},$$
$$Q = \{(p_1, p_2) \in Q_1 \times Q_2 : \ p_2 \in p_1\varphi\},$$

then $\pi(Q)$ is a $\mathcal{Q}$-subdirect product of $S(Q_1)$ and $T(Q_2)$.

Conversely, every $\mathcal{Q}$-subdirect product of $S(Q_1)$ and $T(Q_2)$ can be constructed in this way.

Proof. If $(s_1, t_1),\ (s_2, t_2) \in \pi$, then $t_1 \in s_1\varphi$, $t_2 \in s_2\varphi$. By (2), $t_1t_2 \in (s_1s_2)\varphi$, and so $(s_1s_2, t_1t_2) \in \pi$. Hence π is a subsemigroup of $S \times T$. If $(s, t) \in \pi$, then $t \in s\varphi$. By (4), there exist $s^+ \in V_{Q_1}(s)$ and $t^+ \in V_{Q_2}(t)$ such that $t^+ \in s^+\varphi$, and so $(s^+, t^+) \in \pi$. It is easy to see that $(s^+, t^+) \in V(s, t)$. Thus π is a regular semigroup. Let

$$V_Q(s,t) = \{(s^+, t^+) \in V_{Q_1}(s) \times V_{Q_2}(t) :\ t^+ \in s^+\varphi\}.$$

If $(s, t) \in \pi$, then $t \in s\varphi$. By (4), $V_Q(s, t) \neq \varnothing$. For any $(s, t) \in \pi$, $(s^+, t^+) \in V_Q(s, t)$,

$$(s,t)(s^+,t^+) = (ss^+, tt^+),$$

where $s^+ \in V_{Q_1}(s), t^+ \in V_{Q_2}(t),\ t^+ \in s^+\varphi$. Since $S(Q_1)$, $T(Q_2)$ are $\mathcal{Q}$-regular, it follows that $ss^+ \in V_{Q_1}(ss^+), tt^+ \in V_{Q_2}(tt^+)$. And $tt^+ \in (ss^+)\varphi$, hence

$$(s,t)(s^+,t^+) \in V_Q(ss^+, tt^+) = V_Q((s,t)(s^+,t^+)).$$

Similarly, $(s^+, t^+)(s, t) \in V_Q((s^+, t^+)(s, t))$.

On the other hand, let $(s_1, t_1), (s_2, t_2) \in \pi$. For any

$$((s_1s_2)^+, (t_1t_2)^+) \in V_Q(s_1s_2, t_1t_2),$$

since $S(Q_1)$ and $T(Q_2)$ are all $\mathcal{Q}$-regular semigroups, there exist $s_1^+ \in V_{Q_1}(s_1), s_2^+ \in V_{Q_1}(s_2),\ t_1^+ \in V_{Q_2}(t_1),\ t_2^+ \in V_{Q_2}(t_2)$ such that

$$((s_1s_2)^+, (t_1t_2)^+) = (s_2^+s_1^+, t_2^+t_1^+) = (s_2^+, t_2^+)(s_1^+, t_1^+).$$

Since $t_2^+t_1^+ \in (s_2^+s_1^+)\varphi$, by (6), $t_2^+ \in s_2^+\varphi,\ t_1^+ \in s_1^+\varphi$. And so

$$(s_2^+, t_2^+) \in V_Q(s_2, t_2), \qquad (s_1^+, t_1^+) \in V_Q(s_1, t_1).$$

Hence,

$$V_Q(s_1s_2, t_1t_2) \subseteq V_Q(s_2, t_2)V_Q(s_1, t_1).$$

that is, $\pi(Q)$ is a $\mathcal{Q}$-regular semigroup.

For any

$$(s_1^+, t_1^+) \in V_{Q_1}(s_1) \times V_{Q_2}(t_1), \qquad (s_2^+, t_2^+) \in V_{Q_1}(s_2) \times V_{Q_2}(t_2).$$

If $(s_1^+, t_1^+)(s_2^+, t_2^+) \in \pi$, namely $(s_1^+s_2^+, t_1^+t_2^+) \in \pi$, then $t_1^+t_2^+ \in (s_1^+s_2^+)\varphi$. By (6), $t_1^+ \in s_1^+\varphi,\ t_2^+ \in s_2^+\varphi$, and so $(s_1^+, t_1^+),\ (s_2^+, t_2^+) \in \pi$. Hence $\pi(Q)$ is a $\mathcal{Q}$-inverse filter of $S(Q_1) \times T(Q_2)$.

By (1) and (5), the projection $f_1 : \pi \to S,\ (s, t) \mapsto s$ is a surjective $\mathcal{Q}$-homomorphism. By (3) and (5), the projection $f_2 :\ \pi \to T,\ (s, t) \mapsto t$ is a surjection $\mathcal{Q}$-homomorphism. Therefore, $\pi(Q)$ is a $\mathcal{Q}$-subdirect product of $S(Q_1)$ and $T(Q_2)$.

Conversely, let $H(Q)$ be a $\mathcal{Q}$-subdirect product of $S(Q_1)$ and $T(Q_2)$. Let

$$\varphi :\ S \to 2^T,\ s \mapsto \{t \in T :\ (s,t) \in H\}.$$

Since $H(Q)$ is a $\mathcal{Q}$-subdirect product of $S(Q_1)$ and $T(Q_2)$, there exists $t \in T$ such that $(s, t) \in H$ for any $s \in S$. Thus $t \in s\varphi$. Hence (1) holds. Since H is a subsemigroup of $S(Q_1) \times T(Q_2)$, (2) holds. Since the projection $f_2 :\ H \to T$ is surjective, (3) holds. If $t \in s\varphi\ (s \in S, t \in T)$, then $(s, t) \in H$. Since $H(Q)$ is $\mathcal{Q}$-regular, there exists

$$(s^+, t^+) \in (V_{Q_1}(s) \times V_{Q_2}(t)) \cap H.$$

Hence $s^+ \in V_{Q_1}(s),\ t^+ \in V_{Q_2}(t)$, and $t^+ \in s^+\varphi$, so (4) holds. For any $p_1 \in Q_1$, there exists $(p_1, p_2) \in Q = H \cap (Q_1 \times Q_2)$, and so $p_2 \in Q_2$ and $p_2 \in p_1\varphi$. Additionally, since f_2 is a surjective $\mathcal{Q}$-homomorphism, there exists

$$(p_1, p_2) \in Q = H \cap (Q_1 \times Q_2)$$

for any $p_2 \in Q_2$. Hence $p_1 \in Q_1$ and $p_2 \in p_1\varphi$. Thus (5) holds. Since $H(Q)$ is a $\mathcal{Q}$-inverse filter of $S(Q_1)\times T(Q_2)$, (6) holds. By the proof above, φ is a surjective $\mathcal{Q}$-subhomomorphism of $S(Q_1)$ onto $T(Q_2)$. Obviously, $H = \pi(S, T, \varphi)$. □

4 *E*-unitary covers for $\mathcal{Q}$-regular semigroups

McAlister and Reilly [8] have given E-unitary covers of inverse semigroups by using surjective subhomomorphisms of inverse semigroups. Mitsch [10] has given some sufficient conditions on E-unitary, E-inverse covers of E-inversive semigroups.

In this section, we introduce the concept of the E-unitary $\mathcal{Q}$-regular covers of $\mathcal{Q}$-regular semigroups, in particular for those whose maximum group homomorphic image is a given group.

Let S be a regular semigroup. A subset H of S is said to be
(1) *full* if $E(S) \subseteq H$;
(2) *self-conjugate* if $aHa' \subseteq H$ and $a'Ha \subseteq H$ for all $a \in S$ and all $a' \in V(a)$.
Let U be the minimum full and self-conjugate subsemigroup of S.

Lemma 4.1 ([15]). *Let S be a regular semiroup. The minimum group congruence σ on S is given by*

$$\sigma = \{(a, b) \in S \times S : (\exists x, y \in U)\ xa = by\}.$$

In [12], a subset A of a semigroup S is called right unitary, if

$$(\forall a \in A)(\forall s \in S)\ sa \in A \Rightarrow s \in A.$$

A is called left unitary, if

$$(\forall a \in A)(\forall s \in S)\ as \in A \Rightarrow s \in A.$$

A right and left unitary subset is called unitary.

A regular semigroup S is called E-unitary, if the set $E(S)$ of idempotents of S is unitary.

Definition 4.2. *Let $T(Q_1)$ and $S(Q)$ be two $\mathcal{Q}$-regular semigroups. $T(Q_1)$ is called an E-unitary $\mathcal{Q}$-regular cover of $S(Q)$, if $T(Q_1)$ is E-unitary, and there exists an idempotent separating $\mathcal{Q}$-homomorphism of $T(Q_1)$ onto $S(Q)$.*

A group G is a $\mathcal{Q}$-regular semigroup whose C-set is {1}, where 1 is the identity element of G.

Definition 4.3. *Let $S(Q)$ be a $\mathcal{Q}$-regular semigroup and G be a group. A surjective $\mathcal{Q}$-subhomomorphism φ of $S(Q)$ onto G is called unitary, if*

$$(\forall s \in S) 1 \in s\varphi \Rightarrow s \in E(S),$$

where 1 *is the identity element of G.*

Theorem 4.4. *Let $S(Q)$ be a $\mathcal{Q}$-regular semigroup and G be a group. If there exists a unitary surjetive $\mathcal{Q}$-subhomomorphism φ of $S(Q)$ onto G, then $\pi(S, G, \varphi)$ is an E-unitary $\mathcal{Q}$-regular cover of $S(Q)$.*

Proof. By Theorem 3.7, $T(Q_1) = \pi(S, G, \varphi)$ is $\mathcal{Q}$-regular whose C-set is

$$Q_1 = \{(p, 1) \in S \times G :\ p \in Q,\ 1 \in p\varphi\}.$$

For any $(s, g) \in T$, $(e, 1) \in E_T$, if

$$(s,g)(e,1)\in E_T\subseteq\{(t,1)\in S\times G: t\in E_S\},$$

then $(se,g)\in E_T$, and so $g=1$, and $(s,1)\in T(Q_1)$. By the definition of $T(Q_1)$, $1\in s\varphi$. Since φ is unitary, $s\in E_S$. Hence $(s,g)\in E_T$, so $T(Q_1)$ is E-unitary.

Define $f: T(Q_1)\to S(Q), (s,g)\mapsto s$. Then f is a surjective $\mathcal{Q}$-homomorphism. Obviously, f is idempotent separating. Hence $\pi(S,G,\varphi)$ is an E-unitary $\mathcal{Q}$-regular cover of $S(Q)$. □

Definition 4.5. *A $\mathcal{Q}$-regular semigroup $T(Q_1)$ is called an E-unitary $\mathcal{Q}$-regular cover of $S(Q)$ through G, if*
(1) $T(Q_1)$ is an E-unitary $\mathcal{Q}$-regular cover of $S(Q)$;
(2) $T/\sigma\overset{\mathcal{Q}}{\cong}G$, where σ is the minimum group congruence on $T(Q_1)$.

Lemma 4.6. *Let $S(Q_1)$, $T(Q_2)$ be two $\mathcal{Q}$-regular semigroups and $f: S(Q_1)\to T(Q_2)$ be a $\mathcal{Q}$-homomorphism. Then $S(Q_1)/kerf\overset{\mathcal{Q}}{\cong}S(Q_1)f$.*

Proof. It is easy to see that the mapping $\phi : S(Q_1)/kerf \to S(Q_1)f$ is an isomorphism. Since f is a $\mathcal{Q}$-homomorphism, we have

$$(Q_1kerf)\phi=Q_1f=Q_2\cap S(Q_1)f=Q_2\cap(S(Q_1)kerf)\phi.$$

Thus ϕ is also a $\mathcal{Q}$-homomorphism and $S(Q_1)/kerf\overset{\mathcal{Q}}{\cong}S(Q_1)f$. □

Theorem 4.7. *If there exists a unitary surjective $\mathcal{Q}$-subhomomorphism φ of $S(Q)$ onto G, then $T(Q_1)=\pi(S,G,\varphi)$ is an E-unitary $\mathcal{Q}$-regular cover of $S(Q)$ through G.*

Proof. By Theorem 4.4, $T(Q_1)=\pi(S,G,\varphi)$ is an E-unitary $\mathcal{Q}$-regular cover of $S(Q)$. We prove that $T/\sigma\overset{\mathcal{Q}}{\cong}G$ as follow.

It follows from Theorem 3.7 that $T(Q_1)$ is a $\mathcal{Q}$-subdirect product of $S(Q)$ and G. Hence G is the $\mathcal{Q}$-homomorphic image of $T(Q_1)$ under the projection $f_2: T(Q_1)\to G, (a,g)\mapsto g$. Since G is a group, $kerf_2$ is a group congruence on $T(Q_1)$, and so $\sigma\subseteq kerf_2$. Conversely, if $(a,g)\in(b,g)kerf_2$, then $(a,g)f_2=(b,h)f_2$, so that $g=h$. Since $T(Q_1)$ is $\mathcal{Q}$-regular, for any $(b,g)=(b,h)\in T(Q_1)$ there exists $(x,y)\in V_{Q_1}(b,g)$ such that

$$(b,g)(x,y)=(bx,gy)\in E_T\subseteq\{(e,1)\in T: e\in E_S\}.$$

Hence $bx\in E_S$ and $gy=1$, that is $g^{-1}=y$. So $(x,g^{-1})\in T(Q_1)$ and $(bx,1)\in Q_1$. Now $(xa,1)=(x,y)(a,g)\in T$. Hence, by the definition of T, $1\in(xa)\varphi$. Notice that φ is unitary, we have $xa\in E_S$. Thus $(xa,1)\in E_T$, and so

$$(bx,1)(a,g)=(bxa,g)=(b,h)(xa,1).$$

Since $(bx,1),(xa,1)\in E_T\subseteq U$, by Lemma 4.1, $(a,g)\sigma(b,h)$. Thus $kerf_2\subseteq\sigma$. Hence $\sigma=kerf_2$. Therefore, it follows from lemma 4.6 that $T/\sigma=T/kerf_2\overset{\mathcal{Q}}{\cong}G$. □

Remark 4.8. *As we know, the class of $\mathcal{Q}$-regular semigroups properly contains the class of V-regular semigroups. In fact, if we restrict the C-set Q of a $\mathcal{Q}$-regular semigroup S to the set of idempotents , then the subdirect products of V-regular semigroups can be constructed in a similar way and we can also use this contruction to study the E-unitary cover for V-regular semigroups.*

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