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Oscillation and non-oscillation of half-linear differential equations with coefficients determined by functions having mean values

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Abstract: The paper belongs to the qualitative theory of half-linear equations which are located between linear and non-linear equations and, at the same time, between ordinary and partial differential equations. We analyse the oscillation and non-oscillation of second-order half-linear differential equations whose coefficients are given by the products of functions having mean values and power functions. We prove that the studied very general equations are conditionally oscillatory. In addition, we find the critical oscillation constant.

Keywords: Half-linear equation, Oscillation theory, Riccati technique, Oscillation constant, Conditional oscillation

MSC: 34C10, 34C15

1 Introduction

The aim of this paper is to contribute to the rapidly developing theory of conditionally oscillatory equations. The topic of our research belongs to the qualitative theory concerning the oscillatory behaviour of the half-linear differential equation

$$[R(t)\Phi(x')]' + S(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1, \tag{1}$$

where coefficients $R > 0$, S are continuous functions. Function Φ is the so-called one dimensional p-Laplacian which connects half-linear equations with partial differential equations. Of course, some results obtained for equations of type (1) can be transferred or generalized to (elliptic) PDEs (see, e.g., the last section of [1]).

We recall some basic facts about the treated topic, a short historical background, and the motivation of our research. First of all, we point out that one of the biggest disadvantages of the research in the field of half-linear equations is the lack of the additivity of the solution space (it is the reason for the used nomenclature). Nevertheless, Sturm's separation and comparison theorems remain valid (see, e.g., [2, 3]). Therefore, we can classify half-linear equations as oscillatory and non-oscillatory as well as linear equations. More precisely, Sturm's separation theorem guarantees that if one non-zero solution is oscillatory (i.e., its zero points tend to infinity), then every solution is oscillatory and Eq. (1) is called oscillatory. There exist important equations whose oscillatory properties can be determined simply by measuring (in some sense) their coefficients. Searching for such equations is based on the study of the so-called conditional oscillation.

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On behalf of clarity, we consider the equations of the form

$$[R(t)\Phi(x')]' + \gamma C(t)\Phi(x) = 0, \quad \gamma \in \mathbb{R}. \tag{2}$$

We say that Eq. (2) is conditionally oscillatory if there exists a constant Γ such that Eq. (2) is oscillatory for $\gamma > \Gamma$ and non-oscillatory for $\gamma < \Gamma$. The constant Γ is usually called the critical oscillation constant of Eq. (2). Note that such a critical oscillation constant depends on coefficients $R > 0$ and C and it is non-negative—this observation comes directly from Sturm's comparison theorem (see Theorem 2.3 in Section 2 below). The conditionally oscillatory equations are very useful as testing and comparing equations.

The first conditionally oscillatory half-linear equation was found in [4], where the critical oscillation constant $\Gamma = (p-1)^p/p^p$ was revealed for the equation

$$[\Phi(x')]' + \frac{\gamma}{t^p}\Phi(x) = 0.$$

Then, motivated by results about linear equations (see [5]), the equation

$$[R(t)\Phi(x')]' + \frac{\gamma D(t)}{t^p}\Phi(x) = 0 \tag{3}$$

with positive periodic functions R, D was studied in [6] and it turned out that Eq. (3) is conditionally oscillatory as well. Later, it was proved that the critical oscillation constant can be found even in the case of Eq. (3) with coefficients having mean values (see [1]).

As a follow-up of the above mentioned results, a natural question arose, whether it is possible to remove t^{-p} from the potential of Eq. (3) and to preserve the conditionally oscillatory behaviour of the considered equation. Concerning this research direction, we mention papers [7–9] which are, together with paper [1], the main motivations of the results presented here. We should emphasize that, although the first motivation comes from the linear case, our research follows a path in half-linear equations and the linear case remains a special case for $p = 2$. Hence, our result is new even for linear equations which is demonstrated in Corollary 3.5 at the end of this paper.

To conclude this introductory section, we mention some books and papers that are connected to the treated topic. The theory of half-linear equations is thoroughly described in the already mentioned books [2, 3]. The direction of research which leads to perturbed equations is treated in many papers. We mention at least papers [10–15]. Half-linear equations are close to non-linear equations, where the p-Laplacian is replaced by more general functions. For results concerning such a type of equations, we refer to [16–21]. We should not forget to mention the discrete counterparts of results mentioned in this section. The theory of conditionally oscillatory difference equations is not as developed as the continuous one. Nevertheless, some results are already available in the literature (see [22, 23]). Some basic results are known even for dynamic equations on time scales which connect and generalize the continuous and discrete case. For such results, see [24, 25].

The rest of this paper is divided into two sections. The next section contains the description of the used transformation, where we derive the so-called adapted Riccati equation and we state preparatory lemmas. The last section is devoted to our results. We also mention a corollary concerning linear equations (to demonstrate the novelty of the main result and its impact to linear equations) and an illustrative simple example.

2 Preparations

First of all, we recall the definition of mean values for continuous functions.

Definition 2.1. *Let a continuous function $f : [T, \infty) \to \mathbb{R}$ be such that the limit*

$$\overline{f} := \lim_{a\to\infty} \frac{1}{a} \int_t^{t+a} f(s)\,\mathrm{d}s$$

is finite and exists uniformly with respect to $t \in [T, \infty)$. *The number* $\overline{f}$ *is called the mean value of* f.

Obviously, the mean value of any β-periodic continuous function f is

$$\overline{f} = \frac{1}{\beta} \int_{\tau}^{\tau+\beta} f(s)\,\mathrm{d}s, \tag{4}$$

where τ is arbitrary.

To prove our results, we use the generalized Riccati technique which is described below. We consider the second-order half-linear differential equation

$$\left[t^{\alpha} r(t)\Phi(x')\right]' + t^{\alpha-p} s(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1, \tag{5}$$

where $\alpha \le 0$ and r, s are continuous functions such that the mean values of functions $r^{1/(1-p)}$ and s exist, the mean value $\overline{s}$ is positive, and

$$0 < r^{-} := \inf\{r(t),\ t \in \mathbb{R}\} \le r^{+} := \sup\{r(t),\ t \in \mathbb{R}\} < \infty. \tag{6}$$

We denote by q the number conjugated with the given number $p > 1$, i.e.,

$$p + q = pq. \tag{7}$$

Immediately, we obtain the inverse function to Φ in the form $\Phi^{-1}(x) = |x|^{q-1}\operatorname{sgn} x$.

The basis of our method is the transformation to the Riccati half-linear equation which can be introduced as follows. We consider a non-zero solution x of Eq. (5) and we define

$$w(t) = t^{\alpha} r(t)\Phi\left(\frac{x'(t)}{x(t)}\right). \tag{8}$$

Considering Eq. (5), the differentiation of (8) leads to the Riccati half-linear equation

$$w'(t) + t^{\alpha-p} s(t) + (p-1)\left(t^{\alpha} r(t)\right)^{1-q} |w(t)|^{q} = 0. \tag{9}$$

The form of Eq. (9) is not sufficient enough for our method. Hence, we apply the transformation $\zeta(t) = -t^{p-\alpha-1} w(t)$ which leads to the equation

$$\zeta'(t) = \frac{1}{t}\left[(p-\alpha-1)\zeta(t) + s(t) + (p-1) r^{1-q}(t)|\zeta(t)|^{q}\right]. \tag{10}$$

Eq. (10) is called the adapted generalized Riccati equation for the consistency with similar cases in the literature.

Further, in this section, we formulate auxiliary results that will be needed in the following section within the proof of Theorem 3.1 below. We begin with properties of functions having mean values.

Lemma 2.2. *Let a continuous function* $f : [T, \infty) \subset (0, \infty) \to \mathbb{R}$ *have mean value* $\overline{f}$. *For an arbitrarily given* $a > 0$, *there exists a constant* $M(f) > 0$ *such that*

$$\left| \int_{t}^{t+b} f(\tau)\,\mathrm{d}\tau \right| < M(f) \tag{11}$$

and

$$\left| \int_{t}^{t+b} \frac{f(\tau)}{\tau}\,\mathrm{d}\tau \right| < \frac{M(f)}{t} \tag{12}$$

for all $t \ge T$, $b \in [0, a]$.

Proof. The statement of the lemma follows from the beginning of the proof of [1, Theorem 8] (see directly inequalities (48) and (51) in [1]). □

Next, we recall the well-known Sturm half-linear (also called the Sturm–Picone) comparison theorem.

Theorem 2.3. *Let $\hat{r}, \tilde{r}, \hat{s}, \tilde{s}$ be continuous functions satisfying $\hat{r}(t) \geq \tilde{r}(t) > 0$ and $\tilde{s}(t) \geq \hat{s}(t)$ for all sufficiently large t. Consider the pair of equations*

$$\left[\hat{r}(t)\Phi(x')\right]' + \hat{s}(t)\Phi(x) = 0, \tag{13}$$

$$\left[\tilde{r}(t)\Phi(x')\right]' + \tilde{s}(t)\Phi(x) = 0. \tag{14}$$

(I) If Eq. (14) *is non-oscillatory, then Eq.* (13) *is non-oscillatory.*
(II) If Eq. (13) *is oscillatory, then Eq.* (14) *is oscillatory.*

Proof. See, e.g., [3, Theorem 1.2.4]. □

Now, we formulate the condition which is equivalent to the non-oscillation of Eq. (5).

Theorem 2.4. *Eq.* (5) *is non-oscillatory if and only if there exists $a \in \mathbb{R}$ and a solution $w\colon [a, \infty) \to \mathbb{R}$ of Eq.* (9) *satisfying either*

$$w(t) = \int_t^\infty \tau^{\alpha-p} s(\tau)\,\mathrm{d}\tau + (p-1)\int_t^\infty \left(\tau^\alpha r(\tau)\right)^{1-q} |w(\tau)|^q\,\mathrm{d}\tau \geq 0 \tag{15}$$

or

$$w(t) = \int_t^\infty \tau^{\alpha-p} s(\tau)\,\mathrm{d}\tau + (p-1)\int_t^\infty \left(\tau^\alpha r(\tau)\right)^{1-q} |w(\tau)|^q\,\mathrm{d}\tau \leq 0 \tag{16}$$

for all $t \geq a$.

Proof. See, e.g., [3, Theorems 2.2.4 and 2.2.5], where it suffices to consider that the used divergence of $\int^\infty (\tau^\alpha r(\tau))^{1-q}\,\mathrm{d}\tau$ follows from $\alpha \leq 0$ and from (6) and the used convergence of $\int^\infty \tau^{\alpha-p} s(\tau)\,\mathrm{d}\tau$ is proved as (20) in [1]. □

The upcoming lemma describes the connection between the behaviour of solutions of Eq. (5) and the adapted Riccati equation (10).

Lemma 2.5. *If Eq.* (5) *is non-oscillatory, then there exists a solution ζ of the associated adapted generalized Riccati equation* (10) *such that $\zeta(t) \leq 0$ for all large $t \in \mathbb{R}$.*

Proof. We apply Theorem 2.4. From the positivity of $\overline{s}$ (see also [1]), it is seen that (16) cannot be valid for all large t, i.e., we obtain (15). Hence, there exists a non-negative solution w of Eq. (9) on some interval $[T, \infty)$, i.e., there exists a non-positive solution ζ of Eq. (10) on $[T, \infty)$. □

The last lemma contains the opposite implication to the one in Lemma 2.5.

Lemma 2.6. *If there exists a solution ζ of Eq.* (10) *for all large $t \in \mathbb{R}$, then Eq.* (5) *is non-oscillatory.*

Proof. The statement of the lemma follows directly from the half-linear version of the Reid roundabout theorem (see [3, Theorem 1.2.2] and also [1, Lemma 4]). □

3 Results

In this section, we formulate and prove our results. For reader's convenience, we slightly modify Eq. (5) as follows. We consider the equation

$$\left[t^\alpha r^{-\frac{p}{q}}(t)\Phi(x')\right]' + t^{\alpha-p} s(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1, \tag{17}$$

where $t \in \mathbb{R}$ is sufficiently large, q is the number conjugated with p (see (7)), $\alpha \le 0$, and r, s are continuous functions having mean values such that (6) is satisfied. Note that $\overline{s}$ can be non-positive. The only difference between Eq. (5) and Eq. (17) is the power $-p/q$ of r. The reason for this modification is purely technical (it leads to more transparent calculations below) and it does not mean any restriction (consider (6)).

Theorem 3.1. *Let us consider Eq.* (17).
(I) If

$$\left(\frac{p}{p-\alpha-1}\right)^p \overline{s}\,\overline{r}^{p-1} > 1, \tag{18}$$

then Eq. (17) *is oscillatory.*
(II) If

$$\left(\frac{p}{p-\alpha-1}\right)^p \overline{s}\,\overline{r}^{p-1} < 1, \tag{19}$$

then Eq. (17) *is non-oscillatory.*

Proof. In the both parts of the proof, we will consider such a number $a > 2$ for which (see Definition 2.1 together with (18) and (19))

$$\left(\frac{p}{p-\alpha-1}\right)^p \left(\frac{1}{a}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau\right)\left(\frac{1}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{p-1} > 1+\varepsilon \tag{20}$$

or

$$\left(\frac{p}{p-\alpha-1}\right)^p \left(\frac{1}{a}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau\right)\left(\frac{1}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{p-1} < 1-\varepsilon \tag{21}$$

for all considered t and for some $\varepsilon \in (0,1)$. We can rewrite (20) and (21) into the following forms

$$\frac{1}{a}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau > (1+\varepsilon)\left(\frac{p-\alpha-1}{p}\right)^p \left(\frac{1}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p} \tag{22}$$

and

$$\frac{1}{a}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau < (1-\varepsilon)\left(\frac{p-\alpha-1}{p}\right)^p \left(\frac{1}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p}. \tag{23}$$

Using (6), from (22) and (23), we obtain the existence of $L > 0$, for which

$$\frac{1}{a}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau - \left(\frac{p-\alpha-1}{p}\right)^p \left(\frac{1}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p} > L \tag{24}$$

and

$$\frac{1}{a}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau - \left(\frac{p-\alpha-1}{p}\right)^p \left(\frac{1}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p} < -L \tag{25}$$

for all considered t. Indeed, one can put

$$L := \varepsilon\left(\frac{p-\alpha-1}{p}\right)^p \left[r^+\right]^{1-p}$$

in the both cases.

We will consider the associated adapted generalized Riccati equation in the form of Eq. (10) which corresponds Eq. (17), i.e., the equation

$$\zeta'(t) = \frac{1}{t}\left[(p-\alpha-1)\zeta(t) + s(t) + (p-1)r(t)|\zeta(t)|^q\right]. \tag{26}$$

At first, we show that any solution ζ of Eq. (26) defined for $t \geq T$ is bounded from below, i.e., we show that there exists $K > 1$ satisfying

$$\zeta(t) > -K, \quad t \geq T. \tag{27}$$

On the contrary, let us assume that

$$\liminf_{t\to\infty} \zeta(t) = -\infty \tag{28}$$

or

$$\liminf_{t\to T_0^-} \zeta(t) = -\infty \tag{29}$$

for some $T_0 \in (T, \infty)$. For given a and function s, let us consider $M(s)$ from Lemma 2.2. Let $P > 0$ be an arbitrary number such that

$$(p-1)r^- y^q - (p-\alpha-1)y > M(s), \quad y \geq P. \tag{30}$$

In particular, considering $\inf_{t\geq T} \zeta(t) = -\infty$, the continuity of ζ implies the existence of an interval $[t_1, t_2]$ such that $\zeta(t_1) \leq -P$, $\zeta(t) < -P$ for all $t \in (t_1, t_2]$, and $t_2 - t_1 \in (0, a]$. Without loss of generality, we consider $T > 2$. Using (12) in Lemma 2.2, the form of Eq. (26), and (30), we have

$$\begin{aligned}
\zeta(t_2) - \zeta(t_1) &= \int_{t_1}^{t_2} \zeta'(\tau)\,\mathrm{d}\tau \\
&= \int_{t_1}^{t_2} \frac{(p-\alpha-1)\zeta(\tau) + s(\tau) + (p-1)r(\tau)|\zeta(\tau)|^q}{\tau}\,\mathrm{d}\tau \\
&> \int_{t_1}^{t_2} \frac{M(s)+s(\tau)}{\tau}\,\mathrm{d}\tau \geq \int_{t_1}^{t_2} \frac{M(s)}{\tau}\,\mathrm{d}\tau - \left|\int_{t_1}^{t_2} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau\right| \\
&> 0 - \frac{M(s)}{t_1} \geq -\frac{M(s)}{T} > -M(s).
\end{aligned} \tag{31}$$

This inequality proves that (29) cannot be valid for any $T_0 \in \mathbb{R}$ (consider that $a > 2$ is given). Therefore, there exist arbitrarily long intervals, where $\zeta(t) \leq -P$. Let $I = [t_3, t_4]$ be such an interval whose length is at least 2 (i.e., $t_4 - t_3 \geq 2$) and $t_4 - t_3 \leq a$. As in (31) (consider that $t_3 > 2$), we obtain

$$\begin{aligned}
\zeta(t_4) - \zeta(t_3) &= \int_{t_3}^{t_4} \zeta'(\tau)\,\mathrm{d}\tau > \int_{t_3}^{t_4} \frac{M(s)}{\tau}\,\mathrm{d}\tau - \left|\int_{t_3}^{t_4} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau\right| \\
&\geq M(s)\log\frac{t_4}{t_3} - \frac{M(s)}{t_3} \geq M(s)\left(\log\frac{t_3+2}{t_3} - \frac{1}{t_3}\right) > 0.
\end{aligned} \tag{32}$$

Of course, (32) means that $\zeta(t_4) > \zeta(t_3)$. In fact, (31) and (32) guarantee that

$$\liminf_{t\to\infty} \zeta(t) > -\infty,$$

which contradicts (28). Hence, (27) is valid.

In the both parts of the proof, we will also apply the estimation

$$\left|\int_t^{t+a} \frac{s(\tau)}{t}\,\mathrm{d}\tau - \int_t^{t+a} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau\right| \leq \frac{a}{t^2} M(s) \tag{33}$$

for all large t and $M(s)$ from Lemma 2.2. We use the mean value theorem of the integral calculus to get this estimation. More precisely, considering $t \in [t_1, t_2]$, where t_1 is sufficiently large, since s is integrable and $x(t) = t^{-1}$ is monotone for $t \in [t_1, t_2]$, there exists $t_3 \in [t_1, t_2]$ such that

$$\int_{t_1}^{t_2} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau = \frac{1}{t_1}\int_{t_1}^{t_3} s(\tau)\,\mathrm{d}\tau + \frac{1}{t_2}\int_{t_3}^{t_2} s(\tau)\,\mathrm{d}\tau. \tag{34}$$

Immediately, from (34), we obtain (see (11) in Lemma 2.2)

$$\begin{aligned}\left|\int_t^{t+a}\left(\frac{1}{t}-\frac{1}{\tau}\right)s(\tau)\,\mathrm{d}\tau\right| &= \left|\int_t^{t+a}\frac{s(\tau)}{t}\,\mathrm{d}\tau-\frac{1}{t}\int_t^{t+b}s(\tau)\,\mathrm{d}\tau-\frac{1}{t+a}\int_{t+b}^{t+a}s(\tau)\,\mathrm{d}\tau\right| \\ &= \left|\frac{1}{t}\int_{t+b}^{t+a}s(\tau)\,\mathrm{d}\tau-\frac{1}{t+a}\int_{t+b}^{t+a}s(\tau)\,\mathrm{d}\tau\right| = \frac{a}{t(t+a)}\left|\int_{t+b}^{t+a}s(\tau)\,\mathrm{d}\tau\right| \le \frac{a}{t^2}M(s),\end{aligned} \tag{35}$$

where $b\in[0,a]$. It is seen that (35) gives (33).

Part (I). Evidently (see (18)), $\overline{s}>0$. By contradiction, let us suppose that Eq. (17) is non-oscillatory. From Lemma 2.5, we know that there exists a non-positive solution ζ of Eq. (26) on some interval $[T,\infty)$. For this solution ζ, we introduce the averaging function ζ_{ave} by

$$\zeta_{\mathrm{ave}}(t):=\frac{1}{a}\int_t^{t+a}\zeta(\tau)\,\mathrm{d}\tau,\quad t\ge T. \tag{36}$$

We know that (see (27))

$$\zeta_{\mathrm{ave}}(t)\in(-K,0],\quad t\ge T. \tag{37}$$

For $t>T$ (see Eq. (26)), we have

$$\zeta'_{\mathrm{ave}}(t)=\frac{1}{a}\int_t^{t+a}\zeta'(\tau)\,\mathrm{d}\tau=\frac{1}{a}\int_t^{t+a}\frac{1}{\tau}\left[(p-\alpha-1)\zeta(\tau)+s(\tau)+(p-1)r(\tau)|\zeta(\tau)|^q\right]\mathrm{d}\tau. \tag{38}$$

For $t>T$, we also have (see (27) and (33))

$$\begin{aligned}&\left|\int_t^{t+a}\frac{1}{\tau}\left[(p-\alpha-1)\zeta(\tau)+s(\tau)+(p-1)r(\tau)|\zeta(\tau)|^q\right]\mathrm{d}\tau\right. \\ &\qquad\left.-\frac{1}{t}\int_t^{t+a}\left[(p-\alpha-1)\zeta(\tau)+s(\tau)+(p-1)r(\tau)|\zeta(\tau)|^q\right]\mathrm{d}\tau\right| \\ &\le\int_t^{t+a}\left(\frac{1}{t}-\frac{1}{\tau}\right)\left[(p-\alpha-1)K+(p-1)r^+K^q\right]\mathrm{d}\tau+\left|\int_t^{t+a}\frac{s(\tau)}{t}\,\mathrm{d}\tau-\int_t^{t+a}\frac{s(\tau)}{\tau}\,\mathrm{d}\tau\right| \\ &\le\frac{a^2}{t(t+a)}\left[(p-\alpha-1)K+(p-1)r^+K^q\right]+\frac{a}{t^2}M(s)<\frac{Na}{t^2},\end{aligned} \tag{39}$$

where

$$N:=a(p-\alpha-1)K+a(p-1)r^+K^q+M(s). \tag{40}$$

For $t>T$, we obtain (see (38), (39), and (40))

$$\zeta'_{\mathrm{ave}}(t)\ge\frac{1}{at}\int_t^{t+a}\left[(p-\alpha-1)\zeta(\tau)+s(\tau)+(p-1)r(\tau)|\zeta(\tau)|^q-\frac{N}{t}\right]\mathrm{d}\tau. \tag{41}$$

If we put

$$X(t):=\frac{(p-\alpha-1)^p}{pt}\left(\frac{p}{a}\int_t^{t+a}r(\tau)\,\mathrm{d}\tau\right)^{-\frac{p}{q}},\quad Y(t):=\frac{|\zeta_{\mathrm{ave}}(t)|^q}{qt}\left(\frac{p}{a}\int_t^{t+a}r(\tau)\,\mathrm{d}\tau\right) \tag{42}$$

for $t>T$, then we have (see (41))

$$\begin{aligned}\zeta'_{\mathrm{ave}}(t)&\ge\frac{1}{at}\int_t^{t+a}(p-\alpha-1)\zeta(\tau)\,\mathrm{d}\tau+X(t)+Y(t)-\frac{N}{t^2} \\ &\quad+\frac{1}{at}\int_t^{t+a}s(\tau)\,\mathrm{d}\tau-X(t)+\frac{1}{at}\int_t^{t+a}(p-1)r(\tau)|\zeta(\tau)|^q\,\mathrm{d}\tau-Y(t),\quad t>T.\end{aligned} \tag{43}$$

Taking into account (43), for large t, we will show the inequalities

$$\frac{N}{t^2} \le \frac{L}{3t}, \tag{44}$$

$$\frac{1}{at}\int_t^{t+a} (p-\alpha-1)\zeta(\tau)\,\mathrm{d}\tau + X(t) + Y(t) \ge 0, \tag{45}$$

$$\frac{1}{at}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau - X(t) \ge \frac{L}{t}, \tag{46}$$

$$\left|\frac{1}{at}\int_t^{t+a} (p-1)r(\tau)|\zeta(\tau)|^q\,\mathrm{d}\tau - Y(t)\right| \le \frac{L}{3t}. \tag{47}$$

The first inequality (44) is valid for all $t \ge 3N/L$. Hence, we can approach to (45). It holds (see (36) and (42))

$$\begin{aligned}
&\frac{1}{at}\int_t^{t+a} (p-\alpha-1)\zeta(\tau)\,\mathrm{d}\tau + X(t) + Y(t)\\
&\quad = \frac{1}{t}\left[(p-\alpha-1)\zeta_{\mathrm{ave}}(t) + \frac{(p-\alpha-1)^p}{p}\left(\frac{p}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{-\frac{p}{q}}\right.\\
&\qquad\qquad \left. + \frac{|\zeta_{\mathrm{ave}}(t)|^q}{q}\left(\frac{p}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)\right].
\end{aligned} \tag{48}$$

We recall the well-known Young inequality which says that

$$\frac{A^p}{p} + \frac{B^q}{q} - AB \ge 0 \tag{49}$$

holds for all non-negative numbers A, B. We take $A = (ptX(t))^{1/p}$ and $B = (qtY(t))^{1/q}$. Hence,

$$\frac{A^p}{p} = tX(t) = \frac{(p-\alpha-1)^p}{p}\left(\frac{p}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{-\frac{p}{q}},$$

$$\frac{B^q}{q} = tY(t) = \frac{|\zeta_{\mathrm{ave}}(t)|^q}{q}\left(\frac{p}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right),$$

and (see (37))

$$\begin{aligned}
AB &= (ptX(t))^{\frac{1}{p}}\,(qtY(t))^{\frac{1}{q}}\\
&= (p-\alpha-1)\left(\frac{p}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{-\frac{1}{q}} |\zeta_{\mathrm{ave}}(t)| \left(\frac{p}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{q}} = -(p-\alpha-1)\zeta_{\mathrm{ave}}(t).
\end{aligned}$$

Finally, considering (48) and (49), we have

$$\frac{1}{at}\int_t^{t+a} (p-\alpha-1)\zeta(\tau)\,\mathrm{d}\tau + X(t) + Y(t) = \frac{1}{t}\left[\frac{A^p}{p} + \frac{B^q}{q} - AB\right] \ge 0,$$

which proves (45).

Next, (46) is valid. Indeed, we have (see (24) and (42))

$$\frac{1}{at}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau - X(t) = \frac{1}{t}\left[\frac{1}{a}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau - \frac{(p-\alpha-1)^p}{p}\left(\frac{p}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{-\frac{p}{q}}\right]$$
$$= \frac{1}{t}\left[\frac{1}{a}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau - \left(\frac{p-\alpha-1}{p}\right)^p\left(\frac{1}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p}\right] > \frac{L}{t}$$

for all considered t.

To prove (47), we use the form of Eq. (26) together with (6), (12) from Lemma 2.2, and with (27) which immediately give

$$\begin{aligned}\left|\int_{t+i}^{t+j}\zeta'(\tau)\,\mathrm{d}\tau\right| &\le \left|\int_{t+i}^{t+j}\frac{1}{\tau}\left[(p-\alpha-1)\zeta(\tau)+s(\tau)+(p-1)r(\tau)|\zeta(\tau)|^q\right]\mathrm{d}\tau\right| \\ &\le \int_{t+i}^{t+j}\frac{1}{\tau}\left[(p-\alpha-1)K+(p-1)r^+K^q\right]\mathrm{d}\tau + \left|\int_{t+i}^{t+j}\frac{s(\tau)}{\tau}\,\mathrm{d}\tau\right| \\ &\le \frac{a}{t}\left[(p-\alpha-1)K+(p-1)r^+K^q\right] + \frac{M(s)}{t+i} \le \frac{Q}{t}\end{aligned} \tag{50}$$

for $t > T$ and $i, j \in [0, a]$, $i \le j$, where

$$Q := a\left[(p-\alpha-1)K+(p-1)r^+K^q\right] + M(s).$$

Hence, we have

$$|\zeta(t+j)-\zeta(t+i)| \le \frac{Q}{t}, \quad t > T,\ i, j \in [0, a],$$

which implies (see (36))

$$|\zeta(\tau)-\zeta_{\mathrm{ave}}(t)| \le \frac{Q}{t} \tag{51}$$

for all $t > T$ and $\tau \in [t, t+a]$.

Further, since the function $x(t) = |t|^q$ is continuously differentiable on $[-K, 0]$, there exists $C > 0$ for which

$$\left||y|^q - |z|^q\right| \le C|y-z|, \quad y, z \in [-K, 0]. \tag{52}$$

Thus, we have (see (6), (7), (27), (37), (42), (51), and (52))

$$\begin{aligned}&\left|\frac{1}{at}\int_t^{t+a}(p-1)r(\tau)|\zeta(\tau)|^q\,\mathrm{d}\tau - Y(t)\right| \\ &= \frac{1}{t}\left|\frac{1}{a}\int_t^{t+a}(p-1)r(\tau)|\zeta(\tau)|^q\,\mathrm{d}\tau - \frac{|\zeta_{\mathrm{ave}}(t)|^q}{q}\left(\frac{p}{a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)\right| \\ &\le \frac{p-1}{at}\int_t^{t+a}\left||\zeta(\tau)|^q - |\zeta_{\mathrm{ave}}(t)|^q\right| r(\tau)\,\mathrm{d}\tau \le \frac{(p-1)r^+}{at}\int_t^{t+a}\left||\zeta(\tau)|^q - |\zeta_{\mathrm{ave}}(t)|^q\right|\mathrm{d}\tau \\ &\le \frac{(p-1)r^+}{at}\int_t^{t+a} C|\zeta(\tau)-\zeta_{\mathrm{ave}}(t)|\,\mathrm{d}\tau \le \frac{(p-1)r^+CQ}{t^2}\end{aligned}$$

for $t > T$ which gives (47) for all sufficiently large t.

Altogether, (43) together with (44), (45), (46), and (47) guarantee

$$\zeta'_{\mathrm{ave}}(t) \ge 0 - \frac{L}{3t} + \frac{L}{t} - \frac{L}{3t} = \frac{L}{3t} \tag{53}$$

for all large t. From (53) it follows that $\lim_{t\to\infty} \zeta_{\mathrm{ave}}(t) = \infty$. In particular (see (36)), ζ is positive at least in one point which is a contradiction. The proof of part (I) is complete.

Part (II). Without loss of generality, we can assume that $\overline{s} > 0$. Indeed, for $\overline{s} \le 0$, it suffices to replace function s by function $s + k$ for a constant $k > 0$ such that $\overline{s} + k > 0$ and

$$\left(\frac{p}{p-\alpha-1}\right)^p (\overline{s}+k)\,\overline{r}^{p-1} < 1$$

and to use Theorem 2.3.

Let t_0 be a sufficiently large number. We denote (see (6))

$$Z := \left(\frac{p\,r^-}{p-\alpha-1}\right)^{1-p}.$$

Let ζ be the solution of the adapted generalized Riccati equation (26) satisfying

$$\zeta(t_0) = -\left(\frac{p}{(p-\alpha-1)a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p} \in (-2Z, 0)\,. \tag{54}$$

Based on Lemma 2.6, it suffices to prove that this solution exists for all $t \in [t_0, \infty)$. Let $[t_0, T)$ be the maximal interval, where the solution ζ exists. Note that $T \in (t_0, \infty) \cup \{\infty\}$. In fact, considering the continuity and the boundedness from below of ζ (see (27)), it suffices to show that $\zeta(t) < 0$ for all $t \in [t_0, T)$.

Let us consider an interval $J := [t_0, t_1)$ such that $\zeta(t) \in (-2Z, 0)$ for $t \in J$. For any $t_2, t_3 \in J \cap [t_0, t_0 + a]$, $t_2 \le t_3$, we have (see (6) and (12) in Lemma 2.2, cf. (50))

$$\begin{aligned}
|\zeta(t_3) - \zeta(t_2)| &= \left|\int_{t_2}^{t_3} \zeta'(\tau)\,\mathrm{d}\tau\right| \\
&= \left|\int_{t_2}^{t_3} \frac{1}{\tau}\left[(p-\alpha-1)\zeta(\tau) + s(\tau) + (p-1)r(\tau)|\zeta(\tau)|^q\right]\mathrm{d}\tau\right| \\
&\le \int_{t_2}^{t_3} \frac{1}{\tau}\left[(p-\alpha-1)2Z + (p-1)r^+(2Z)^q\right]\mathrm{d}\tau + \left|\int_{t_2}^{t_3} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau\right| \\
&\le \frac{1}{t_0}\int_{t_0}^{t_0+a} \left[(p-\alpha-1)2Z + (p-1)r^+(2Z)^q\right]\mathrm{d}\tau + \frac{M(s)}{t_2} \le \frac{\tilde{Q}}{t_0},
\end{aligned} \tag{55}$$

where

$$\tilde{Q} := a\left[(p-\alpha-1)2Z + (p-1)r^+(2Z)^q\right] + M(s).$$

Since t_0 is given as a sufficiently large number, from (55), we see that

$$\zeta(t) \in (-2Z, 0), \quad t \in [t_0, t_0 + a]. \tag{56}$$

In addition, (55) means that

$$|\zeta(t_0) - \zeta(\tau)| < \frac{\tilde{Q}}{t_0}, \quad \tau \in [t_0, t_0 + a]. \tag{57}$$

Similarly as in the first part of the proof, we introduce

$$\zeta_{\mathrm{ave}}(t) := \frac{1}{a}\int_t^{t+a} \zeta(\tau)\,\mathrm{d}\tau \tag{58}$$

for t from a neighbourhood of t_0. It holds (see (56), (57), and (58))

$$\zeta_{\mathrm{ave}}(t_0) \in (-2Z, 0), \qquad |\zeta_{\mathrm{ave}}(t_0) - \zeta(t_0)| < \frac{\tilde{Q}}{t_0}. \tag{59}$$

We have (see Eq. (26), cf. (38))

$$\zeta'_{\text{ave}}(t_0) = \frac{1}{a}\int_{t_0}^{t_0+a} \zeta'(\tau)\,\mathrm{d}\tau = \frac{1}{a}\int_{t_0}^{t_0+a} \frac{1}{\tau}\left[(p-\alpha-1)\zeta(\tau) + s(\tau) + (p-1)r(\tau)|\zeta(\tau)|^q\right]\mathrm{d}\tau. \tag{60}$$

In addition, as in the first part of the proof (see (39)), we have

$$\begin{aligned}\left|\int_{t_0}^{t_0+a} \frac{1}{\tau}\left[(p-\alpha-1)\zeta(\tau) + s(\tau) + (p-1)r(\tau)|\zeta(\tau)|^q\right]\mathrm{d}\tau\right.\\ \left. -\frac{1}{t_0}\int_{t_0}^{t_0+a}\left[(p-\alpha-1)\zeta(\tau) + s(\tau) + (p-1)r(\tau)|\zeta(\tau)|^q\right]\mathrm{d}\tau\right| < \frac{Na}{t_0^2},\end{aligned} \tag{61}$$

where N is defined in (40) (we can also put $K = 2Z$).

Hence, from (60) and (61), we obtain (cf. (41))

$$\zeta'_{\text{ave}}(t_0) \le \frac{1}{at_0}\int_{t_0}^{t_0+a}\left[(p-\alpha-1)\zeta(\tau) + s(\tau) + (p-1)r(\tau)|\zeta(\tau)|^q + \frac{N}{t_0}\right]\mathrm{d}\tau. \tag{62}$$

If we put (cf. (42))

$$X(t_0) := \frac{(p-\alpha-1)^p}{p}\left(\frac{p}{a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p}, \tag{63}$$

$$Y(t_0) := \frac{(p-\alpha-1)^p}{q}\left(\frac{p}{a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p}, \tag{64}$$

then we have (see (62))

$$\begin{aligned}\zeta'_{\text{ave}}(t_0) \le{}& \frac{1}{at_0}\int_{t_0}^{t_0+a} s(\tau) - X(t_0)\,\mathrm{d}\tau + \frac{1}{at_0}\int_{t_0}^{t_0+a}(p-1)r(\tau)|\zeta(\tau)|^q - Y(t_0)\,\mathrm{d}\tau\\ &+ \frac{1}{at_0}\int_{t_0}^{t_0+a}(p-\alpha-1)\zeta(\tau)\,\mathrm{d}\tau + \frac{X(t_0)}{t_0} + \frac{Y(t_0)}{t_0} + \frac{N}{t_0^2}.\end{aligned} \tag{65}$$

The aim of our process is to prove that $\zeta'_{\text{ave}}(t_0) < 0$. To obtain this inequality, it suffices to show (see (65))

$$\frac{1}{a}\int_{t_0}^{t_0+a} s(\tau) - X(t_0)\,\mathrm{d}\tau \le -L, \tag{66}$$

$$\left|\frac{1}{a}\int_{t_0}^{t_0+a}(p-1)r(\tau)|\zeta(\tau)|^q - Y(t_0)\,\mathrm{d}\tau\right| \le \frac{L}{4}, \tag{67}$$

$$\frac{1}{a}\int_{t_0}^{t_0+a}(p-\alpha-1)\zeta(\tau)\,\mathrm{d}\tau + X(t_0) + Y(t_0) \le \frac{L}{4}, \tag{68}$$

$$\frac{N}{t_0} \le \frac{L}{4}. \tag{69}$$

Evidently, (69) is valid, because t_0 is sufficiently large. We show that (66) is valid. We have (see (25), (63))

$$\begin{aligned}\frac{1}{a}\int_{t_0}^{t_0+a} s(\tau) - X(t_0)\,\mathrm{d}\tau &= \frac{1}{a}\int_{t_0}^{t_0+a} s(\tau)\,\mathrm{d}\tau - \frac{(p-\alpha-1)^p}{p}\left(\frac{p}{a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p}\\ &= \frac{1}{a}\int_{t_0}^{t_0+a} s(\tau)\,\mathrm{d}\tau - \left(\frac{p-\alpha-1}{p}\right)^p\left(\frac{1}{a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p} < -L,\end{aligned}$$

i.e., we obtain (66).

Now we prove (67). Let $D > 0$ be such that (cf. (52))

$$\big||y|^q - |z|^q\big| \le D|y - z|, \qquad y, z \in [-2Z, 0]. \tag{70}$$

Using (6), (7), (54), (56), (57), (64), and (70), we have

$$\begin{aligned}
&\left|\frac{1}{a}\int_{t_0}^{t_0+a}(p-1)r(\tau)|\zeta(\tau)|^q - Y(t_0)\,\mathrm{d}\tau\right| \\
&= \left|\frac{1}{a}\int_{t_0}^{t_0+a}(p-1)r(\tau)|\zeta(\tau)|^q\,\mathrm{d}\tau - \frac{(p-\alpha-1)^p}{q}\left(\frac{p}{a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p}\right| \\
&= \left|\frac{1}{a}\int_{t_0}^{t_0+a}(p-1)r(\tau)|\zeta(\tau)|^q\,\mathrm{d}\tau - \frac{p}{q}|\zeta(t_0)|^q\left(\frac{1}{a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)\right| \\
&\le \frac{p-1}{a}\int_{t_0}^{t_0+a} r(\tau)\big||\zeta(\tau)|^q - |\zeta(t_0)|^q\big|\,\mathrm{d}\tau \le \frac{(p-1)r^+}{a}\int_{t_0}^{t_0+a}\big||\zeta(\tau)|^q - |\zeta(t_0)|^q\big|\,\mathrm{d}\tau \\
&\le \frac{(p-1)r^+}{a}\int_{t_0}^{t_0+a} D\,|\zeta(\tau) - \zeta(t_0)|\,\mathrm{d}\tau \le \frac{(p-1)r^+}{a}\int_{t_0}^{t_0+a} D\,\frac{\tilde{Q}}{t_0}\,\mathrm{d}\tau = \frac{D\tilde{Q}(p-1)r^+}{t_0}.
\end{aligned} \tag{71}$$

For a sufficiently large number t_0, inequality (67) follows from (71).

It remains to prove (68). We have (see (58), (63), and (64))

$$\begin{aligned}
&\frac{1}{a}\int_{t_0}^{t_0+a}(p-\alpha-1)\zeta(\tau)\,\mathrm{d}\tau + X(t_0) + Y(t_0) = (p-\alpha-1)\zeta_{\mathrm{ave}}(t_0) \\
&\qquad + \frac{(p-\alpha-1)^p}{p}\left(\frac{p}{a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p} + \frac{(p-\alpha-1)^p}{q}\left(\frac{p}{a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p}.
\end{aligned} \tag{72}$$

Let us assume that (see (54))

$$\zeta_{\mathrm{ave}}(t_0) = \zeta(t_0) = -\left(\frac{p}{(p-\alpha-1)a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p}. \tag{73}$$

Then, (72) gives

$$\begin{aligned}
&\frac{1}{a}\int_{t_0}^{t_0+a}(p-\alpha-1)\zeta(\tau)\,\mathrm{d}\tau + X(t_0) + Y(t_0) \\
&= -(p-\alpha-1)\left(\frac{p}{(p-\alpha-1)a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p} \\
&+ \frac{(p-\alpha-1)^p}{p}\left(\frac{p}{a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p} + \frac{(p-\alpha-1)^p}{q}\left(\frac{p}{a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p} \\
&= (p-\alpha-1)^p\left(\frac{p}{a}\int_{t_0}^{t_0+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p}\left(-1 + \frac{1}{p} + \frac{1}{q}\right) = 0.
\end{aligned} \tag{74}$$

Using (59) (consider (73)), one can see that (74) gives (68) for large t_0.

Finally, applying (66), (67), (68), and (69) in (65), we have

$$\zeta'_{\mathrm{ave}}(t_0) \le \frac{1}{t_0}\left(-L + \frac{L}{4} + \frac{L}{4} + \frac{L}{4}\right) = -\frac{L}{4t_0} < 0,$$

i.e., we have (see (58))

$$\zeta'_{\text{ave}}(t_0) = \frac{\zeta(t_0+a)-\zeta(t_0)}{a} < 0,$$

i.e., $\zeta(t_0+a) < \zeta(t_0)$. Since we can replace t_0 by an arbitrary number $t > t_0$ in the process above, we obtain $\zeta(t+a) < \zeta(t)$ and $\zeta(\tau) < 0$ for all $\tau \in [t, t+a]$ if

$$\zeta(t) = -\left(\frac{p}{(p-\alpha-1)a}\int_t^{t+a} r(\tau)\,\mathrm{d}\tau\right)^{1-p}$$

for some $t \in (t_0, \infty)$. Considering the fact that a can be arbitrarily large (see Definition 2.1 and (6)) together with (55) and $\zeta'_{\text{ave}}(t) < 0$, we obtain that $\zeta(t) < 0$ for all $t > t_0$. The proof is complete. □

Remark 3.2. *Now we describe the connection of Theorem 3.1 and our motivation. We repeat that our basic motivation comes from papers [1, 7–9]. To the best of our knowledge, the strongest known results about non-perturbed conditionally oscillatory half-linear differential equations are proved just in those articles. In [1, 8, 9], only the case $\alpha = 0$ is analysed. Note that, in [8], the considered type of equations differs from Eq.* (17)*. In [7], the general form of Eq.* (17) *is treated. The process in [7] (and also in [8, 9]) is based on the modified Prüfer transformation. Hence, it is entirely different from the method used in this paper which enables us to cover new types of equations. The coefficients of equations considered in [7] has to be restricted in a certain sense. This restriction is removed in the presented results.*

Remark 3.3. *Theorem 3.1 does not cover the case when*

$$\left(\frac{p}{p-\alpha-1}\right)^p \overline{s}\,\overline{r}^{p-1} = 1. \tag{75}$$

It is known (see any of papers [10–15]) that this case is not generally solvable. More precisely, for

$$r(t) \equiv 1, \qquad s(t) \equiv \left(\frac{p-\alpha-1}{p}\right)^p,$$

Eq. (17) *is non-oscillatory (see [7]) and, at the same time, there exist continuous functions r, s satisfying* (75) *for which Eq.* (17) *is oscillatory (for $\alpha = 0$, see again papers [10–15]). We conjecture that Eq.* (17) *is non-oscillatory in the situation given by* (75)*, where the coefficients r, s are periodic functions (see also* (4)*). Our conjecture is based on results of [26, 27], but it remains an open problem.*

To illustrate Theorem 3.1, we mention the following example of a simple equation whose oscillatory behaviour does not follow from any previously known result (also for $p = 2$, i.e., in the case of linear equations). In fact, as far as we know, such equations with general α are not studied in the literature.

Example 3.4. *Let $c, d \neq 0$ be arbitrarily given. We define the function $s : [1, \infty) \to \mathbb{R}$ by the formula*

$$s(t) := \begin{cases} c + 4n\sqrt{n}d\,(t-n), & t \in \left[n, n+\frac{1}{4n}\right); \\ c + 4n\sqrt{n}d\left(n+\frac{2}{4n}-t\right), & t \in \left[n+\frac{1}{4n}, n+\frac{3}{4n}\right); \\ c + 4n\sqrt{n}d\left(t-n-\frac{4}{4n}\right), & t \in \left[n+\frac{3}{4n}, n+\frac{4}{4n}\right); \\ \quad\vdots & \quad\vdots \\ c + 4n\sqrt{n}d\left(t-n-\frac{4(n-1)}{4n}\right), & t \in \left[n+\frac{4(n-1)}{4n}, n+\frac{4(n-1)+1}{4n}\right); \\ c + 4n\sqrt{n}d\left(n+\frac{4(n-1)+2}{4n}-t\right), & t \in \left[n+\frac{4(n-1)+1}{4n}, n+\frac{4(n-1)+3}{4n}\right); \\ c + 4n\sqrt{n}d\,(t-n-1), & t \in \left[n+\frac{4(n-1)+3}{4n}, n+1\right), \end{cases}$$

where $n \in \mathbb{N}$. One can easily show that the mean value of this function exists and that $\overline{s} = c$. We consider the equation

$$\left[t^\alpha \Phi(x')\right]' + t^{\alpha-p} s(t) \Phi(x) = 0, \tag{76}$$

where $p > 1$ and $\alpha \le 0$ are arbitrary. From Theorem 3.1, we know that Eq. (76) *is oscillatory for $c > (p-\alpha-1)^p/p^p$ and non-oscillatory for $c < (p-\alpha-1)^p/p^p$.*

We repeat that we obtain new results even for linear equations. With regard to the importance of this fact, we formulate the corresponding consequence of Theorem 3.1 as the corollary below.

Corollary 3.5. *Let us consider the equation*

$$\left[\frac{x'}{t^\alpha r(t)}\right]' + \frac{s(t)}{t^{\alpha+2}}\,x = 0, \tag{77}$$

where $t \in \mathbb{R}$ is sufficiently large, $\alpha \ge 0$, and r, s are continuous functions having mean values $\overline{r}$, $\overline{s}$ such that (6) *is satisfied.*
(I) If $4\,\overline{r}\,\overline{s} > (1+\alpha)^2$, then Eq. (77) *is oscillatory.*
(II) If $4\,\overline{r}\,\overline{s} < (1+\alpha)^2$, then Eq. (77) *is non-oscillatory.*

Proof. The corollary follows directly from Theorem 3.1. □

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