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Topological entropy for positively weak measure expansive shadowable maps

Abstract: In this paper, we consider positively weak measure expansive homeomorphisms and flows with the shadowing property on a compact metric space X. Moreover, we prove that if a homeomorphism (or flow) has a positively weak expansive measure and the shadowing property on its nonwandering set, then its topological entropy is positive.

1 Introduction

The main goal of the study on dynamical systems is to understand the structure of the orbits for homeomorphisms or flows on a compact metric space. To describe the dynamics on the underlying space, it is common to study the dynamic properties such as *shadowing property*, *expansiveness*, *entropy*, etc. It has close relations with stable or chaotic and sensitive properties of a given system.

Recently, Morales [1] has introduced the notion of *measure expansiveness*, generalizing the concept of expansiveness, and Lee *et al.* [2] has introduced a notion of *weak measure expansiveness* for flows which is really weaker than measure expansive flows in [3]. The concept of *positively measure-expansiveness* is introduced by [1] as a generalization of the notion of positively expansiveness, and positively measure expansive continuous maps of a compact metric space are studied from the measure theoretical point of view. Also Morales [4] proved that every homeomorphism exhibiting positively expansive measures has positive topological entropy, and its restriction to the nonwandering set has the shadowing property. Based on this, we consider the shadowing property and entropy for the positively weak measure expansive homeomorphisms and flows, respectively.

In this paper, we show that if a homeomorphism (or flow) has a positively weak expansive measure and the shadowing property on its nonwandering set, then its topological entropy is positive. This is a slight generalization of the main result in [4]. We also consider a relationship between the weak measure expansivity with shadowing property and topological entropy.

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1.1 Basics for positively weak measure expansive homeomorphisms

As pointed out by Morales [1], a notion generalizing the concept of expansiveness is called *measure expansiveness*. Lee *et al.* [2] introduced a notion of *weak measure expansive homeomorphism* which is weaker than the notion of measure expansive homeomorphism. From this, we study the various properties of weak measure expansive homeomorphisms, such as sensitivity, equicontinuity, shadowing property, and topological entropy.

Let (X, d) be a compact metric space and f be a homeomorphism on X. A homeomorphism $f : X \to X$ is called *expansive* if there is $\delta > 0$ such that for any distinct points $x, y \in X$ there exists $i \in \mathbb{Z}$ such that $d(f^i(x), f^i(y)) > \delta$. Given $x \in X$ and $\delta > 0$, we define the *dynamic δ-ball of f at x*,

$$\Phi_\delta^f(x) = \{y \in X : d(f^i(x), f^i(y)) \le \delta \text{ for all } i \in \mathbb{Z}\}.$$

(Denote $\Phi_\delta(x)$ by $\Phi_\delta^f(x)$ for simplicity if there is no confusion.) Then we see that f is expansive if there is $\delta > 0$ such that $\Phi_\delta(x) = \{x\}$ for all $x \in X$.

Let β be the Borel σ-algebra on X. Denote by $\mathcal{M}(X)$ the set of Borel probability measures on X endowed with weak* topology. Let $\mathcal{M}^*(X) = \{\mu \in \mathcal{M}(X) : \mu \text{ be nonatomic}\}$. A homeomorphism $f : X \to X$ is said to be μ-*expansive* if there is $\delta > 0$ (called an *expansive constant* of μ with respect to f) such that $\mu(\Phi_\delta(x)) = 0$ for all $x \in X$. In the case, we say that f *has expansive measure* μ. Note that $\Phi_\delta(x) = \cap_{i \in \mathbb{Z}} f^{-i}(B[f^i(x), \delta])$, where $B[x, \delta] = \{y \in X : d(x, y) \le \delta\}$.

Now we first introduce the notions of a finite partition P of X and a dynamical P-ball of a homeomorphism f on X. We say that a finite collection $P = \{A_1, A_2, \cdots, A_n\}$ of subsets of X is a *finite δ-partition* ($\delta > 0$) of X if each A_i is disjoint, measurable, int$A_i \ne \varnothing$, diam $A_i \le \delta$, and $\cup_{i=1}^n A_i = X$. For a homeomorphism f on X, a finite δ-partition P of X and $x \in X$, we denote $\Phi_P^f(x)$ by

$$\Phi_P^f(x) = \{y \in X : f^i(y) \in P(f^i(x)) \text{ for all } i \in \mathbb{Z}\},$$

and it is called by the *dynamical P-ball* of f centered at x, where $P(x)$ denotes the element of P containing x. Denote $\Phi_P(x)$ by $\Phi_P^f(x)$ for simplity if there is no confusion. Then it is easy to check that $\Phi_P(x)$ is measurable, $\Phi_P(x) = \cap_{i=1}^n f^{-i}(P(f^i(x)))$, and $f(\Phi_P(x)) \subset \Phi_P(f(x))$.

Definition 1.1. *A homeomorphism f on X is said to be weak μ-expansive ($\mu \in \mathcal{M}(X)$) if there exists a constant $\delta > 0$ and finite δ-partition* $P = \{A_1, A_2, \ldots, A_n\}$ *of X such that*

$$\mu(\Phi_P^f(x)) = 0 \text{ for all } x \in X.$$

We say that f is weak measure expansive iff f is weak μ-expansive for all $\mu \in \mathcal{M}^*(X)$. In the case, we say that f has weak expansive measure μ.

We can also define the *positively weak measure expansiveness for homeomorphisms* by defining the positive dynamical P-ball

$$\Gamma_P(x) = \{y \in X : f^n(y) \in P(f^n(x)) \text{ for all } n \in \mathbb{N} \cup \{0\}\}.$$

Definition 1.2. *A homeomorphism f on X is said to be positively weak μ-expansive ($\mu \in \mathcal{M}(X)$) if $\mu(\Gamma_P(x)) = 0$ for all $x \in X$. We say that f is positively weak measure expansive iff f is positively weak μ-expansive for all $\mu \in \mathcal{M}^*(X)$. In the case, we say that f has positively weak expansive measure μ.*

It follows easily from the definitions that any weak measure expansive homeomorphism f is positively weak measure expansive.

We give some definitions and notations for our works. Recall that (X, d) is a compact metric space and $f : X \to X$ is a homeomorphism. The *f-orbit* $\{x, f(x), f^2(x), \cdots\}$ of a point $x \in X$ is denoted by $\mathcal{O}_f(x)$. The *ω-limit set* $\omega_f(x)$ of a point $x \in X$ is the set of limit points of $\overline{\mathcal{O}_f(x)}$. We say that a point $x \in X$ is *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$, *recurrent* if there exists $n \in \mathbb{N}$ such that $f^n(x) \in U$ for any neighborhood U and

V of x, and *non-wandering* if there exists $n \in \mathbb{N}$ such that $U \cap f^{-n}(V) \neq \varnothing$ for any neighborhood U of x. Let $P(f)$, $R(f)$ and $\Omega(f)$ denote the sets of periodic, recurrent, and non-wandering points of f, respectively. Then we have

$$P(f) \subset R(f) \subset \Omega(f).$$

A point $x \in X$ is a *sensitive point* if there is $\epsilon > 0$ with the property that for any neighborhood U of x, we have $\operatorname{diam}[f^n(U)] > \epsilon$ for some $n \in \mathbb{N}$. Let $Sen(f)$ denote the set of sensitive points of f. We say that f is *sensitive* if $Sen(f) = X$ and if there is $\epsilon > 0$ that works for all x. By the compactness of X, we see that $Sen(f) = \varnothing$ if and only if for any $\epsilon > 0$ there is $\delta > 0$ such that

$$d(f^n(x), f^n(y)) < \epsilon$$

for all $n \in \mathbb{Z}$ whenever $x, y \in X$ with $d(x, y) < \delta$. If this condition holds, we say f is *equicontinuous*. If $x \notin Sen(f)$ then we say that f is equicontinuous at x, or x is an equicontinuity point for f.

For $\delta > 0$, a *δ-pseudo orbit* of f in X is a finite or infinite sequence of points $\{x_n\}_{n=0}^p$ such that

$$d(x_{n+1}, f(x_n)) < \delta$$

for $p \in \mathbb{N} \cup \{\infty\}$ and every $n < p$. We say that a δ-pseudo orbit $\{x_n\}_{n=0}^p$ is *ϵ-traced* by a point $y \in X$ if $d(x_n, f^n(y)) < \epsilon$ for every $n < p$. And f is said to have the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that every infinite δ-pseudo orbit $\{x_n\}_{n=0}^\infty$ of f in X is ϵ-traced by some point in X. By the compactness of X, f has shadowing property if and only if for every $\epsilon > 0$ there is $\delta > 0$ such that every finite δ-pseudo orbit $\{x_n\}_{n=0}^p$ $(p \in \mathbb{N})$ of f in X is ϵ-traced by some point in X.

Let us recall the *topological entropy* for a homeomorphism f on a closed set([6]). Let $n \in \mathbb{N}$, $\epsilon > 0$, and K be a compact subset of X. A subset E of K is said to be *(n, ϵ)-separated with respect to f*, if $x \neq y \in E$ implies

$$\max_{0 \leq i \leq n} d(f^i(x), f^i(y)) > \epsilon.$$

And let $s_n(\epsilon, K)$ denote the largest cardinality of any *(n, ϵ)-separated subset of K with respect to f*. Put

$$s(\epsilon, K, f) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(\epsilon, K).$$

So, *topological entropy of f on K* is defined as the number

$$h(f, K) = \lim_{\epsilon \to 0} \left\{ \limsup_{n \to \infty} \frac{\log s_n(\epsilon, K)}{n} \right\}.$$

The *topological entropy* of f on X is defined as $h(f) = h(f, X)$. We say that $x \in X$ is an *entropy point* for f if $h(f, \overline{U}) > 0$ for any neighborhood U of x. Let $Ent(f)$ denote the set of entropy points of f. Then $Ent(f)$ is a closed f-invariant set and $Ent(f) \neq \varnothing$ if and only if $h(f) > 0$.

1.2 Basics for positively weak measure expansive flows

Many dynamic results for homeomorphisms can be extended to the case of vector fields, but not always. Bowen and Walters [5], inspired by the notion of expansiveness for discrete dynamical systems, introduced a definition of expansiveness for continuous flows. Studying the dynamics of expansive continuous flows (or vector fields) is challenging. In this section, we begin to study the expansive flows from the measure theoretical view point.

Let (X, d) be a compact metric space. A *flow* on X is a continuous map $\phi : X \times \mathbb{R} \longrightarrow X$ satisfying $\phi(x, 0) = x$ and $\phi(\phi(x, s), t) = \phi(x, s + t)$ for $x \in X$ and $s, t \in \mathbb{R}$. For convenience, we will denote by

$$\phi(x, s) = \phi_s(x) \text{ and } \phi_{(a,b)}(x) = \{\phi_t(x) : t \in (a, b)\}.$$

The set $\phi_\mathbb{R}(x)$ is called by the *orbit of ϕ through $x \in X$* and will be denoted by $\mathcal{O}_\phi(x)$.

Let $\mathcal{M}(X)$ be the set of all Borel probability measures μ on X, and denote by $\mathcal{M}_\phi^*(X)$ the set of μ in $\mathcal{M}(X)$ vanishing along the orbits of the flow ϕ on X. More precisely, we let

$$\mathcal{M}_\phi^*(X) = \{\mu \in \mathcal{M}(X) : \mu(\mathcal{O}_\phi(x)) = 0 \text{ for all } x \in X\}.$$

Then we have $\mathcal{M}_\phi^*(X) \subset \mathcal{M}(X)$. For any subset $B \subset X$ (Borel measurable or not) we write $\mu(B) = 0$ if $\mu(A) = 0$ for any Borel subset $A \subset B$.

More general extension, which is called measure expansivity for flows using Borel measures on a compact metric space, was introduced by Carrasco-Olivera *et al.* in [3]. For any flow ϕ on X, $x \in X$ and $\delta > 0$, we denote $\Phi_\delta^\phi(x)$

$$\Phi_\delta^\phi(x) = \{y \in X : d(\phi_t(x), \phi_{h(t)}(y)) \leq \delta \text{ for some } h \in \mathcal{H} \text{ and all } t \in \mathbb{R}\}$$

and it is called by *the dynamical δ-ball* of ϕ centered at $x \in X$. Note that

$$\Gamma_\delta^\phi(x) = \bigcup_{h \in \mathcal{H}} \bigcap_{t \in \mathbb{R}} \phi_{-h(t)}(B[\phi_t(x), \delta]).$$

For any $\mu \in \mathcal{M}(X)$, we say that ϕ is *μ-expansive* if there exists a constant $\delta > 0$ such that $\mu(\Phi_\delta^\phi(x)) = 0$ for all $x \in X$. Such a δ is called an *expansivity constant* of ϕ with respect to μ. Note that if ϕ is μ-expansive for some $\mu \in \mathcal{M}(X)$ then μ vanishes along the orbits of ϕ (see $(A4)$ of Theorem 1 in [3]). A flow ϕ on X is said to be *measure expansive* if ϕ is μ-expansive for any $\mu \in \mathcal{M}_\phi^*(X)$. In the case, we say that ϕ *has expansive measure μ*.

Now we recall that the notions of a finite δ-partition P of X and a dynamical P-ball of a homeomorphism f on X as before. For a flow ϕ on X, a finite δ-partition P of X and $x \in X$, the *dynamical P-ball* of ϕ centered at x, $\Phi_P^\phi(x)$, is defined by

$$\{y \in X : \phi_{h(t)}(y) \in P(\phi_t(x)) \text{ for some } h \in \mathcal{H} \text{ and all } t \in \mathbb{R}\},$$

where $\mathcal{H}$ denotes the set of increasing continuous maps $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ and P(x) denotes the element of P containing x.

Definition 1.3. *A flow ϕ on X is said to be* weak μ-expansive *($\mu \in \mathcal{M}(X)$) if there exists a finite δ-partition P of X such that*

$$\mu(\Phi_P^\phi(x)) = 0 \text{ for all } x \in X.$$

We say that ϕ is weak measure expansive *if ϕ is weak μ-expansive for all $\mu \in \mathcal{M}_\phi^*(X)$. In the case, we say that ϕ has weak expansive measure μ.*

We can also define the *positively weak measure expansiveness for flows* by defining the positive dynamical P-ball

$$\Gamma_P^\phi(x) = \{y \in X : \phi_{h(t)}(y) \in P(\phi_t(x)) \text{ for some } h \in \mathcal{H} \text{ and all } t \geq 0\}.$$

Definition 1.4. *A flow ϕ on X is said to be* positively weak μ-expansive *($\mu \in \mathcal{M}(X)$) if there exists a finite δ-partition P of X such that*

$$\mu(\Gamma_P^\phi(x)) = 0 \text{ for all } x \in X.$$

We say that ϕ is positively weak measure expansive *if ϕ is positively weak μ-expansive for all $\mu \in \mathcal{M}_\phi^*(X)$. In the case, we say that ϕ has positively weak expansive measure μ.*

Similarly, we can define periodic, recurrent, non-wandering and sensitive points for flows. A point $x \in X$ is called *nonwandering* if for any neighborhood U of x, there is $T > 0$ such that for all $t \geq T$ $\phi_t(U) \cap U \neq \varnothing$. The set of all nonwandering points of ϕ_t is called the *nonwandering set* of ϕ_t, denoted by $\Omega(\phi)$. By *non-trivial recurrence* of a flow ϕ on a compact metric space X we mean a non-periodic point x_0 which is *recurrent* in the sense that $x_0 \in \omega(x_0)$, where

$$\omega(x) = \{y \in X : y = \lim_{n \to \infty} \phi_{t_n}(x) \text{ for some sequence } t_n \to \infty\}$$

for any $x \in X$. The set of all recurrent points of ϕ_t is called the *recurrent set* of ϕ_t, denoted by $R(\phi)$.

Let ϕ be a continuous flow on a compact metric space X. Given real numbers δ, $a > 0$, we say that a *finite* (δ, a)-*chain*, is a pair of sequences $\{(x_i, t_i) : i = 0, \dots, k\}$ such that $t_i \geq a$ and $d(\phi_{t_i}(x_i), x_{i+1}) < \delta$. An infinite (δ, a)-chain is a pair of doubly infinite sequences $\{(x_i, t_i) : i \in \mathbb{Z}\}$ such that $t_i \geq a$ and $d(\phi_{t_i}(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. The definition of a *finite(infinite)* (δ, a)-*pseudo orbit* is the same as that of a finite(infinite) (δ, a)-chain. According to standard notation let

$$s_0 = 0, \ s_n = \sum_{i=0}^{n-1} t_i, \ \text{and} \ s_{-n} = \sum_{i=-n}^{-1} t_i$$

for every sequence $\{t_i : i \in \mathbb{Z}\}$ of real numbers.

Let $\epsilon > 0$ be given. A reparametrization $h \in \tilde{\mathcal{H}}$ satisfying $h : \mathbb{R} \to \mathbb{R}$ is a monotone increasing homeomorphism with $h(0) = 0$ and

$$\left| \frac{h(s) - h(t)}{s - t} - 1 \right| \leq \epsilon \ \text{for every } s, \ t \in \mathbb{R}.$$

A finite(infinite) (δ, a)-pseudo orbit $\{(x_i, t_i) : i \in \mathbb{Z}\}$ is ϵ-*traced by an orbit* $(\phi_t(z))_{t \in \mathbb{R}, z \in X}$ if there exists $h \in \tilde{\mathcal{H}}$ such that

$$\begin{cases} d(\phi_{h(t)}(z), \phi_{t-s_i}(x_i)) < \epsilon, & \text{if } t \geq 0, \ s_i \leq t < s_{i+1}, \\ d(\phi_{h(t)}(z), \phi_{t+s_{-i}}(x_{-i})) < \epsilon, & \text{if } t \leq 0, \ -s_i \leq t < -s_{-i+1}. \end{cases}$$

for $i = 0, 1, \cdots$. For every $a > 0$, the flow ϕ on X has the *shadowing property (or pseudo-orbit tracing property)* with respect to time $a > 0$ if and only if ϕ has the shadowing property (that is with respect to time 1).

For a flow ϕ, given any ϕ-invariant probability measure μ on X, we denote by $h_\mu(\phi)$ the measure theoretic entropy of ϕ with respect to μ. The topological entropy, denoted by $h_{\text{top}}(\phi)$, can be defined using the variational principle [9] by ;

$$h_{\text{top}}(\phi) = \sup\{h_\mu(\phi) : \mu \text{ is a } \phi\text{-invariant probability measure}\}.$$

The topological entropy is always non-negative and finite.

For $E, F \subset X$ we say E is a (t, δ)-*separate* subset of F with respect to ϕ if for any $x, y \in E$ with $x \neq y$ we have

$$\max_{0 \leq s \leq t} d(\phi_s(x), \phi_s(y)) > \delta.$$

Let $s_t(F, \delta) = s_t(F, \delta, \phi)$ denote the maximum cardinality of a set which is a (t, δ)-separated subset of F. If F is compact then [9] shows that $s_t(F, \delta) < \infty$. We define

$$\bar{s}_\phi(F, \delta) = \limsup_{t \to \infty} \frac{1}{t} \log s_t(F, \delta)$$

and *topological entropy* by

$$h(\phi, F) = \lim_{\delta \to 0} \bar{s}_\phi(F, \delta).$$

By Lemma 1 in [6] these limits exists and are equal. The topological entropy of ϕ is defined as $h(\phi) = h(\phi, X)$. We say that $x \in X$ is an entropy point for ϕ if $h(\phi, \overline{U}) > 0$ for any neighborhood U of x. Denote by $\text{Ent}(\phi)$ the set of entropy points of ϕ. Then $\text{Ent}(\phi)$ is a closed ϕ-invariant set and $\text{Ent}(\phi) \neq \varnothing$ if and only if $h(\phi) > 0$.

2 Main Theorems

2.1 Topological entropy for positively weak measure expansive homeomorphisms

Before we state the main theorems, we recall some results from [1] and [7]. Given a map $f : X \to X$, $x \in X$, $\delta > 0$ and $n \in \mathbb{N}$, we define

$$V[x, n, \delta] = \{y \in X : d(f^i(x), f^i(y)) \leq \delta \text{ for all } 0 \leq i \leq n\}.$$

That is, $V[x, n, \delta] = \bigcap_{i=0}^n f^{-i}(B[f^i(x), \delta])$, where $B[x, \delta]$ denotes the closed δ-ball centered at x. It is clear that $\Gamma_\delta(x) = \bigcap_{n \in \mathbb{Z}} V[x, n, \delta]$ and $V[x, m, \delta] \subset V[x, n, \delta]$ for $n \leq m$. Consequently, $\mu(\Gamma_\delta(x)) = \lim_{l \to \infty} \mu(V[x, k_l, \delta])$ for every $x \in X$, $\delta > 0$, every Borel probability measure μ of X, and every sequence $k_l \to \infty$.

Based on this, we can construct the weak measure expansive set, and we will use the set for the proof of the main theorems. Let

$$V_P[x, n, \delta] = \{y \in X : f^i(y) \in P(f^i(x)) \text{ for all } 0 \leq i \leq n\}$$

then $V_P[x, n, \delta] = \bigcap_{i=0}^n f^{-i}(P(f^i(x)))$ and $\Gamma_P(x) = \bigcap_{i \in \mathbb{N}} f^{-i}(P(f^i(x)))$. Similarly,

$$\mu(\Gamma_P(x)) = \lim_{l \to \infty} \mu(V_P[x, k_l, \delta]) \cdots\cdots(*).$$

Given a measure $\mu \in \mathcal{M}^*(X)$ and a homeomorphism $f : X \to X$, we denote $f_*(\mu)$ the *pullback measure* of μ denoted by $f_*(\mu)(A) = \mu(f^{-1}(A))$ for all Borel set A of X. We say that a Borel measure is *invariant* for f if $\mu = \mu \circ f^{-1}$.

Lemma 2.1. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. If $\mu \in \mathcal{M}^*(X)$ is a positively weak expansive measure with expansive constant δ of f, then so does $f_*^{-1}\mu$.*

Proof. By the definition of $\Gamma_P(x)$, we can check that
(i) $f(\Gamma_P(x)) \subset \Gamma_P(f(x))$ and (ii) $\Gamma_P(x) \subset \Gamma_P(f^{-1}(x))$.

So we show that if $\mu(\Gamma_P(x)) = 0$ then $\mu(\Gamma_P(f^{-1}(x))) = 0$ for all $x \in X$, by (i) and (ii). □

Lemma 2.2. *Let $f : X \to X$ be a homeomorphism of a metric space X. Then every invariant measure of f which is the limit with respect to weak* topology of a sequence of μ with a common expansivity constant is positively weak expansive.*

Proof. As in the proof of Lemma 7 in [4], we let δ_x and $W[x, n]$. Then we can check that

$$V_P[x, n, \frac{\delta}{2}] \subset W[x, n] \subset V_P[x, n, \delta]$$

for all $x \in X$, $n \in \mathbb{N}$. Similarly, we verify that

$$\lim_{n \to \infty} \inf \mu(V_P[x, n, \frac{\delta}{2}]) = 0$$

by the above fact of $(*)$. So, μ is positively weak expansive measure. □

Lemma 2.3. *If a homeomorphism f of a compact metric space X has positively weak expansive measure then it has positively weak expansive invariant measures.*

Proof. Let μ be a positively weak expansive measure with expansive constant δ of $f : X \to X$. By Lemma 2.1, we know that $f_*^{-1}\mu$ is a positively weak expansive measure with positive expansive constant δ of f. And so, $f_*^{-i}\mu$ is a positively weak expansive measure with positively expansive constant δ of f for all $i \in \mathbb{N}$, we can consider a sequence of positively weak expansive measures with uniform expansive constant δ,

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^{-i}\mu, \quad \text{for all } n \in \mathbb{N}.$$

Since X is compact there is a subsequence μ_{n_k} such that $\mu_{n_k} \to \mu$ as $n_k \to \infty$. Since μ is invariant for f^{-1} and f are homeomorphisms, we have that μ is also an invariant measure of f. So, we conclude that μ is a positively weak expansive measure of f, by applying Lemma 2.2. □

From the above facts, we can state the first main theorem as following.

Theorem A. *If a homeomorphism f on X has a positively weak expansive measure and the shadowing property on its nonwandering set, then its topological entropy is positive.*

The following lemma is a particular case of Corollary 6 in [8].

Lemma 2.4. *If f is a homeomorphism with the shadowing property of a compact metric space X and $h(f) = 0$, then $f|_{\Omega(f)}$ is equicontinuous.*

Proof. See Lemma 9 in [4]. □

Lemma 2.5. *Let $f : X \to X$ be a continuous map having the shadowing property on a compact metric space X. Let $Y \subset X$ be an f-invariant closed set, $g = f|_Y$, and consider g in Y. If g is not equicontinuous then $h(f) > 0$.*

Proof. It is easy to prove this lemma from the next section Lemma 2.8. For more details, see Theorem 3 in [8]. □

We know that if $h(f) = 0$ and f has the shadowing property, then $\Omega(f)$ is totally disconnected and $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$ is an equicontinuous map. That is, an equicontinuous map of a compact metric space has zero topological entropy (for more details, Corollary 6 in [8]). The following lemma improves this result. First of all, let $\mathcal{M}_f^*(X) = \{\mu \in \mathcal{M}^*(X) : \mu \ be \ f\text{-invariant}\}$.

Lemma 2.6. *Let $f : X \to X$ be positively weak μ-expansive. Then f is not equicontinuous.*

Proof. Let f be a homeomorphism of a compact metric space X. Suppose that f is equicontinuous. Since f is weak μ-expansive, there exist $\delta > 0$ and a finite δ-partition $P = \{A_i : i = 1, \cdots, n\}$ such that $\mu(\Gamma_P(x)) = 0$ for all $x \in X$. By the definition of equicontinuous, we obtain $\delta' > 0$ such that $B[x, \delta'] \subset \Gamma_P(x)$ for all $x \in X$. From this, we get $\mu(B[x, \delta']) = 0$ for all $x \in \Omega(f)$. Since X is compact, there are finitely many points $x_1, x_2, \cdots, x_n$ such that $X = \bigcup_{i=1}^n B[x_n, \delta']$. Then

$$\mu(X) \le \sum_{i=1}^n \mu(B[x_i, \delta']) = 0.$$

This is a contradiction which completes the proof. □

End of the Proof of Theorem A. Suppose that f is positively weak μ-expansive but $h(f) = 0$. Then by Lemma 2.4, $f|_{\Omega(f)}$ is equicontinuous. By Lemma 2.6, f is not positively weak measure expansive. This is a contradiction which completes the proof. □

Example 2.7. *It is well-known that the horseshoe map has the shadowing property, expansive property and positive topological entropy. If a map is expansive then it has positively weak expansive measure. That is, the horseshoe map has positively weak expansive measure. So, we can conclude that this map is an example of applying Theorem A.*

2.2 Topological entropy for positively weak measure expansive flows

Let X and $\mathcal{M}_\phi^*(X)$ be as before. We consider that weak measure expansive flows with the shadowing property is an extension for flows of the Theorem A.

Theorem B. *If a flow ϕ has a positively weak expansive measure and the shadowing property on its nonwandering set, then its topological entropy is positive.*

Now we consider a relationship between equicontinuity and topological entropy for a flow. We say that a flow ϕ is *equicontinuous* if for any $\epsilon > 0$ there is $\delta > 0$ such that for any $y \in X$ if $d(x, y) < \delta$ then $d(\phi_t(x), \phi_t(y)) < \epsilon$ for all $t \in \mathbb{R}$.

Lemma 2.8. *Let X be a compact metric space and $\phi : X \times \mathbb{R} \to X$ be a continuous flow having the finite shadowing property. Let $Y \subset X$ and $\psi = \phi|_Y$. If ψ is not equicontinuous then ϕ has positive topological entropy.*

Proof. Since ψ is not equicontinuous, there exist $z \in Sen(\psi)$ with $(z,z) \in \text{int}\overline{[R(\psi \times \psi)]}$. Let U be a neighborhood of z in X. We have to show that $h(\phi, \overline{U}) > 0$. Choose $\epsilon > 0$ with $B(z, 2\epsilon) \subset U$ by taking ϵ small enough. We may also assume that for any neighborhood V of z in X, there exists $t \in \mathbb{R}$ with $\text{diam}[\psi_t(V \cap Y)] > 3\epsilon$. Using the shadowing property of ϕ, choose $\delta \in (0, \epsilon)$ so that every $(\delta, 1)$-pseudo orbit in X is ϵ-traced by some point in X.

Since V is a neighborhood of z in X with $(V \cap Y) \times (V \cap Y) \subset \text{int}\overline{[R(\psi \times \psi)]}$ and $\text{diam}(V) < \frac{\delta}{2}$. Then there exist $T \in \mathbb{R}$ and $(x_0, y_0) \in R(\psi \times \psi) \cap (V \times V)$ such that $d(\psi_T(x_0), \psi_T(y_0)) > 3\epsilon$. Since $(x_0, y_0) \in R(\psi \times \psi)$, there is $\tau > T$ with

$$d(x_0, \psi_\tau(x_0)) < \frac{\delta}{2} \text{ and } d(y_0, \psi_\tau(y_0)) < \frac{\delta}{2}.$$

Now we claim that

$$h(\phi, \overline{U}) \geq \frac{\log 2}{\tau}.$$

It is enough to show that $s_t(\overline{U}, \delta, \phi) \geq 2^n$, and we take $t = 1$, for simplicity. For every $\delta \in (0, \epsilon)$ and all $t \in \mathbb{R}$, $s_t(\overline{U}, \delta, \phi)$ is the maximum cardinality of $(\overline{U}, \delta, \phi)$-separated set for ϕ. Let

$$A = \{(x_i, t_i) : t_i \geq 1, \ i = 0, 1, \cdots, n-1\} \text{ and}$$
$$B = \{(y_i, t_i) : t_i \geq 1, \ i = 0, 1, \cdots, n-1\}$$

satisfying

$$d(\phi_{t_i}(x_i), x_{i+1}) < \delta, \text{ for any } i = 0, 1, \cdots, n-1 \text{ and}$$
$$d(\phi_{t_i}(y_i), y_{i+1}) < \delta, \text{ for any } i = 0, 1, \cdots, n-1.$$

Also, there is $j \in \mathbb{N}$ with $d(\phi_{t_j}(x_0), \phi_{t_j}(y_0)) > 3\epsilon$. Since

$$d(x_0, y_0) < \frac{\delta}{2}, \ d(x_0, \psi_{t_j}(x_0)) < \frac{\delta}{2} \text{ and } d(y_0, \psi_{t_j}(x_0)) < \frac{\delta}{2},$$

we can take $C = C_1 \cdots C_n \in \{A, B\}^n$ for any $n \in \mathbb{N}$. Then C is a δ-pseudo orbit for ϕ consisting of n^2-elements. For $C \in \{A, B\}^n$ let $w_C \in X$ be a point ϵ-tracing the δ-pseudo orbit C. If $y \in \{x_0, y_0\} \subset V$ is the starting element of C then

$$d(z, w_C) \leq d(z, y) + d(y, w_C) < \delta + \epsilon.$$

So, $w_C \in U$. If $C, D \in \{A, B\}^n$ are distinct then for some $k \in \{0, 1, \cdots, n-1\}$, the k-th elements of the pseudo orbits C and D are more than 3ϵ apart. Therefore by the triangle inequality,

$$d(\phi_{t_k}(w_C), \phi_{t_k}(w_D)) > \epsilon \text{ for some } k \in \{0, 1, \cdots, n-1\}.$$

This means that the set $\{w_C : C \in \{A, B\}^n\}$ is $(\overline{U}, \epsilon, \phi)$-separated and hence $(\overline{U}, \delta, \phi)$-separated for ϕ and for any $\delta \in (0, \epsilon)$. That is,

$$|\{w_C : C \in \{A, B\}^n\}| = 2^n.$$

So, we complete the proof. □

Now we introduce the notion of $V_P[\phi, x, T, \delta]$ which is a flow case of $V_P[x, n, \delta]$. Let $V_P[\phi, x, T, \delta] = \{y \in X : \phi_{h(t)}(y) \in P(\phi_t(x)) \text{ for some } h \in \tilde{\mathcal{H}} \text{ and } -T \leq t \leq T\}$. Then

$$V_P[\phi, x, T, \delta] = \bigcup_{h \in \tilde{\mathcal{H}}} \bigcap_{-T \leq t \leq T} \phi_{-h(t)}(B[\phi_t(x), \delta]).$$

Similarly, $\mu(\Gamma_P^\phi(x)) = \lim_{t \to \infty} \mu(V_P[\phi, x, t, \delta])$.

Lemma 2.9. *Let ϕ be positively weak μ-expansive. Then ϕ is not equicontinuous.*

Proof. Let ϕ be an equicontinuous flow of a compact metric space X. Suppose by contradiction that ϕ is a weak μ-expansive for any $\mu \in \mathcal{M}_\phi^*(X)$. Then there exist a constant $\delta' > 0$ and a finite δ'-partition $P = \{A_i : i = 1, \ldots, n\}$ of X such that $\mu(\Gamma_P^\phi(x)) = 0$.

Letting it in the definition of the equicontinuity, we obtain $\delta > 0(\delta < \delta')$ such that $B[x, \delta] \subset \Gamma_P^\phi(x)$ for any $x \in X$. From this, we get $\mu(B[x, \delta]) = 0$ for any $x \in \Omega(\phi)$. Since X is compact, so there are finitely many points $x_1, x_2, \cdots, x_n$ such that $X = \bigcup_{i=1}^n B[x_i, \delta]$. Then

$$\mu(X) \le \sum_{i=1}^n \mu(B[x_i, \delta]) = 0.$$

This is a contradiction, so we complete the proof. $\square$

The following lemma is an extension for a flow case of Corollary 6 in [8].

Lemma 2.10. *If ϕ is a flow with the shadowing property on a compact metric space X and $h(\phi) = 0$, then $\phi|_{\Omega(\phi)}$ is equicontinuous.*

Proof. By Lemma 2.8, we know that if ϕ is weak measure expansive then ϕ is not equicontinuous. By Lemma 2.9, if ϕ is not equicontinuous then ϕ has positive topological entropy. $\square$

Finally, we can see that a equicontinuous positively weak measure expansive flow of a compact metric space has zero topological entropy.

End of the Proof of Theorem B. Suppose that ϕ is positively weak μ-expansive ($\mu \in \mathcal{M}_\phi^*(X)$) but $h(\phi) = 0$. Then by Lemma 2.10, $\phi|_{\Omega(\phi)}$ is equicontinuous, and so by Lemma 2.9, ϕ is not positively weak measure expansive. This is a contradiction. $\square$

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