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Francesc Tugores*

Recursive interpolating sequences

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Abstract: This paper is devoted to pose several interpolation problems on the open unit disk $\mathbb{D}$ of the complex plane in a recursive and linear way. We look for interpolating sequences (z_n) in $\mathbb{D}$ so that given a bounded sequence (a_n) and a suitable sequence (w_n), there is a bounded analytic function f on $\mathbb{D}$ such that $f(z_1) = w_1$ and $f(z_{n+1}) = a_n f(z_n) + w_{n+1}$. We add a recursion for the derivative of the type: $f'(z_1) = w'_1$ and $f'(z_{n+1}) = a'_n\,[(1-|z_n|^2)/(1-|z_{n+1}|^2)]\,f'(z_n)+w'_{n+1}$, where (a'_n) is bounded and (w'_n) is an appropriate sequence, and we also look for zero-sequences verifying the recursion for f'. The conditions on these interpolating sequences involve the Blaschke product with zeros at their points, one of them being the uniform separation condition.

Keywords: Interpolating sequence, Uniformly separated sequence, Bounded analytic function

MSC: 30E05, 30H05, 30J10

1 Introduction

Interpolation problems on the unit disk $\mathbb{D}$ are a classical branch of complex analysis. Several types of interpolating sequences for different classes of analytic functions have been addressed since the middle of the last century, beginning with the celebrated works of W.K. Hayman [1], D.J. Newman [2] and L. Carleson [3] about the so-called "universal" interpolation problem, which consists in characterizing the sequences (z_n) in $\mathbb{D}$ verifying that for any bounded sequence (w_n), there is a bounded analytic function f on $\mathbb{D}$ such that $f(z_n) = w_n$.

On the other hand, recursion appears in many areas of mathematics: formulas, algorithms, optimization..., providing alternative definition procedures. Since there exists a specific theory for recursive numerical sequences and interpolating sequences have not been studied from a recursive perspective, we think it is interesting to pose recursive-type interpolation problems.

We want to emphasize that our approach converts the universal interpolation problem and other problems related to it into trivial cases of those that we introduce. Furthermore, most conditions involved are new and depend not only on the separation of the points of the sequence in $\mathbb{D}$, but also on the sequences that we employ to define recursion.

We begin with the necessary notation. Let H^∞ be the space of all analytic functions f on $\mathbb{D}$ such that $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty$ and let l^∞ be the Banach space of all sequences of complex numbers (w_n) such that $\|(w_n)\|_\infty = \sup_n |w_n| < \infty$. We put $Z = (z_n)$ for any sequence of different points in $\mathbb{D}$ verifying the Blaschke condition $\sum_n (1-|z_n|) < \infty$, which characterizes the zero-sequences of functions in H^∞. For two points z and w in $\mathbb{D}$, we write

$$\psi(z,w) = \frac{z-w}{1-\bar{z}w},$$

***Corresponding Author: Francesc Tugores:** Department of Mathematics, University of Vigo, Ourense 32003, Spain, E-mail: ftugores@uvigo.es

so that $\rho = |\psi|$ is their pseudo-hyperbolic distance. Let B be the Blaschke product with zeros at Z, that is,

$$B(z) = \prod_n \frac{|z_n|}{z_n}\,\psi(z_n, z).$$

If E is a subsequence of Z, we put B_E for the Blaschke product with zeros at E and for a fixed $m \in \mathbb{N}$, we denote $B_{Z\setminus\{z_m\}}$ by B_m. We write c for strictly positive constants that may change from one occurrence to the next one.

First, we recall that Z is *interpolating* if given any $(w_n) \in l^\infty$, there exists $f \in H^\infty$ such that $f(z_n) = w_n$. Interpolating sequences are characterized by the well-known Carleson's theorem:

Theorem 1.1 ([3]). *Z is interpolating if and only if*

$$|B_m(z_m)| \geq c \quad \forall m \in \mathbb{N}. \tag{1}$$

Sequences satisfying (1) are called *uniformly separated (u.s.)*. From the Schwarz lemma, it follows that

$$|f(z) - f(w)| \leq c\,\|f\|_\infty\,\rho(z, w) \tag{2}$$

and thus, it is said that Z is *interpolating in differences* if given (w_n) verifying $|w_i - w_j| \leq c\,\rho(z_i, z_j)$, there is $f \in H^\infty$ such that $f(z_n) = w_n$. These sequences are the union of two u.s. [4], characterized as follows.

Lemma 1.2 ([5]). *For a sequence Z, the following are equivalent*
(a) *Z is the union of two u.s. sequences.*
(b) *For each z_i, there exists z_j such that $|B_i(z_i)| \geq c\,\rho(z_i, z_j)$.*
(c) *Z is either u.s. or it can be rearranged $Z = (\alpha_n) \cup (\beta_n)$, where (α_n) and (β_n) are u.s. sequences, $\rho(\alpha_n, \beta_n) = \rho(\alpha_n, Z \setminus \{\alpha_n\}) \to 0$, and $\rho(\alpha_n, z_i) \geq c$, if $z_i \neq \beta_n$.*

Finally, since $|f'(z)|(1 - |z|^2) \leq c\,\|f\|_\infty$, it is said that Z is *double interpolating* if given $(w_n) \in l^\infty$ and (w'_n) satisfying $(w'_n(1 - |z_n|^2)) \in l^\infty$, there is $f \in H^\infty$ such that $f(z_n) = w_n$ and $f'(z_n) = w'_n$. It is proved in [6] that these sequences are also the u.s. ones.

Next we consider the following three quantities for the terms of a sequence $T = (t_n) \in l^\infty$:

$$\begin{aligned}
\Gamma(t_m) &= \rho(z_m, z_{m+1}) + |1 - t_m| \\
\Pi(t_m) &= \|T\|_\infty\,\rho(z_m, z_{m+1}) + \rho(z_{m+1}, z_{m+2}) + |t_m - t_{m+1}| \\
\Lambda(t_m) &= \rho(z_m, z_{m+1})\,\{\rho(z_m, z_{m+1}) + |1 + t_m|\}.
\end{aligned}$$

We need them to take suitable target spaces in our interpolation problems and they also appear in the results we get (Section 2).

We write $\Gamma(t_m) = O(\Gamma(t_{m+1}))$ (resp. $\Lambda(t_m) = O(\Lambda(t_{m+1}))$) if there exists a constant $c_{T,Z} > 0$ such that $\Gamma(t_m) \leq c_{T,Z}\,\Gamma(t_{m+1})$ (resp. $\Lambda(t_m) \leq c_{T,Z}\,\Lambda(t_{m+1})$) for all $m \in \mathbb{N}$. We write $\Pi(t_m) \sim \Pi(t_{m+1})$ if $\Pi(t_m) \leq c_{T,Z}\,\Pi(t_{m+1})$ and $\Pi(t_{m+1}) \leq c'_{T,Z}\,\Pi(t_m)$ for some constants $c_{T,Z}, c'_{T,Z} > 0$ and for all $m \in \mathbb{N}$.

From now on $A = (a_n)$ and $A' = (a'_n)$ will denote sequences in l^∞. Our purpose is to examine the following distinguished sequences of $\mathbb{D}$:

Definition 1.3. *We say that Z is A-interpolating if given (w_n) verifying*

$$|(\sum_{i=1}^{n} a_i\,a_{i+1}\ldots a_n w_i) + w_{n+1}| \leq c \tag{3}$$

and

$$|w_{n+1}| \leq c\,\Gamma(a_n), \tag{4}$$

there exists $f \in H^\infty$ such that

$$\begin{cases} f(z_1) = w_1 \\ f(z_{n+1}) = a_n f(z_n) + w_{n+1}. \end{cases} \tag{5}$$

Clearly if $\|A\|_\infty < 1$, then the sum in (3) is bounded by $\dfrac{\|(w_n)\|_\infty}{1-\|A\|_\infty}$.

Definition 1.4. *We say that Z is A-interpolating in differences if given (w_n) satisfying* (3) *and*

$$|w_{n+1} - w_{n+2}| \le c\, \Pi(a_n), \tag{6}$$

there is $f \in H^\infty$ verifying recursion (5).

Definition 1.5. *We say that Z is (A, A')-interpolating if given (w_n) satisfying* (3) *and* (4) *and (w'_n) verifying*

$$|\sum_{i=1}^{n} a'_i\, a'_{i+1} \dots a'_n (1-|z_i|^2) w'_i + (1-|z_{n+1}|^2) w'_{n+1}| \le c \tag{7}$$

and

$$|w'_{n+1}|(1-|z_{n+1}|^2) \le c\, \Gamma(a'_n), \tag{8}$$

there exists $f \in H^\infty$ satisfying recursion (5) *and*

$$\begin{cases} f'(z_1) = w'_1 \\ f'(z_{n+1}) = a'_n \dfrac{1-|z_n|^2}{1-|z_{n+1}|^2} f'(z_n) + w'_{n+1}. \end{cases} \tag{9}$$

Definition 1.6. *We say that Z is zero and A'-interpolating if given (w'_n) verifying* (7),

$$|w'_1|(1-|z_1|^2) \le c\, \Lambda(a'_1) \tag{10}$$

and

$$|w'_{n+1}|(1-|z_{n+1}|^2) \le c\, \Lambda(a'_n), \tag{11}$$

there is $f \in H^\infty$ vanishing on Z and satisfying recursion (9).

Recursion (5) is equivalent to $f(z_n) = \mu_n$, where

$$\begin{cases} \mu_1 = w_1 \\ \mu_{n+1} = (\sum_{i=1}^{n} a_i\, a_{i+1} \dots a_n w_i) + w_{n+1}, \end{cases} \tag{12}$$

and recursion (9) is equivalent to $f'(z_n) = \mu'_n$, with

$$\begin{cases} \mu'_1 = w'_1 \\ \mu'_{n+1} = \dfrac{\sum_{i=1}^{n} a'_i\, a'_{i+1} \dots a'_n (1-|z_i|^2) w'_i}{1-|z_{n+1}|^2} + w'_{n+1}. \end{cases} \tag{13}$$

Thus, we must have (3) and (7) to state that sequences (μ_n) and $(\mu'_n(1-|z_n|^2))$ are bounded. We impose that data sequences (w_n) and (w'_n) verify (4), (6), (8) and (11), because they are intrinsic to recursions (a technical reason justifies (10)). In effect, (4) and (6) are obtained taking into account (2) in the inequalities

$$|w_{n+1}| \le |f(z_n) - f(z_{n+1})| + |f(z_n)|\,|1-a_n|$$

and

$$|w_{n+1} - w_{n+2}| \le |a_n|\,|f(z_n) - f(z_{n+1})| + |f(z_{n+1}) - f(z_{n+2})| + |f(z_{n+1})|\,|a_n - a_{n+1}|,$$

respectively. On the other hand,

$$|f'(z)(1-|z|^2) - f'(w)(1-|w|^2)| \le c\, \|f\|_\infty\, \rho(z,w), \tag{14}$$

$$|f(z) - f(w) + f'(w)(1-|w|^2)\psi(z,w)| \le c\, \|f\|_\infty\, \rho(z,w)^2 \tag{15}$$

and

$$|f(z)-f(w)-[f'(z)(1-|z|^2)+f'(w)(1-|w|^2)]\frac{\psi(z,w)}{2}|\le c\,\|f\|_\infty\,\rho(z,w)^3. \tag{16}$$

See [7] for (14), [8] for (15) and [9] for (16). Thus, (8) is obtained using (14) in the inequality

$$|w'_{n+1}|(1-|z_{n+1}|^2)\le |f'(z_{n+1})(1-|z_{n+1}|^2)-f'(z_n)(1-|z_n|^2)|+|f'(z_n)|(1-|z_n|^2)|1-a'_n|.$$

If on the right of this last inequality, we put

$$|f'(z_{n+1})(1-|z_{n+1}|^2)+f'(z_n)(1-|z_n|^2)|+|f'(z_n)|(1-|z_n|^2)|1+a'_n|,$$

then (11) is obtained using (16) for the first summand and (15) for the second one.

We introduce these interpolating sequences because they provide a generalization of the usual interpolation problems, in the sense that (0)-interpolating sequences, (0)-interpolating in differences and $((0),(0))$-interpolating are interpolating, interpolating in differences and double interpolating, respectively.

Extending recursion to an arbitrary order or increasing the degree of derivability are projects certainly cumbersome, so that we confine ourselves to order one and the first derivative. Nevertheless, we think it would be interesting to consider these types of sequences for other spaces of analytic functions, such as the Lipschitz class and the Bloch space, for which the pseudo-hyperbolic distance in (2) is replaced by the Euclidean and hyperbolic distance, respectively (interpolating sequences for these spaces are characterized in [10] and [11]).

While the proofs of results turn out to be rather standard (Carleson's theorem is used repeatedly), we appreciate the following separation conditions, which are consistent with the problems posed and appear in a natural way.

Definition 1.7. *We say that* (Z,A) *satisfies condition* (S) *if*

$$|B_{m+1}(z_{m+1})|\ge c\,\Gamma(a_m)\quad \forall m\in\mathbb{N},$$

and condition (D) *if*

$$|B_{m+1}(z_{m+1})|\ge c\,\Pi(a_m)\quad \forall m\in\mathbb{N}.$$

We say that (Z,A') *satisfies condition* (M) *if*

$$|B_{m+1}(z_{m+1})|^2\ge c\,\Lambda(a'_m)\quad \forall m\in\mathbb{N}.$$

We name (S), (D) and (M) to the above conditions because these are the initials of simple, differences and mixed, respectively, and we will see in the next section that condition (S) is related to interpolating sequences in a simple sense; (D), in a differences sense, and (M), in a mixed sense (zero and interpolating).

Since $\rho(z_m,z_{m+1})>|B_{m+1}(z_{m+1})|$, it follows that if (Z,A') verifies (M), then $(Z,-A')$ satisfies (S). All conditions imply (b) in Lemma 1.2 so that Z is the union of two u.s. sequences. Note that if $\|A\|_\infty<1$ (resp. $\|A'\|_\infty<1$), then (S) (resp. (M)) $\Rightarrow$ u.s.

2 Statement of results

Our results are the following ones.

Proposition 2.1.

(i) *If* Z *is* A*-interpolating and* $\Gamma(a_n)=O(\Gamma(a_{n+1}))$*, then* (Z,A) *verifies* (S).
(ii) *If* Z *is* A*-interpolating in differences and* $\Pi(a_n)\sim\Pi(a_{n+1})$*, then* (Z,A) *verifies* (D).
(iii) *If* Z *is* (A,A')*-interpolating,* $\Gamma(a_n)=O(\Gamma(a_{n+1}))$ *and* $\Gamma(a'_n)=O(\Gamma(a'_{n+1}))$*, then* Z *is u.s.*
(iv) *If* Z *is zero and* A'*-interpolating and* $\Lambda(a'_n)=O(\Lambda(a'_{n+1}))$*, then* (Z,A') *verifies* (M).

Proposition 2.2.

(i) *If* (Z, A) *verifies* (S), *then* Z *is* A*-interpolating.*

(ii) *If* (Z, A) *verifies* (D) *and* A *is such that*

$$|a_{n+1}| \le r\,|a_n| \tag{17}$$

for some $r \in (0, 1)$, *then* Z *is* A*-interpolating in differences.*

(iii) *If* Z *is u.s., then* Z *is* (A, A')*-interpolating for any sequences* A *and* A'.

(iv) *If* (Z, A') *verifies* (M), $\Lambda(a'_n) = O(\Lambda(a'_{n+1}))$ *and*

$$|a'_1\,a'_2\cdots a'_n|\Lambda(a'_1) + \sum_{i=2}^{n} |a'_i\,a'_{i+1}\cdots a'_n|\Lambda(a'_{i-1}) \le c\,\Lambda(a'_{n-1}) \quad \forall n \ge 2, \tag{18}$$

then Z *is zero and* A'*-interpolating.*

3 Proof of results

Proof of Proposition 2.1. (i) For a fixed $m \in \mathbb{N}$, let (w_n) be defined by $w_{m+1} = \Gamma(a_m)$, $w_{m+2} = -a_{m+1}w_{m+1}$ and $w_n = 0$ if $n \ne m+1, m+2$. Since $\Gamma(a_n) = O(\Gamma(a_{n+1}))$, it follows that

$$|w_{m+2}| \le \|A\|_\infty\,\Gamma(a_m) \le c_{A,Z}\,\|A\|_\infty\,\Gamma(a_{m+1})$$

and w_{m+2} also verifies (4). Since the operator

$$\mathcal{R} : H^\infty \longrightarrow \{(w_n) \,/\, (w_n) \text{ verifies (3) and (4)}\}$$

defined by $\mathcal{R}(f) = (u_n)$, where $u_1 = f(z_1)$ and $u_{n+1} = f(z_{n+1}) - a_n f(z_n)$, is linear and onto, a standard argument using the closed graph theorem gives the existence of $f_m \in H^\infty$ such that $\mathcal{R}(f_m) = (w_n)$ and $\|f_m\|_\infty \le c\,\|(w_n)\|_\infty = c$. Since $f_m(z_{m+1}) = w_{m+1}$ and $f_m(z_n) = 0$ if $n \ne m+1$, there is $g_m \in H^\infty$ such that $f_m = g_m B_{m+1}$ and $\|g_m\|_\infty = \|f_m\|_\infty$. Thus,

$$\Gamma(a_m) = |f_m(z_{m+1})| \le c\,|B_{m+1}(z_{m+1})|$$

and (S) holds.

(ii) For a fixed $m \in \mathbb{N}$, let (w_n) be defined by $w_{m+1} = \Pi(a_m)$ and the other terms as in (i). This sequence satisfies (6), because taking into account that $\Pi(a_n) \sim \Pi(a_{n+1})$,

$$\begin{aligned} |w_{m+1} - w_{m+2}| &= |1 + a_{m+1}|\,w_{m+1} \le (1 + \|A\|_\infty)\,\Pi(a_m), \\ |w_m - w_{m+1}| &= w_{m+1} = \Pi(a_m) \le c'_{A,Z}\,\Pi(a_{m-1}) \text{ if } m \ge 2, \\ |w_{m+2} - w_{m+3}| &= w_{m+2} \le \|A\|_\infty\,\Pi(a_m) \le c_{A,Z}\,\|A\|_\infty\,\Pi(a_{m+1}). \end{aligned}$$

Condition (D) is obtained proceeding exactly as in the proof of (i).

(iii) Let (w_n) be defined as in (i) and let (w'_n) be defined by $w'_{m+1} = \dfrac{\Gamma(a'_m)}{1 - |z_{m+1}|^2}$, $w'_{m+2} = -a'_{m+1}\,\dfrac{\Gamma(a'_m)}{1 - |z_{m+2}|^2}$ and $w'_n = 0$ if $n \ne m+1, m+2$. We have that w'_{m+2} also satisfies (8), because

$$|w'_{m+2}|(1 - |z_{m+2}|^2) \le \|A'\|_\infty\,\Gamma(a'_m) \le c_{A',Z}\,\|A'\|_\infty\,\Gamma(a'_{m+1}).$$

Proceeding as in (i), there exists $g_m \in H^\infty$ such that $f_m = g_m B_{m+1}^2$. Since $\rho(z_m, z_{m+1}) \le \Gamma(a_m)$ (see definition of Γ) and $|B_{m+1}(z_{m+1})| < \rho(z_m, z_{m+1})$, we have

$$\rho(z_m, z_{m+1}) \le \Gamma(a_m) = |f_m(z_{m+1})| \le c\,|B_{m+1}(z_{m+1})|^2 < c\,|B_{m+1}(z_{m+1})|\,\rho(z_m, z_{m+1}).$$

Thus, it follows that $Z \setminus \{z_1\}$ is u.s. and so is Z.

(iv) Let (w'_n) be defined as in (iii) replacing Γ by Λ. Since $\Lambda(a'_n) = O(\Lambda(a'_{n+1}))$, then w'_{m+2} verifies (11). Proceeding as in (i), there is $g_m \in H^\infty$ such that $f_m = g_m B B_{m+1}$. A simple calculation gives

$$|B'(z_{m+1})| = \frac{|B_{m+1}(z_{m+1})|}{1 - |z_{m+1}|^2} \tag{19}$$

and then,

$$\frac{\Lambda(a'_m)}{1-|z_{m+1}|^2} = |f'_m(z_{m+1})| = |(g_m B' B_{m+1})(z_{m+1})| \le c\,\frac{|B_{m+1}(z_{m+1})|^2}{1-|z_{m+1}|^2}.$$

Thus, (M) holds. □

Proof of Proposition 2.2. Let (μ_n) be as in (12) and (μ'_n) as in (13).

(i) We situate ourselves in (c) of Lemma 1.2. If Z is u.s., then Carleson's theorem provides f performing the interpolation $f(z_n) = \mu_n$. Otherwise, Z is the union of $E = (z_{2m})$ and $F = (z_{2m+1})$, both u.s. Let $g \in H^\infty$ such that $g(z_{2m}) = \mu_{2m}$. We look for a function $h \in H^\infty$ such that $h(z_{2m+1}) = \lambda_m$, where

$$\lambda_m = \frac{\mu_{2m+1} - g(z_{2m+1})}{B_E(z_{2m+1})},$$

because then $f_1 = g + B_E h$ is in H^∞ and performs the above interpolation on $Z \setminus \{z_1\}$. Since $|B_E(z_{2m+1})| > |B_{2m+1}(z_{2m+1})|$ and $|B_{2m+1}(z_{2m+1})| \ge c\,\Gamma(a_{2m})$ (condition (S)), we have $|B_E(z_{2m+1})| \ge c\,\Gamma(a_{2m})$. Then,

$$|\lambda_m| \le c\,\frac{|\mu_{2m+1} - \mu_{2m}| + |g(z_{2m}) - g(z_{2m+1})|}{\Gamma(a_{2m})}.$$

Taking into account that

$$\mu_{2m+1} = a_{2m}\,\mu_{2m} + w_{2m+1} \tag{20}$$

and using (3) and (4),

$$|\mu_{2m+1} - \mu_{2m}| \le |1 - a_{2m}|\,|\mu_{2m}| + |w_{2m+1}| \le c\,\Gamma(a_{2m}).$$

By (2)

$$|g(z_{2m}) - g(z_{2m+1})| \le c\,\rho(z_{2m}, z_{2m+1}) \le c\,\Gamma(a_{2m}).$$

Thus, $(\lambda_n) \in l^\infty$ and Carleson's theorem provides h. Putting

$$f(z) = f_1(z) + (\mu_1 - f_1(z_1))\,\frac{B_1(z)}{B_1(z_1)},$$

it follows that f is in H^∞ and performs the above interpolation on Z.

(ii) The proof is as in (i). Now we have

$$|\lambda_m| \le c\,\frac{|\mu_{2m+1} - \mu_{2m+2}| + |g(z_{2m+2}) - g(z_{2m+1})|}{\Pi(a_{2m})}.$$

By (20) and using (3) and (6),

$$|\mu_{2m+1} - \mu_{2m+2}| \le |w_{2m+1} - w_{2m+2}| + |\mu_{2m+1}|\,|a_{2m} - a_{2m+1}| + |a_{2m}|\,(|\mu_{2m+1}| + |\mu_{2m}|) \le c\,\{\Pi(a_{2m}) + |a_{2m}|\}$$

and by (17), it turns out that $|a_{2m}| \le \dfrac{|a_{2m} - a_{2m+1}|}{1-r} \le c\,\Pi(a_{2m})$. On the other hand, by (2)

$$|g(z_{2m+2}) - g(z_{2m+1})| \le c\,\rho(z_{2m+1}, z_{2m+2}) \le c\,\Pi(a_{2m}).$$

Thus, $(\lambda_n) \in l^\infty$ and the proof continues as in (i).

(iii) Since Z is u.s., then Z is double interpolating and there is f in H^∞ performing $f(z_n) = \mu_n$ and $f'(z_n) = \mu'_n$.

(iv) In case that $Z = E \cup F$, where $E = (z_{2m})$ and $F = (z_{2m+1})$ are u.s., we know that there is $g \in H^\infty$ vanishing on E and satisfying $g'(z_{2m}) = \mu'_{2m}$. We look for a function $h_1 \in H^\infty$ such that $h_1(z_{2m+1}) = \lambda_m$, where

$$\lambda_m = \frac{-g(z_{2m+1})}{B_E^2(z_{2m+1})},$$

because then $h_2 = g + B_E^2 h_1$ is in H^∞, vanishes on $Z \setminus \{z_1\}$ and $h'_2(z_{2m}) = \mu'_{2m}$. We have

$$|\lambda_m| \le c\,\frac{|g(z_{2m+1})|}{\Lambda(a'_{2m})}.$$

It follows from (13), taking into account (10) and (11), that

$$(1-|z_2|^2)|\mu_2'| \le |a_1'|(1-|z_1|^2)|w_1'| + (1-|z_2|^2)|w_2'| \le c\,\Lambda(a_1'). \tag{21}$$

For $n \ge 2$, by (13), (10), (11) and (18)

$$\begin{aligned}(1-|z_{n+1}|^2)|\mu_{n+1}'| &\le c\,\{|a_1'\,a_2'\dots a_n'|\Lambda(a_1') + \sum_{i=2}^{n}|a_i'\,a_{i+1}'\dots a_n'|\Lambda(a_{i-1}')\}\\ &+ (1-|z_{n+1}|^2)|w_{n+1}'| \le c\,\{\Lambda(a_{n-1}') + \Lambda(a_n')\} \le c\,\Lambda(a_n').\end{aligned} \tag{22}$$

Then, using (15), (21), (22) and $\Lambda(a_n') = O(\Lambda(a_{n+1}'))$,

$$\begin{aligned}|g(z_{2m+1})| &= |g(z_{2m}) - g(z_{2m+1}) - g'(z_{2m})(1-|z_{2m}|^2)\psi(z_{2m},z_{2m+1})| + (1-|z_{2m}|^2)|\mu_{2m}'|\rho(z_{2m},z_{2m+1})\\ &\le c\,\{\rho(z_{2m},z_{2m+1})^2 + \Lambda(a_{2m}')\rho(z_{2m},z_{2m+1})\}.\end{aligned}$$

Thus,

$$|\lambda_m| \le c\left(\frac{\rho(z_{2m},z_{2m+1})^2}{\Lambda(a_{2m}')} + \rho(z_{2m},z_{2m+1})\right) \le c$$

and Carleson's theorem provides h_1 performing the above interpolation. We replace h_2 by h_{2c}, where h_{2c} is given by

$$h_{2c}(z) = h_2(z) - h_2(z_1)\,\frac{B_1^2(z)}{B_1^2(z_1)},$$

because then h_{2c} also vanishes on z_1. Finally, we look for $h_3 \in H^\infty$ such that $h_3(z_{2m+1}) = \lambda_m'$, where

$$\lambda_m' = \frac{\mu_{2m+1}' - h_{2c}'(z_{2m+1})}{(B'B_E)(z_{2m+1})},$$

because then $f = h_{2c} + BB_E h_3$ is in H^∞ and performs the desired interpolation on $Z \setminus \{z_1\}$. Using (19),

$$|\lambda_m'| \le c\,\frac{\{|\mu_{2m+1}' + \mu_{2m}'| + |h_{2c}'(z_{2m}) + h_{2c}'(z_{2m+1})|\}\,(1-|z_{2m+1}|^2)}{\Lambda(a_{2m}')}.$$

An easy computation shows that

$$\frac{|z-w|}{1-|z|^2} < 4\,\rho(z,w), \tag{23}$$

if $\rho(z,w) < \min(|z|, \frac{1}{2})$; then, using (21), (22), (23) and $\Lambda(a_n') = O(\Lambda(a_{n+1}'))$,

$$|\mu_{2m}'|\,||z_{2m}|^2 - |z_{2m+1}|^2| \le c\,\Lambda(a_{2m-1}')\,\frac{|z_{2m}-z_{2m+1}|}{1-|z_{2m}|^2} \le c\,\Lambda(a_{2m}')\rho(z_{2m},z_{2m+1}) \tag{24}$$

and taking into account (21), (22), (24) and $\Lambda(a_n') = O(\Lambda(a_{n+1}'))$,

$$\begin{aligned}|\mu_{2m+1}' + \mu_{2m}'|(1-|z_{2m+1}|^2) &\le |\mu_{2m+1}'|(1-|z_{2m+1}|^2) + |\mu_{2m}'|(1-|z_{2m}|^2)\\ &+|\mu_{2m}'|\,||z_{2m}|^2 - |z_{2m+1}|^2| \le c\,\{\Lambda(a_{2m}') + \Lambda(a_{2m}')\rho(z_{2m},z_{2m+1})\}.\end{aligned}$$

On the other hand, by (16) and (24)

$$\begin{aligned}&|h_{2c}'(z_{2m}) + h_{2c}'(z_{2m+1})|\,(1-|z_{2m+1}|^2)\\ &\le |h_{2c}'(z_{2m})(1-|z_{2m}|^2) + h_{2c}'(z_{2m+1})(1-|z_{2m+1}|^2)|\\ &+|\mu_{2m}'|\,||z_{2m}|^2 - |z_{2m+1}|^2| \le c\,\{\rho(z_{2m},z_{2m+1})^2 + \Lambda(a_{2m}')\rho(z_{2m},z_{2m+1})\}.\end{aligned}$$

Then,

$$|\lambda_m'| \le c\left(1 + \frac{\rho(z_{2m},z_{2m+1})^2}{\Lambda(a_{2m}')} + \rho(z_{2m},z_{2m+1})\right) \le c$$

and Carleson's theorem provides h_3 performing the above interpolation. We replace f by f_c defined by

$$f_c(z) = f(z) - (f'(z_1) - \mu_1')\,\frac{(BB_1)(z)}{(B'B_1)(z_1)},$$

because then $f_c'(z_1) = \mu_1'$ and so f_c performs the desired interpolation on Z. □

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