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Biderivations of the higher rank Witt algebra without anti-symmetric condition

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Abstract: The Witt algebra $\mathfrak{W}_d$ of rank $d(\geq 1)$ is the derivation algebra of Laurent polynomial algebras in d commuting variables. In this paper, all biderivations of $\mathfrak{W}_d$ without anti-symmetric condition are determined. As an applications, commutative post-Lie algebra structures on $\mathfrak{W}_d$ are obtained. Our conclusions recover and generalize results in the related papers on low rank or anti-symmetric cases.

Keywords: Biderivation, Higher rank Witt algebra, Anti-symmetric, Post-Lie algebra

MSC: 17B05, 17B40

1 Introduction

Let $\mathbb{F}$ be a field of characteristic zero. We denote by $\mathbb{Z}$ the sets of all integers. We fix a positive integer $d \geq 1$ and denote by $\mathfrak{W}_d$ the derivation Lie algebra of the Laurent polynomial algebra $A = \mathbb{F}\left[z_1^{\pm 1}, \cdots, z_d^{\pm 1}\right]$ in d commuting variables $z_1, \cdots, z_d$ over $\mathbb{F}$. It is well known that the infinite-dimensional Lie algebra $\mathfrak{W}_d$ is called the Witt algebra of rank d. Its representations have attracted a lot of attention from many mathematicians [1–5]. According to their notations, the Witt algebra can be described as follows.

For $i \in \{1, 2, \ldots, d\}$, set $\partial_i = z_i \frac{\partial}{z_i}$. For any $n \in \mathbb{Z}^d$ (considered as row vectors, i.e., $n = (n_1, \cdots, n_d)$ where $n_i \in \mathbb{Z}$), set $z^n = z_1^{n_1} z_2^{n_2} \cdots z_d^{n_d}$. We fix the vector space $\mathbb{F}^d$ of $1 \times d$ matrices. Denote the standard basis by $e_1, e_2, \ldots, e_d$, which are the row vectors of the identity matrix I_d. Let $(\cdot, \cdot)$ be the standard symmetric bilinear form such that $(u, v) = uv^T \in \mathbb{F}$, where v^T is the matrix transpose. For $u \in \mathbb{F}^d$ and $r \in \mathbb{Z}^d$, we denote $D(u, r) = z^r \sum_{i=1}^d u_i \partial_i$. Then we have

$$[D(u, r), D(v, s)] = D(w, r + s), \ u, v \in \mathbb{F}^d, \ r, s \in \mathbb{Z}^d, \tag{1}$$

where $w = (u, s)v - (v, r)u$. Therefore, the Witt algebra of rank d is the $\mathbb{F}$-linear space

$$\mathfrak{W}_d = \mathrm{span}_{\mathbb{F}}\{D(u, r) | u \in \mathbb{F}^d, \ r \in \mathbb{Z}^d\}$$

with brackets determined by (1). Note that $D(u, r)$ is linear only with respect to the first component u. It is clear that $\mathfrak{W}_d$ has a basis as

$$\{D(e_1, r), D(e_2, r), \cdots, D(e_d, r) | r \in \mathbb{Z}^d\}.$$

Recall that

$$\mathfrak{h} = \left\{D(u, 0) | u \in \mathbb{F}^d\right\} = \mathrm{span}_{\mathbb{F}}\{D(e_1, 0), D(e_2, 0), \cdots, D(e_d, 0)\}$$

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is the Cartan subalgebra of $\mathfrak{W}_d$.

It is well-known that derivations and generalized derivations are very important subjects in the research of both algebras and their generalizations. In recent years, biderivations have interested a great number of authors, see [6–16]. In [7], Brešar et al. showed that all biderivations on commutative prime rings are inner biderivations, and determined the biderivations of semiprime rings. The notion of biderivations of Lie algebras was introduced in [15]. Since then, biderivations of Lie algebras have been studied by many authors. It may be useful and interesting for computing the biderivations of some important Lie algebras. In particular, the authors in [11] determined anti-symmetric biderivations for all $\mathfrak{W}_d$. All biderivations of $\mathfrak{W}_1$ without anti-symmetric condition were later obtained in [14]. In the present paper, we shall use the methods of [14] to determine all biderivations of $\mathfrak{W}_d$ for all $d \geq 1$.

Next, let us introduce the definition of biderivation. For an arbitrary Lie algebra L, a bilinear map $f : L \times L \to L$ is called a biderivation of L if it is a derivation with respect to both components. Namely, for each $x \in L$, both linear maps ϕ_x and ψ_x form L into itself given by $\phi_x = f(x, \cdot)$ and $\psi_x = f(\cdot, x)$ are derivations of L, i.e.,

$$\begin{aligned} f([x, y], z) &= [x, f(y, z)] + [f(x, z), y], \\ f(x, [y, z]) &= [f(x, y), z] + [y, f(x, z)] \end{aligned} \tag{2}$$

for all $x, y, z \in L$. Denote by $B(L)$ the set of all biderivations of L. For $\lambda \in \mathbb{C}$, it is easy to verify that the bilinear map $f : L \times L \to L$ given by $f(x, y) = \lambda[x, y]$ for all $y \in L$ is a biderivation of L. Such biderivation is said to be inner. Recall that f is anti-symmetric if $f(x, y) = -f(y, x)$ for all $x, y \in L$.

In this paper, we will prove that every biderivation of $\mathfrak{W}_d$ without anti-symmetric condition is inner. As an application, we characterize the commutative post-Lie algebra structures on $\mathfrak{W}_d$.

2 Biderivations of the Witt algebras

We first give some lemmas which will be useful for our proof.

Lemma 2.1 ([17]). *Every derivation of $\mathfrak{W}_d$ is inner.*

Lemma 2.2. *Suppose that $f \in B(\mathfrak{W}_d)$. Then there are linear maps ϕ and ψ from $\mathfrak{W}_d$ into itself such that*

$$f(x, y) = [\phi(x), y] = [x, \psi(y)]$$

for all $x, y \in \mathfrak{W}_d$.

Proof. Since f is a biderivation of $\mathfrak{W}_d$, then for a fixed element $x \in \mathfrak{W}_d$ the map $\phi_x : L \to L$ given by $\phi_x(y) = f(x, y)$ is a derivation of $\mathfrak{W}_d$ by (2). Therefore, from Lemma 2.1 we know that ϕ_x is an inner derivation of $\mathfrak{W}_d$. Therefore, there is a map $\phi : \mathfrak{W}_d \to \mathfrak{W}_d$ such that $\phi_x = \mathrm{ad}\phi(x)$, i.e., $f(x, y) = [\phi(x), y]$. Since f is bilinear, it is easy to verify that ϕ is linear. Similarly, if we define a map ψ_z from $\mathfrak{W}_d$ into itself given by $\psi_z(y) = f(y, z)$ for all $y \in \mathfrak{W}_d$, then one can obtain a linear map ψ from $\mathfrak{W}_d$ into itself such that $f(x, y) = \mathrm{ad}(-\psi(y))(x) = [x, \psi(y)]$. The proof is completed. □

Lemma 2.3. *Let $f \in B(\mathfrak{W}_d)$ and ϕ, ψ be determined by Lemma 2.2. For any $i, j \in \{1, \cdots, d\}$ and $r, s \in \mathbb{Z}^d$, we assume that*

$$\phi(D(e_i, r)) = \sum_{k=1}^{d} \sum_{n \in \mathbb{Z}^d} a_{k,n}^{(i,r)} D(e_k, n), \tag{3}$$

$$\psi(D(e_j, s)) = \sum_{k=1}^{d} \sum_{n \in \mathbb{Z}^d} b_{k,n}^{(j,s)} D(e_k, n) \tag{4}$$

where $a_{k,n}^{(i,r)}, b_{k,n}^{(j,s)} \in \mathbb{F}$. *Then*

$$\begin{aligned}&\sum_{k=1}^{d}\sum_{n\in\mathbb{Z}^d} a_{k,n}^{(i,r)} s_k D(e_j, n+s) - \sum_{k=1}^{d}\sum_{n\in\mathbb{Z}^d} a_{k,n}^{(i,r)} n_j D(e_k, n+s)\\ &= \sum_{k=1}^{d}\sum_{n\in\mathbb{Z}^d} b_{k,n}^{(j,s)} n_i D(e_k, n+r) - \sum_{k=1}^{d}\sum_{n\in\mathbb{Z}^d} b_{k,n}^{(j,s)} r_k D(e_i, n+r).\end{aligned} \tag{5}$$

Proof. Lemma 2.2 tells us that

$$f(D(e_i,r), D(e_j,s)) = [\phi(D(e_i,r)), D(e_j,s)] = [D(e_i,r), \psi(D(e_j,s))]$$

for all $i, j \in \{1, \cdots, d\}$ and $r, s \in \mathbb{Z}^d$. From (3) and (4), the conclusion follows by direct computations. □

Lemma 2.4. *Let* $f \in B(\mathfrak{W}_d)$ *and* ϕ, ψ *be determined by Lemma 2.2. Then the Cartan subalgebra* $\mathfrak{h}$ *is an invariant subspace of both maps* ϕ *and* ψ.

Proof. Note that $D(e_i, 0), i = 1, \cdots d$ span the Cartan subalgebra $\mathfrak{h}$, so it is enough to prove that $\phi(D(e_i,0)), \psi(D(e_i,0)) \in \mathfrak{h}$ for each $i \in \{1, 2, \cdots, d\}$. For any fixed $i \in \mathbb{Z}$, applying (3) for $r = 0$ we have that

$$\phi(D(e_i,0)) = \sum_{k=1}^{d}\sum_{n\in\mathbb{Z}^d} a_{k,n}^{(i,0)} D(e_k, n). \tag{6}$$

We will prove that $a_{k,n}^{(i,0)} = 0$ in (6) for all $n \in \mathbb{Z}^d \setminus \{0\}$, and so that $\phi(D(e_i,0)) = \sum_{k=1}^{d} a_{k,0}^{(i,0)} D(e_k, 0) \in \mathfrak{h}$. The proof of $\psi(D(e_i,0)) \in \mathfrak{h}$ is similar.

Now for an arbitrary $s \in \mathbb{Z}^d \setminus \{0\}$, we assume that $s_j \neq 0$ for some $j \in \{1, 2, \cdots, d\}$. It follows by letting $r = 0$ in (5) that

$$\sum_{k=1}^{d}\sum_{n\in\mathbb{Z}^d} a_{k,n}^{(i,0)} s_k D(e_j, n+s) - \sum_{k=1}^{d}\sum_{n\in\mathbb{Z}^d} a_{k,n}^{(i,0)} n_j D(e_k, n+s) = \sum_{k=1}^{d}\sum_{n\in\mathbb{Z}^d} b_{k,n}^{(j,s)} n_i D(e_k, n). \tag{7}$$

It is clear that the right-hand side of (7) does not contain any non-zero elements in $\mathfrak{h}$, thereby the left-hand side is so. From this, one has that

$$\sum_{k=1}^{d} a_{k,-s}^{(i,0)} s_k D(e_j, 0) + \sum_{k=1}^{d} a_{k,-s}^{(i,0)} s_j D(e_k, 0) = 0,$$

which implies that

$$(2a_{j,-s}^{(i,0)} s_j + \sum_{\substack{k=1\\k\neq j}}^{d} a_{k,-s}^{(i,0)} s_k) D(e_j, 0) + \sum_{\substack{k=1\\k\neq j}}^{d} a_{k,-s}^{(i,0)} s_j D(e_k, 0) = 0. \tag{8}$$

Thanks to $s_j \neq 0$, we have by (8) that $a_{k,-s}^{(i,0)} = 0$ for every $k \neq j$. Once again applying (8), we see that $2a_{j,-s}^{(i,0)} s_j = 0$, i.e., $a_{j,-s}^{(i,0)} = 0$. In other words, $a_{k,-s}^{(i,0)} = 0$ for all $k = 1, \cdots, d$. Notice the arbitrariness of s, the proof is completed. □

Lemma 2.5. *Let* $f \in B(\mathfrak{W}_d)$ *and* ϕ, ψ *be determined by Lemma 2.2. Then we have*

$$\phi(D(e_i,r)) \equiv a_{i,r}^{(i,r)} D(e_i, r) \mod \mathfrak{h}, \tag{9}$$

$$\psi(D(e_j,s)) \equiv b_{j,s}^{(j,s)} D(e_j, s) \mod \mathfrak{h} \tag{10}$$

for all $i, j \in \{1, \cdots, d\}$ *and* $r, s \in \mathbb{Z}^d \setminus \{0\}$.

Proof. We will only prove (10), the proof for (9) is similar. Continuing the use of the assumptions (3) and (4), we also have that (7) holds. This, together with Lemma 2.4 meaning $a_{k,n}^{(i,0)} = 0$ for all $n \in \mathbb{Z}^d \setminus \{0\}$, yields that

$$(\sum_{k=1}^{d} a_{k,0}^{(i,0)} s_k) D(e_j, s) = \sum_{k=1}^{d} \sum_{\substack{n\in\mathbb{Z}^d \\ n\neq 0}} b_{k,n}^{(j,s)} n_i D(e_k, n). \tag{11}$$

Therefore, from (11) we see that $b_{k,n}^{(j,s)} n_i = 0$ for all $i = 1, \cdots d$ and $(k, n) \neq (j, s)$ with $n \neq 0$. This implies that $b_{k,n}^{(j,s)} = 0$ for all $(k, n) \neq (j, s)$ since $n \neq 0$. It has been obtained that

$$\psi(D(e_j, s)) = \sum_{k=1}^{d} b_{k,0}^{(j,s)} D(e_k, 0) + b_{j,s}^{(j,s)} D(e_j, s),$$

which proves (10). □

Lemma 2.6. *Let $f \in B(\mathfrak{W}_d)$ and ϕ, ψ be determined by Lemma 2.2. Then there is $\lambda \in \mathbb{F}$ such that*

$$\phi(D(e_i, r)) = \lambda D(e_i, r), \quad \psi(D(e_j, s)) = \lambda D(e_j, s)$$

for all $i, j \in \{1, \cdots, d\}$ and $r, s \in \mathbb{Z}^d \setminus \{0\}$.

Proof. We use the assumptions (3) and (4). With Lemmas 2.4 and 2.5, Equation (5) becomes

$$\begin{aligned}
&(\sum_{k=1}^{d} a_{k,0}^{(i,r)} s_k) D(e_j, s) + a_{i,r}^{(i,r)} s_i D(e_j, r+s) - a_{i,r}^{(i,r)} r_j D(e_i, r+s) \\
&= -(\sum_{k=1}^{d} b_{k,0}^{(j,s)} r_k) D(e_i, r) + b_{j,s}^{(j,s)} s_i D(e_j, r+s) - b_{j,s}^{(j,s)} r_j D(e_i, r+s).
\end{aligned}$$

It follows that

$$\sum_{k=1}^{d} a_{k,0}^{(i,r)} s_k = \sum_{k=1}^{d} b_{k,0}^{(j,s)} r_k = 0, \ \forall s, r \in \mathbb{Z}^d \setminus \{0\}, \text{ with } s \neq r, \tag{12}$$

$$a_{i,r}^{(i,r)} s_i = b_{j,s}^{(j,s)} s_i, \quad a_{i,r}^{(i,r)} r_j = b_{j,s}^{(j,s)} r_j, \ \forall s, r \in \mathbb{Z}^d \setminus \{0\}, i \neq j. \tag{13}$$

Although $r \neq 0$, but we still can find a subset $\{\widetilde{s}^{(1)}, \cdots, \widetilde{s}^{(d)}\}$ of $\mathbb{Z}^d$ such that $\widetilde{s}^{(1)}, \cdots, \widetilde{s}^{(d)}$ are $\mathbb{F}$-linearly independent with $\widetilde{s}^{(t)} \neq r$, $t = 1, \cdots, d$. Let s run over the vectors $\widetilde{s}^{(1)}, \cdots, \widetilde{s}^{(d)}$ in (12), then we see that

$$\begin{pmatrix} \widetilde{s}^{(1)} \\ \vdots \\ \widetilde{s}^{(d)} \end{pmatrix} \begin{pmatrix} a_{1,0}^{(i,r)} \\ \vdots \\ a_{d,0}^{(i,r)} \end{pmatrix} = 0,$$

which implies that

$$a_{1,0}^{(i,r)} = \cdots = a_{d,0}^{(i,r)} = 0.$$

Similarly, we have

$$b_{1,0}^{(j,s)} = \cdots = b_{d,0}^{(j,s)} = 0.$$

Next, by taking $s = (1, \cdots, 1) \doteq e$ in (13), we have $a_{i,r}^{(i,r)} = b_{j,e}^{(j,e)}$ for all $i \neq j$. This tells us that $a_{i,r}^{(i,r)} = b_{1,e}^{(1,e)}$ for any $i \neq 1$ and $a_{i,r}^{(i,r)} = b_{2,e}^{(2,e)}$ for any $i \neq 2$. It follows that $a_{i,r}^{(i,r)}$ is a constant denoted by λ for all $i = 1, \cdots, d$ and $r \in \mathbb{Z}^d \setminus \{0\}$. Similarly, we obtain that $b_{j,s}^{(j,s)}$ is a constant denoted by μ for all $j = 1, \cdots, d$ and $s \in \mathbb{Z}^d \setminus \{0\}$. Finally, by $a_{2,e}^{(2,e)} = b_{1,e}^{(1,e)}$ we have $\lambda = \mu$, which completes the proof. □

Lemma 2.7. *Let $f \in B(\mathfrak{W}_d)$, and ϕ, ψ be determined by Lemma 2.2, $\lambda \in \mathbb{F}$ be given by Lemma 2.7. Then*

$$\phi(D(e_i, 0)) = \lambda D(e_i, 0), \quad \psi(D(e_j, 0)) = \lambda D(e_j, 0)$$

for all $i, j \in \{1, \cdots, d\}$.

Proof. We use the assumptions (3) and (4). By Lemma 2.4, we have

$$\phi(D(e_i,0)) = \sum_{k=1}^{d} a_{k,0}^{(i,0)} D(e_k,0),$$
$$\psi(D(e_j,0)) = \sum_{k=1}^{d} b_{k,0}^{(j,0)} D(e_k,0)$$

for all $i, j \in \mathbb{Z}$. Namely, it follows that, in (3) and (4), $a_{k,n}^{(i,0)} = b_{k,n}^{(j,0)} = 0$ for all $n \in \mathbb{Z}^d \setminus \{0\}$. Note that Lemma 2.6 tells us that, in (3) and (4), $a_{k,n}^{(i,r)} = \delta_{i,k}\delta_{n,r}\lambda$ and $b_{k,n}^{(j,s)} = \delta_{j,k}\delta_{n,s}\lambda$ for any $i, j \in \mathbb{Z}$ and $r, s \in \mathbb{Z}^d \setminus \{0\}$. All these together with letting $r = 0$ in (5), deduce that

$$\sum_{k=1}^{d} a_{k,0}^{(i,0)} s_k D(e_j, s) = \lambda s_i D(e_j, s).$$

Then we have

$$s_1 a_{1,0}^{(i,0)} + \cdots + s_i(a_{i,0}^{(i,0)} - \lambda) + \cdots + s_d a_{d,0}^{(i,0)} = 0$$

for all $s \in \mathbb{Z}^d \setminus \{0\}$. Let s run over the vectors $e_1, e_2, \cdots, e_d$, we have $a_{i,0}^{(i,0)} = \lambda$ and $a_{k,0}^{(i,0)} = 0$ for every $k \neq i$. This proves that $\phi(D(e_i,0)) = \lambda D(e_i,0)$. Similarly, we can obtain that $\psi(D(e_j,0)) = \lambda D(e_j,0)$. The proof is completed. □

Our main result is the following.

Theorem 2.8. *Every biderivation of $\mathfrak{W}_d$ without anti-symmetric condition is inner.*

Proof. Suppose that f is a biderivation of $\mathfrak{W}_d$. Let ϕ be determined by Lemma 2.2, $\lambda \in \mathbb{F}$ be given by Lemma 2.7. Note that $\mathfrak{W}_d$ is spanned by $D(u,0)$, $D(u,r)$ for all $u \in \mathbb{F}^d$ and $r \in \mathbb{Z}^d \setminus \{0\}$. Then by Lemmas 2.6 and 2.7, we see that $\phi(x) = \lambda x$ for all $x \in \mathfrak{W}_d$. Now, it follows by Lemma 2.2 that

$$f(x,y) = [\phi(x), y] = [\lambda x, y] = \lambda[x,y]$$

for all $x, y \in \mathfrak{W}_d$, as desired. □

3 An application

The anti-symmetric biderivation can be applied to linear commuting maps, commuting automorphisms and derivations, see [8]. Another application of biderivation without the anti-symmetric condition is the characterization of post-Lie algebra structures. Post-Lie algebras have been introduced by Valette in connection with the homology of partition posets and the study of Koszul operads [18]. As [19] point out, post-Lie algebras are natural common generalization of pre-Lie algebras and LR-algebras in the geometric context of nil-affine actions of Lie groups. Recently, many authors have studied some post-Lie algebras and post-Lie algebra structures [19–23]. In particular, the authors of [19] study the commutative post-Lie algebra structure on Lie algebra. Let us recall the following definition of a commutative post-Lie algebra.

Definition 3.1. *Let $(L, [,])$ be a Lie algebra over $\mathbb{F}$. A commutative post-Lie algebra structure on L is a $\mathbb{F}$-bilinear product $x \circ y$ on L and satisfies the following identities:*

$$x \circ y = y \circ x,$$
$$[x,y] \circ z = x \circ (y \circ z) - y \circ (x \circ z),$$
$$x \circ [y,z] = [x \circ y, z] + [y, x \circ z].$$

for all $x, y, z \in L$. It is also said that $(L, [,], \circ)$ is a commutative post-Lie algebra.

Lemma 3.2 ([14]). *Let $(L, [,], \circ)$ be a commutative post-Lie algebra. If we define a bilinear map $f : L \times L \to L$ given by $f(x, y) = x \circ y$ for all $x, y \in L$, then f is a biderivation of L.*

Theorem 3.3. *Any commutative post-Lie algebra structure on the generalized Witt algebra $\mathfrak{W}_d$ is trivial. Namely, $x \circ y = 0$ for all $x, y \in \mathfrak{W}_d$.*

Proof. Suppose that $(\mathfrak{W}_d, [,], \circ)$ is a commutative post-Lie algebra. By Lemma 3.2 and Theorem 2.8, we know that there is $\lambda \in \mathbb{F}$ such that $x \circ y = \lambda[x, y]$ for all $x, y \in \mathfrak{W}_d$. Since the post-Lie algebra is commutative, so we have $\lambda[x, y] = \lambda[y, x]$. It implies that $\lambda = 0$. The proof is completed. □

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