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On the freeness of hypersurface arrangements consisting of hyperplanes and spheres

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Abstract: Let V be a smooth variety. A hypersurface arrangement $\mathcal{M}$ in V is a union of smooth hypersurfaces, which locally looks like a union of hyperplanes. We say $\mathcal{M}$ is free if all these local models can be chosen to be free hyperplane arrangements. In this paper, we use Saito's criterion to study the freeness of hypersurface arrangements consisting of hyperplanes and spheres, and construct the bases for the derivation modules explicitly.

Keywords: Hypersurface arrangement, Freeness, Hyperplane, Sphere

MSC: 52C35, 32S22

1 Introduction

A hypersurface arrangement $\mathcal{M}$ in a smooth variety V is a reduced divisor D consisting of a union of smooth hypersurfaces, such that at each point D is locally analytically isomorphic to a hyperplane arrangement. For hypersurface arrangements, many researchers made focus on the study of Milnor fibers, higher homotopy groups and Alexander invariants of the hypersurface complements, such as [1–4]. Besides the topological properties, the freeness of a hypersurface arrangement could also be considered. We say a hypersurface arrangement is free if D is itself a free divisor on V. The study of free hyperplane arrangements was initiated by H. Terao in [5], and has been playing the central role in this area. Recently, there have been several studies to determine when a hyperplane arrangement is free, e.g., [6–9] and so on. However, it is still very difficult to determine the freeness. Freeness of hyperplane arrangements implies several interesting geometric and combinatorial properties of the arrangements, for example see [6, 10, 11]. Therefore, there were many works on the freeness of hyperplane arrangements, especially on Coxeter arrangements and the cones over Catalan and Shi arrangements[12–16].

In [17], H. Schenck and S. Tohăneanu studied the freeness of Conic-Line arrangements in P_2 and their results are the first to give an inductive criterion for freeness of nonlinear arrangements. Until now, the papers about the freeness of hypersurface arrangements are few. In this paper, we will consider the freeness of hypersurface arrangements consisting of hyperplanes and spheres, and will construct bases for the derivation modules of hypersurface arrangements.

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The paper is organized as follows: in Section 2, we recall the basic definitions and generalize Saito's criterion to hypersurface arrangements consisting of hyperplanes and spheres. In Section 3, for the hypersurface arrangement consisting of n spheres, the hypersurface arrangement containing a free hyperplane arrangement and n spheres, we present the constructions of bases for the derivation modules respectively.

2 Preliminaries and Notations

We begin with some basic concepts and notations of arrangements, for more information see P. Orlik and H. Terao [18].

Let V be an ℓ-dimensional vector space on $\mathbb{K}$ with a coordinate system $\{x_1, \dots, x_\ell\} \subset V^*$. Let $S = S(V^*)$ be the symmetric algebra of V^* and $\mathrm{Der}_{\mathbb{K}}(S)$ be the module of derivations

$$\mathrm{Der}_{\mathbb{K}}(S) = \{\theta : S \to S |\ \theta(fg) = f\theta(g) + g\theta(f),\ f, g \in S\}.$$

Define $D_i = \partial/\partial x_i$, $1 \le i \le \ell$, then $D_1, \dots, D_\ell$ is a basis for $\mathrm{Der}_{\mathbb{K}}(S)$ over S.

Definition 2.1. *A nonzero element $\theta \in \mathrm{Der}_{\mathbb{K}}(S)$ is of polynomial degree p if $\theta = \sum_{k=1}^{\ell} f_k D_k$ and the maximum of the degrees of coefficient polynomials $f_1, \dots, f_\ell$ (get rid of 0) is p. In this case we write* $\mathrm{pdeg}\theta = p$.

Definition 2.2. *For a hypersurface arrangement $\mathcal{M}$ in V, the derivation module $D(\mathcal{M})$ is defined by*

$$D(\mathcal{M}) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) |\ \theta(\alpha_X) \in \alpha_X S \textit{ for all } X \in \mathcal{M}\},$$

where $X = \ker(\alpha_X)$, $\mathcal{M}$ is called free if $D(\mathcal{M})$ is free.

Definition 2.3. *Let $\mathcal{M}$ be a free hypersurface arrangement and let $\{\theta_1, \dots, \theta_\ell\}$ be a basis for $D(\mathcal{M})$. We call* $\mathrm{pdeg}\theta_1, \dots, \mathrm{pdeg}\theta_\ell$ *the exponents of $\mathcal{M}$ and write*

$$\exp\mathcal{M} = \{\mathrm{pdeg}\theta_1, \dots, \mathrm{pdeg}\theta_\ell\}.$$

Definition 2.4. *Given derivations $\theta_1, \dots, \theta_\ell \in D(\mathcal{M})$, define the coefficient matrix $\mathsf{M}(\theta_1, \dots, \theta_\ell)$ by $\mathsf{M}_{i,j} = \theta_j(x_i)$, thus*

$$\mathsf{M}(\theta_1, \dots, \theta_\ell) = \begin{pmatrix} \theta_1(x_1) & \dots & \theta_\ell(x_1) \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ \theta_1(x_\ell) & \dots & \theta_\ell(x_\ell) \end{pmatrix},$$

and $\theta_j = \sum_{i=1}^{\ell} \mathsf{M}_{i,j} D_i$.

Definition 2.5. *Let $\mathcal{M}$ be a hypersurface arrangement, the product*

$$Q(\mathcal{M}) = \prod_{X \in \mathcal{M}} \alpha_X$$

is called a defining polynomial of $\mathcal{M}$, where $X = \ker(\alpha_X)$.

For hyperplane arrangements, Saito's criterion provides a wonderful method to prove the freeness. Next, we will prove it also holds for $\mathcal{M}$, where $\mathcal{M}$ is a hypersurface arrangement in $\mathbb{R}^\ell$ consisting of linear hyperplanes and spheres.

Lemma 2.6. *If $\theta_1, \dots, \theta_\ell \in D(\mathcal{M})$, then* $\det \mathsf{M}(\theta_1, \dots, \theta_\ell) \in Q(\mathcal{M})S$.

Proof. Let $X \in \mathcal{M}$, and let $X = \ker(\alpha_X)$, then

$$\det \mathsf{M}(\theta_1, \dots, \theta_\ell) = f \det \begin{pmatrix} \theta_1(x_1) & \dots & \theta_\ell(x_1) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \theta_1(\alpha_X) & \dots & \theta_\ell(\alpha_X) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \theta_1(x_\ell) & \dots & \theta_\ell(x_\ell) \end{pmatrix},$$

If $\alpha_X = \sum_{k=1}^{\ell} c_k x_k$, then $f = c_k \in \mathbb{R}$; If $\alpha_X = \sum_{k=1}^{\ell}(x_k - a_k)^2 - r$, then $f = 2(x_k - a_k)$. For any case, $\det \mathsf{M}(\theta_1, \dots, \theta_\ell)$ is divisible by α_X. Since X is arbitrary, $\det \mathsf{M}(\theta_1, \dots, \theta_\ell) \in Q(\mathcal{M})S$. □

Lemma 2.7. *Let M_n be an $n \times n$ matrix with the (p, q) entry as follows:*

$$M_{pq} = \begin{cases} x_p x_q & \text{if } p \neq q, \\ x_p^2 - r & \text{if } p = q, \end{cases}$$

where $1 \le p, q \le n$, $r \in \mathbb{R}$. Therefore,

$$\det M_n = (-r)^{n-1}\Big(\sum_{k=1}^{n} x_k^2 - r\Big).$$

Proof. We will prove the lemma by induction on n.

(1) For the case $n = 1$, $M_1 = x_1^2 - r$, then $\det M_1 = x_1^2 - r$.

(2) We assume that for the case n the result holds, that is $\det M_n = (-r)^{n-1}(\sum_{k=1}^{n} x_k^2 - r)$.

For the case $n + 1$ we have

$$M_{n+1} = \begin{pmatrix} M_n & N \\ N^T & x_{n+1}^2 - r \end{pmatrix},$$

where $N = (x_1 x_{n+1}, \dots, x_n x_{n+1})^T$. Therefore,

$$\begin{aligned} \det M_{n+1} &= \det \begin{pmatrix} M_n & N \\ N^T & x_{n+1}^2 \end{pmatrix} + \det \begin{pmatrix} M_n & O_{n\times 1} \\ N^T & -r \end{pmatrix} \\ &= \det \begin{pmatrix} -rE_n & N \\ O_{1\times n} & x_{n+1}^2 \end{pmatrix} + (-r) \det M_n \\ &= (-r)^n (x_{n+1}^2) + (-r)(-r)^{n-1}\Big(\sum_{i=1}^{n} x_i^2 - r\Big) \\ &= (-r)^n\Big(\sum_{k=1}^{n+1} x_k^2 - r\Big), \end{aligned}$$

where $O_{n\times 1}$, $O_{1\times n}$ are the $n \times 1$ and $1 \times n$ null matrices respectively, and E_n is the $n \times n$ identity matrix. □

Lemma 2.8. *Let*

$$\mathcal{S} = \{(x_1, \dots, x_\ell) \mid \sum_{k=1}^{\ell}(x_k - a_k)^2 = r \in \mathbb{R}^+\}$$

be the $(\ell - 1)$-dimensional sphere in $\mathbb{R}^\ell$ with center $(a_1, a_2, \dots, a_\ell)$ and radius $\sqrt{r}$, define

$$\theta_q = \sum_{p=1}^{\ell} f_{pq} D_p,\ 1 \le q \le \ell,$$

where

$$f_{pq} = \begin{cases} (x_p - a_p)(x_q - a_q) & \text{if } p \neq q, \\ (x_p - a_p)^2 - r & \text{if } p = q. \end{cases}$$

Then $\theta_1, \ldots, \theta_\ell \in D(\{\mathcal{S}\})$ *and* $\det \mathsf{M}(\theta_1, \ldots, \theta_\ell) \doteq Q(\{\mathcal{S}\})$.

Proof. We can see

$$\theta_q = \sum_{p=1}^{\ell} f_{pq} D_p = (x_q - a_q) \sum_{p=1}^{\ell} (x_p - a_p) D_p - r D_q,$$

and

$$\theta_q \Big[\sum_{k=1}^{\ell} (x_k - a_k)^2 - r \Big] = 2(x_q - a_q) \Big[\sum_{k=1}^{\ell} (x_k - a_k)^2 - r \Big] \in \Big[\sum_{k=1}^{\ell} (x_k - a_k)^2 - r \Big] S,$$

Thus $\theta_q \in D(\{\mathcal{S}\})$ for $1 \leq q \leq \ell$.

By Lemma 2.7, we obtain

$$\det \mathsf{M}(\theta_1, \ldots, \theta_\ell) = (-r)^{\ell-1} \Big[\sum_{k=1}^{\ell} (x_k - a_k)^2 - r \Big] \doteq Q(\{\mathcal{S}\}).$$ □

Next, we will show Saito's criterion for hypersurface arrangements.

Theorem 2.9. *Given* $\theta_1, \ldots, \theta_\ell \in D(\mathcal{M})$, *the following two conditions are equivalent:*

(1) $\det \mathsf{M}(\theta_1, \ldots, \theta_\ell) \doteq Q(\mathcal{M})$.

(2) $\theta_1, \ldots, \theta_\ell$ *form a basis for* $D(\mathcal{M})$ *over* S.

Proof. (1)⇒(2) The proof is exactly the same with that of Saito's criterion in [18] .

(2)⇒(1) By Lemma 2.6, we can write $\det \mathsf{M}(\theta_1, \ldots, \theta_\ell) = fQ(\mathcal{M})$ for some $f \in S$. Fix $X \in \mathcal{M}$, if X is a hyperplane, then $\{X\}$ is a free hyperplane arrangement; if X is a sphere, by Lemma 2.8 and (1)⇒(2), $\{X\}$ is a free hypersurface arrangement. Assume $\eta_1, \ldots, \eta_\ell$ is the basis of X, then $Q_X\eta_1, \ldots, Q_X\eta_\ell \in D(\mathcal{M})$, where $Q_X = Q(\mathcal{M})/\alpha_X$. Since each $Q_X\eta_i$ is an S-linear combination of $\eta_1, \ldots, \eta_\ell$, then there exists an $\ell \times \ell$ matrix N with entries in S, such that

$$\mathsf{M}(Q_X\eta_1, \ldots, Q_X\eta_\ell) = \mathsf{M}(\theta_1, \ldots, \theta_\ell)\mathsf{N}.$$

Thus we have

$$Q(\mathcal{M})Q_X^{\ell-1} \doteq \det \mathsf{M}(Q_X\eta_1, \ldots, Q_X\eta_\ell) \in \det \mathsf{M}(\theta_1, \ldots, \theta_\ell)S = fQ(\mathcal{M})S.$$

Therefore f divides $Q_X^{\ell-1}$ for all $X \in \mathcal{M}$. Since the polynomials $\{Q_X^{\ell-1}\}_{X \in \mathcal{M}}$ have no common factor, $f \in \mathbb{R}^*$. □

Corollary 2.10. *If* $\mathcal{S}$ *is an* $(\ell - 1)$*-dimensional sphere in* $\mathbb{R}^\ell$, *then* $\{\mathcal{S}\}$ *is a free hypersurface arrangement with*

$$\exp\{\mathcal{S}\} = \{2, \ldots, 2\},$$

where 2 *appears* ℓ *times.*

Proof. The result is obtained directly from Lemma 2.8 and Theorem 2.9. □

3 Main results

In this section, we will consider the freeness for hypersurface arrangements containing hyperplanes and spheres, and give the explicit bases for the derivation modules of the free ones. First, we show that the hypersurface arrangement having n spheres is free.

Theorem 3.1. *Let $\mathcal{M}_n$ be the hypersurface arrangement consisting of n spheres $\mathcal{S}_1, \dots, \mathcal{S}_n$, where*

$$\mathcal{S}_i = \{(x_1, \dots, x_\ell) \mid \sum_{k=1}^{\ell}(x_k - a_k)^2 = r_i,\ (a_1, a_2, \dots, a_\ell) \in \mathbb{R}^\ell,\ r_i \in \mathbb{R}^+\}.$$

Define derivations $\varphi_1^n, \dots, \varphi_\ell^n$ by

$$\mathsf{M}(\varphi_1^n, \dots, \varphi_\ell^n) = A_n A_{n-1} \cdots A_1,$$

where A_i is an $\ell \times \ell$ matrix and the (p, q) entry of A_i is

$$(A_i)_{pq} = \begin{cases} (x_p - a_p)(x_q - a_q) & \text{if } p \neq q, \\ (x_p - a_p)^2 - r_i & \text{if } p = q. \end{cases}$$

Then $\varphi_1^n, \dots, \varphi_\ell^n$ form a basis for $D(\mathcal{M}_n)$ and $\exp \mathcal{M}_n = \{2n, \dots, 2n\}$, where $2n$ appears ℓ times.

Proof. We will prove this result by Theorem 2.9: Saito's criterion. By Lemma 2.7, we obtain

$$\det A_i = (-r_i)^{\ell-1}\Big[\sum_{k=1}^{\ell}(x_k - a_k)^2 - r_i\Big].$$

Therefore,

$$\begin{aligned} \det \mathsf{M}(\varphi_1^n, \dots, \varphi_\ell^n) &= \prod_{i=1}^{n} \det A_i \\ &\doteq \prod_{i=1}^{n}\Big[\sum_{k=1}^{\ell}(x_k - a_k)^2 - r_i\Big] \\ &= Q(\mathcal{M}_n). \end{aligned}$$

Next, we will prove $\varphi_i^n \in D(\mathcal{M}_n)$ and $\deg \varphi_i^n = 2n$ for any $1 \le i \le \ell$ by induction on n.

The case $n = 1$ is clear according to Lemma 2.8 and Corollary 2.10. For the case $n + 1$, we notice that

$$\begin{aligned} \varphi_i^{n+1} &= \sum_{p=1}^{\ell} \varphi_i^{n+1}(x_p) D_p \\ &= \sum_{p=1}^{\ell}\Big[\sum_{q=1}^{\ell}(A_{n+1})_{pq}\varphi_i^n(x_q)\Big]D_p \\ &= \sum_{p=1}^{\ell}\Big[\sum_{q \neq p}(x_p - a_p)(x_q - a_q)\varphi_i^n(x_q) + [(x_p - a_p)^2 - r_{n+1}]\varphi_i^n(x_p)\Big]D_p \\ &= \sum_{p=1}^{\ell}\Big[(x_p - a_p)\sum_{q=1}^{\ell}(x_q - a_q)\varphi_i^n(x_q)\Big]D_p - r_{n+1}\sum_{p=1}^{\ell}\varphi_i^n(x_p)D_p \\ &= \sum_{q=1}^{\ell}(x_q - a_q)\varphi_i^n(x_q)\sum_{p=1}^{\ell}(x_p - a_p)D_p - r_{n+1}\varphi_i^n \\ &= \frac{1}{2}\varphi_i^n\Big[\sum_{q=1}^{\ell}(x_q - a_q)^2\Big]\sum_{p=1}^{\ell}(x_p - a_p)D_p - r_{n+1}\varphi_i^n. \end{aligned}$$

Therefore, for $1 \le i \le \ell$ and $1 \le j \le n$,

$$\varphi_i^{n+1}\Big[\sum_{k=1}^{\ell}(x_k - a_k)^2 - r_j\Big]$$

$$= \varphi_i^n \Big[\sum_{q=1}^{\ell}(x_q - a_q)^2\Big] \sum_{p=1}^{\ell}(x_p - a_p)^2 - r_{n+1}\varphi_i^n \Big[\sum_{k=1}^{\ell}(x_k - a_k)^2 - r_j\Big]$$

$$= \varphi_i^n \Big[\sum_{q=1}^{\ell}(x_q - a_q)^2\Big] \Big[\sum_{p=1}^{\ell}(x_p - a_p)^2 - r_{n+1}\Big],$$

by induction hypothesis,

$$\varphi_i^n \Big[\sum_{q=1}^{\ell}(x_q - a_q)^2\Big] \in \prod_{j=1}^{n}\Big[\sum_{k=1}^{\ell}(x_k - a_k)^2 - r_j\Big] S,$$

hence,

$$\varphi_i^{n+1} \Big[\sum_{k=1}^{\ell}(x_k - a_k)^2 - r_j\Big] \in \prod_{j=1}^{n+1}\Big[\sum_{k=1}^{\ell}(x_k - a_k)^2 - r_j\Big] S.$$

This means $\varphi_i^{n+1} \in D(\mathcal{M}_{n+1})$ for any $1 \le i \le \ell$, in addition,

$$\operatorname{pdeg}\varphi_i^{n+1} = \operatorname{pdeg}\varphi_i^n + 2 = 2n + 2 = 2(n + 1).$$

We complete the induction, so by Saito's criterion $\varphi_1^n, \dots, \varphi_\ell^n$ form a basis for $\mathcal{M}_n$, and $\exp \mathcal{M}_n = \{2n, \dots, 2n\}$. □

Next, we will study the freeness for the hypersurface arrangements consisting of hyperplanes and spheres, where all the spheres are centered at origin.

Theorem 3.2. *Assume $\mathcal{A}$ is a free hyperplane arrangement with a homogeneous basis $\theta_1, \dots, \theta_\ell$, and $\exp \mathcal{A} = \{d_1, \dots, d_\ell\}$, $\mathcal{S}_i^0$ is the sphere centered at origin:*

$$\mathcal{S}_i^0 = \{(x_1, \dots, x_\ell) \mid \sum_{k=1}^{\ell} x_k^2 = r_i \in \mathbb{R}^+\}, 1 \le i \le n,$$

and

$$\mathcal{M}_n = \mathcal{A} \cup \{\mathcal{S}_1^0, \dots, \mathcal{S}_n^0\},$$

Define derivations $\varphi_1^n, \dots, \varphi_\ell^n$ by

$$\mathsf{M}(\varphi_1^n, \dots, \varphi_\ell^n) = (A_n A_{n-1} \cdots A_1)\mathsf{M}(\theta_1, \dots, \theta_\ell),$$

where A_i is an $\ell \times \ell$ matrix and the (p, q) entry of A_i is

$$(A_i)_{pq} = \begin{cases} x_p x_q & \text{if } p \ne q, \\ x_p^2 - r_i & \text{if } p = q, \end{cases}$$

then $\varphi_1^n, \dots, \varphi_\ell^n$ form a basis for $D(\mathcal{M}_n)$ and $\exp \mathcal{M}_n = \{d_1 + 2n, \dots, d_\ell + 2n\}$.

Proof. By Lemma 2.7, we obtain

$$\det A_i = (-r_i)^{\ell-1}(\sum_{k=1}^{\ell} x_k^2 - r_i).$$

Since $\mathcal{A}$ is a free arrangement with a homogeneous basis $\theta_1, \dots, \theta_\ell$, by Saito's criterion,

$$\det \mathsf{M}(\theta_1, \dots, \theta_\ell) \doteq Q(\mathcal{A}).$$

Therefore,

$$\det \mathsf{M}(\varphi_1^n, \dots, \varphi_\ell^n) = \prod_{i=1}^{n}(\det A_i) \det \mathsf{M}(\theta_1, \dots, \theta_\ell)$$

$$\doteq \prod_{i=1}^{n}(\sum_{k=1}^{\ell} x_k^2 - r_i)Q(\mathcal{A})$$
$$= Q(\mathcal{M}_n).$$

Next, we will prove $\varphi_i^n \in D(\mathcal{M}_n)$ and $\deg \varphi_i^n = d_i + 2n$ for any $1 \leq i \leq \ell$ by induction on n.

For the case $n = 1$,

$$\begin{aligned}
\varphi_i^1 &= \sum_{p=1}^{\ell} \varphi_i^1(x_p)D_p \\
&= \sum_{p=1}^{\ell} \Big[\sum_{q=1}^{\ell}(A_1)_{pq}\theta_i(x_q)\Big]D_p \\
&= \sum_{p=1}^{\ell} \Big[\sum_{q \neq p} x_p x_q \theta_i(x_q) + (x_p^2 - r_1)\theta_i(x_p)\Big]D_p \\
&= \sum_{p=1}^{\ell} \Big[x_p \sum_{q=1}^{\ell} x_q\theta_i(x_q)\Big]D_p - r_1 \sum_{p=1}^{\ell} \theta_i(x_p)D_p \\
&= \sum_{q=1}^{\ell} x_q\theta_i(x_q) \sum_{p=1}^{\ell} x_p D_p - r_1\theta_i \\
&= \sum_{q=1}^{\ell} x_q\theta_i(x_q)\theta_E - r_1\theta_i.
\end{aligned}$$

Since $\theta_E, \theta_i \in D(\mathcal{A})$, we have $\varphi_i^1 \in D(\mathcal{A})$ for any $1 \leq i \leq \ell$. And

$$\mathrm{pdeg}\varphi_i^1 = \mathrm{pdeg}\Big[\sum_{q=1}^{\ell} x_q\theta_i(x_q)\theta_E\Big] = \mathrm{pdeg}\theta_i + 2 = d_i + 2.$$

In addition,

$$\begin{aligned}
\varphi_i^1\Big(\sum_{k=1}^{\ell} x_k^2 - r_1\Big) &= \Big[\sum_{q=1}^{\ell} x_q\theta_i(x_q)\theta_E - r_1\theta_i\Big]\Big(\sum_{k=1}^{\ell} x_k^2 - r_1\Big) \\
&= \sum_{q=1}^{\ell} x_q\theta_i(x_q)\Big(2\sum_{k=1}^{\ell} x_k^2\Big) - 2r_1 \sum_{q=1}^{\ell} x_q\theta_i(x_q) \\
&= 2\Big(\sum_{k=1}^{\ell} x_k^2 - r_1\Big) \sum_{q=1}^{\ell} x_q\theta_i(x_q) \\
&\in \Big(\sum_{k=1}^{\ell} x_k^2 - r_1\Big)S,
\end{aligned}$$

that is, $\varphi_i^1 \in D(\{\mathcal{S}_1^0\})$. Therefore, $\varphi_i^1 \in D(\mathcal{A}) \cap D(\{\mathcal{S}_1^0\}) = D(\mathcal{M}_1)$ for any $1 \leq i \leq \ell$.

For the case $n + 1$, by the similar calculation of φ_i^1, we get

$$\begin{aligned}
\varphi_i^{n+1} &= \sum_{q=1}^{\ell} x_q\varphi_i^n(x_q)\theta_E - r_{n+1}\varphi_i^n \\
&= \frac{1}{2}\varphi_i^n\Big(\sum_{q=1}^{\ell} x_q^2\Big)\theta_E - r_{n+1}\varphi_i^n.
\end{aligned}$$

By induction hypothesis,

$$\varphi_i^n \in D(\mathcal{M}_n) \subseteq D\Big(\bigcup_{i=1}^{n}\{\mathcal{S}_i^0\}\Big),$$

we obtain

$$\varphi_i^n\Big(\sum_{q=1}^{\ell} x_q^2\Big) \in \prod_{j=1}^{n}\Big(\sum_{k=1}^{\ell} x_k^2 - r_j\Big)S.$$

Therefore,

$$\varphi_i^n\Big(\sum_{q=1}^{\ell} x_q^2\Big)\theta_E \in D\Big(\bigcup_{i=1}^{n}\{\mathcal{S}_i^0\}\Big).$$

Combining $\theta_E \in D(\mathcal{A})$ with

$$D(\mathcal{M}_n) = D\Big(\bigcup_{i=1}^{n}\{\mathcal{S}_i^0\}\Big)\bigcap D(\mathcal{A}),$$

we conclude

$$\varphi_i^n\Big(\sum_{q=1}^{\ell} x_q^2\Big)\theta_E \in D(\mathcal{M}_n).$$

Hence, $\varphi_i^{n+1} \in D(\mathcal{M}_n)$ since $\varphi_i^n \in D(\mathcal{M}_n)$.

In addition, we have

$$\begin{aligned}
&\varphi_i^{n+1}\Big(\sum_{k=1}^{\ell} x_k^2 - r_{n+1}\Big)\\
&= \varphi_i^n\Big(\sum_{q=1}^{\ell} x_q^2\Big)\sum_{k=1}^{\ell} x_k^2 - r_{n+1}\varphi_i^n\Big(\sum_{k=1}^{\ell} x_k^2\Big)\\
&= \Big(\sum_{k=1}^{\ell} x_k^2 - r_{n+1}\Big)\varphi_i^n\Big(\sum_{q=1}^{\ell} x_q^2\Big)\\
&\in \Big(\sum_{k=1}^{\ell} x_k^2 - r_{n+1}\Big)S,
\end{aligned}$$

We obtain $\varphi_i^{n+1} \in D(\{\mathcal{S}_{n+1}^0\})$ for any $1 \le i \le \ell$, therefore

$$\varphi_i^{n+1} \in D(\{\mathcal{S}_{n+1}^0\}) \cap D(\mathcal{M}_n) = D(\mathcal{M}_{n+1}),\ \ 1 \le i \le \ell.$$

Moreover,

$$\text{pdeg}\varphi_i^{n+1} = \text{pdeg}\Big[\varphi_i^n(\sum_{q=1}^{\ell} x_q^2)\theta_E\Big] = \text{pdeg}\varphi_i^n + 2 = d_i + 2n + 2 = d_i + 2(n+1),\ \ 1 \le i \le \ell.$$

We complete the induction. □

Corollary 3.3. *Let $\mathcal{M}_n = \mathcal{A} \cup \{\mathcal{S}_1^0, \ldots, \mathcal{S}_n^0\}$ be the hypersurface arrangement defined in Theorem 3.2. Then $\mathcal{A}$ is free if and only if $\mathcal{M}_n$ is free.*

Proof. If $\mathcal{A}$ is free we can obtain that $\mathcal{M}_n$ is free directly from Theorem 3.2. Assume $\mathcal{M}_n$ is free, $\mathcal{A} \subseteq \mathcal{M}_n$, then $D(\mathcal{M}_n) \subseteq D(\mathcal{A})$. Let $\varphi_1, \ldots, \varphi_\ell$ be a basis for $D(\mathcal{M}_n)$, then $\varphi_i \in D(\mathcal{A})$ for $1 \le i \le \ell$. Write $\varphi_i = \sum_{k\ge0}\varphi_i^{(k)}$, where $\varphi_i^{(k)}$ is zero or homogeneous of degree $k \ge 0$. Since $Q(\mathcal{A})S$ is generated by homogeneous polynomial $Q(\mathcal{A})$, each homogeneous component $\varphi_i^{(k)}(Q(\mathcal{A}))$ of $\varphi_i(Q(\mathcal{A}))$ also lies in $Q(\mathcal{A})S$. This shows that $\varphi_i^{(k)} \in D(\mathcal{A})$ for $k \ge 0$. Since $\big[Q(\bigcup_{i=1}^n\{\mathcal{S}_i\})\big](0) \neq 0$, there exist $\varphi_1^{(d_1)}, \ldots, \varphi_\ell^{(d_\ell)}$ such that

$$\det \mathsf{M}(\varphi_1^{(d_1)}, \ldots, \varphi_\ell^{(d_\ell)}) \doteq Q(\mathcal{A}).$$

By Saito's criterion, $\varphi_1^{(d_1)}, \ldots, \varphi_\ell^{(d_\ell)}$ form a basis for $D(\mathcal{A})$. □

Remark 3.4. *In Theorem 3.1 and Theorem 3.2 the preconditions did not impose the restrictions on the size relations of* $r_1, r_2, \ldots, r_n$. *Hence, if* $r_1 = r_2 = \cdots = r_n$, *Theorem 3.1 and Theorem 3.2 also hold. In this case,* $(\mathcal{M}, \mathbf{m})$ *is a hypersurface arrangement with a multiplicity* m_i *for each hypersurface in* $\mathcal{M}$, *we call it hypersurface multiarrangement. As defined by G. Ziegler in [19], the module of derivations consists of* θ *such that* $\theta(\alpha_i) \in \alpha_i^{m_i} S$.

Example 3.5. *Let* $\mathcal{M}$ *be a hypersurface arrangement with the defining polynomial*

$$Q(\mathcal{M}) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1^2 + x_2^2 + x_3^2 - 1).$$

In this case, the hyperplane arrangement $\mathcal{A} \subseteq \mathcal{M}$ *is the Coxeter arrangement of type* A_2, *it is a free arrangement with* $\exp \mathcal{A} = \{0, 1, 2\}$. *By Theorem 3.2,* $\mathcal{M}$ *is a free hypersurface arrangement and* $D(\mathcal{M})$ *has the basis* $\varphi_1, \varphi_2, \varphi_3$ *as follows:*

$$(\varphi_1, \varphi_2, \varphi_3) = (D_1, D_2, D_3) \begin{pmatrix} x_1^2 - 1 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 - 1 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 - 1 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}.$$

That is

$$\begin{aligned} \varphi_1 &= (x_1^2 + x_1x_2 + x_1x_3 - 1)D_1 + (x_1x_2 + x_2^2 + x_2x_3 - 1)D_2 + (x_1x_3 + x_2x_3 + x_3^2 - 1)D_3, \\ \varphi_2 &= x_1(x_1^2 + x_2^2 + x_3^2 - 1)D_1 + x_2(x_1^2 + x_2^2 + x_3^2 - 1)D_2 + x_3(x_1^2 + x_2^2 + x_3^2 - 1)D_3, \\ \varphi_3 &= x_1(x_1^3 + x_2^3 + x_3^3 - x_1)D_1 + x_2(x_1^3 + x_2^3 + x_3^3 - x_2)D_2 + x_3(x_1^3 + x_2^3 + x_3^3 - x_3)D_3. \end{aligned}$$

And $\exp \mathcal{M} = \{\mathrm{pdeg}\varphi_1, \mathrm{pdeg}\varphi_2, \mathrm{pdeg}\varphi_3\} = \{2, 3, 4\}$.

Example 3.6. *Let* $\mathcal{M}$ *be a hypersurface arrangement with the defining polynomial*

$$Q(\mathcal{M}) = x_1x_2x_3(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_1^2 + x_2^2 + x_3^2 - 2).$$

In this case, the hyperplane arrangement $\mathcal{A} \subseteq \mathcal{M}$ *is the Coxeter arrangement of type* B_3, *it is a free arrangement with* $\exp \mathcal{A} = \{1, 3, 5\}$. *By Theorem 3.2,* $\mathcal{M}$ *is a free hypersurface arrangement and* $D(\mathcal{M})$ *has the basis* $\varphi_1, \varphi_2, \varphi_3$ *as follows:*

$$(\varphi_1, \varphi_2, \varphi_3) = (D_1, D_2, D_3) \begin{pmatrix} x_1^2 - 2 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 - 2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 - 2 \end{pmatrix} \begin{pmatrix} x_1^2 - 1 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 - 1 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 - 1 \end{pmatrix} \begin{pmatrix} x_1 & x_1^3 & x_1^5 \\ x_2 & x_2^3 & x_2^5 \\ x_3 & x_3^3 & x_3^5 \end{pmatrix}.$$

And $\exp \mathcal{M} = \{\mathrm{pdeg}\varphi_1, \mathrm{pdeg}\varphi_2, \mathrm{pdeg}\varphi_3\} = \{5, 7, 9\}$.

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References

[1] Dimca A., Maxim L., Multivariable Alexander invariants of hypersurface complements, Trans. Amer. Math. Soc., 2007, 359, 3505-3528

[2] Dimca A., Papadima S., Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements, Ann. of Math., 2003, 158(2), 473-507

[3] Hun J., Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, J. Amer. Math. Soc., 2012, 25(3), 907-927

[4] Liu Y., Maxim L., Characteristic varieties of hypersurface complements, Adv. Math., 2014, 306, 451-493

[5] Terao H., Arrangements of hyperplanes and their freeness I, II., J. Fac. Sci. Univ., 1980, 27, 293-320

[6] Abe T., Roots of characteristic polynomials and and intersection points of line arrangements, J. Singularities, 2014, 8, 100-117

[7] Abe T., Yoshinaga M., Free arrangements and coefficients of characteristic polynomials, Math. Z., 2013, 275(3), 911-919
[8] Yoshinaga M., Characterization of a free arrangement and conjecture of Edelman and Reiner, Invent. Math., 2004, 157(2), 449-454
[9] Yoshinaga M., On the freeness of 3-arrangements, Bull. London Math. Soc., 2005, 37(1), 126-134
[10] Abe T., Chambers of 2-affine arrangements and freeness of 3-arrangements, J. Alg. Combin., 2013, 38(1), 65-78
[11] Terao H., Generalized exponents of a free arrangement of hyperplanes and Shephard-Todd-Brieskorn formula, Invent. math., 1981, 63, 159-179
[12] Abe T., Terao H., Simple-root bases for Shi arrangements, J. Algebra, 2015, 422, 89-104
[13] Gao R., Pei D., Terao H., The Shi arrangement of the type D_ℓ, Proc. Japan Acad. Ser. A Math. Sci., 2012, 88(3), 41-45
[14] Suyama D., A Basis Construction for the Shi Arrangement of the Type B_ℓ or C_ℓ, Comm. Algebra, 2015, 552, 1435-1448
[15] Suyama D., Terao H., The Shi arrangements and the Bernoulli polynomials, Bull. London Math. Soc., 2012, 44, 563-570
[16] Terao H., Multiderivations of Coxeter arrangements, Invent. Math., 2002, 148, 659-674
[17] Schenck H., Tohăneanu S., Freeness of Conic-Line arrangements in P_2, Comment. Math. Helv., 2009, 84, 235-258
[18] Orlik P., Terao H., Arrangements of Hyperplanes, Grundlehren der Mathematischen Wissenschaften, 1992, 300, Berlin: Springer-Verlag
[19] Ziegler G., Multiarrangements of hyperplanes and their freeness, In Singularities (Iowa City, IA, 1986), Contemp. Math., Amer. Math. Soc., Providence, RI, 1989, 90, 345-359