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Generalizations of Steffensen's inequality via the extension of Montgomery identity

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Abstract: In this paper, we obtained new generalizations of Steffensen's inequality for n -convex functions by using extension of Montgomery identity via Taylor's formula. Since 1-convex functions are nondecreasing functions, new inequalities generalize Steffensen's inequality. Related Ostrowski type inequalities are also provided. Bounds for the reminders in new identities are given by using the Chebyshev and Grüss type inequalities.

Keywords: Steffensen's inequality, n -convex functions, Montgomery identity, Ostrowski-type inequality, Grüss-type inequality

MSC: 26D15, 26A51

1 Introduction

In [1], the authors obtain the following extension of Montgomery identity using Taylor's formula:

Theorem 1.1. Let $f : I \rightarrow \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Then the following identity holds

$$f(x) = \frac{1}{b-a} \int_a^b f(t) dt - \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} + \frac{1}{(n-1)!} \int_a^b T_n(x, s) f^{(n)}(s) ds \quad (1)$$

where

$$T_n(x, s) = \begin{cases} \frac{-1}{n(b-a)} (a-s)^n, & a \leq s \leq x; \\ \frac{-1}{n(b-a)} (b-s)^n, & x < s \leq b. \end{cases}$$

Remark 1.2. The last identity holds also for $n = 1$. In this special case, we assume that $\sum_{i=0}^{n-2} \dots$ is an empty sum. Thus (1) reduces to well-known **Montgomery identity** (e.g. [2])

$$f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \int_a^b T_1(x, s) f'(s) ds$$

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where the Peano kernel is

$$T_1(x, s) = \begin{cases} \frac{s-a}{b-a}, & a \leq s \leq x; \\ \frac{s-b}{b-a}, & x < s \leq b. \end{cases}$$

The aim of this paper is to obtain some new generalizations of Steffensen's inequality using above extension of Montgomery identity. The Steffensen's inequality was first given and proved by Steffensen in 1918 ([3]):

Theorem 1.3. Suppose that f is nonincreasing and g is integrable on $[a, b]$ with $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t) dt$. Then we have

$$\int_{b-\lambda}^b f(t) dt \leq \int_a^b f(t)g(t) dt \leq \int_a^{a+\lambda} f(t) dt. \quad (2)$$

The inequalities are reversed for f nondecreasing.

In [4] Jakšetić and Pečarić generalized Steffensen's inequality for positive measures.

Mitrinović stated in [5] that the inequalities in (2) follow from the identities which will be the starting point for our generalizations of Steffensen's inequality.

First, let us recall the definition of n -convex functions.

Let f be a real-valued function defined on the segment $[a, b]$. The **divided difference** of order n of the function f at distinct points $x_0, \dots, x_n \in [a, b]$, is defined recursively (see [6, 7]) by

$$f[x_i] = f(x_i), \quad (i = 0, \dots, n)$$

and

$$f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0, \dots, x_n]$ is independent of the order of the points x_0, \dots, x_n .

The definition may be extended to include the case when some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x, \dots, x}_{j\text{-times}}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \quad (3)$$

The notion of n -convexity goes back to Popoviciu ([8]). We follow the definition given by Karlin ([9]):

Definition 1.4. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be n -convex on $[a, b]$, $n \geq 0$, if for all choices of $(n+1)$ distinct points in $[a, b]$, n -th order divided difference of f satisfies

$$f[x_0, \dots, x_n] \geq 0.$$

Note that, 1-convex functions are nondecreasing functions. If $f^{(n)}$ exists, then f is n -convex iff $f^{(n)} \geq 0$.

The paper is organized as follows. After this Introduction, in Section 2 we obtain new identities related to Steffensen's inequality. Using these new identities we generalize Steffensen's inequality for n -convex functions. Further, in Section 3 we give Ostrowski-type inequalities related to our new generalizations. We conclude this paper with some new bounds for our identities, using the Chebyshev and Grüss type inequalities.

Throughout the paper, it is assumed that all integrals under consideration exist and that they are finite.

2 Generalizations of Steffensen's inequality via the extension of Montgomery identity

In this section we obtain generalizations of Steffensen's inequality for n -convex functions using identity (1).

Theorem 2.1. Let $f : I \rightarrow \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a, b] \rightarrow \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t) dt = \int_a^b g(t)p(t) dt$ and let the function G_1 be defined by

$$G_1(x) = \begin{cases} \int_a^x (1-g(t))p(t) dt, & x \in [a, a+\lambda], \\ \int_x^b g(t)p(t) dt, & x \in [a+\lambda, b]. \end{cases} \quad (4)$$

Then

$$\begin{aligned} & \int_a^{a+\lambda} f(t)p(t) dt - \int_a^b f(t)g(t)p(t) dt \\ & + \int_a^b G_1(x) \left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right) dx \\ & = -\frac{1}{(n-2)!} \int_a^b \left(\int_a^b G_1(x) T_{n-1}(x, s) dx \right) f^{(n)}(s) ds. \end{aligned} \quad (5)$$

Proof. Using identity

$$\int_{[a, a+\lambda]} f(t)p(t) dt - \int_{[a, b]} f(t)g(t)p(t) dt = \int_{[a, a+\lambda]} f(t)(1-g(t))p(t) dt - \int_{(a+\lambda, b]} f(t)g(t)p(t) dt$$

and integration by parts we have

$$\begin{aligned} & \int_a^{a+\lambda} f(t)p(t) dt - \int_a^b f(t)g(t)p(t) dt \\ & = \int_a^{a+\lambda} [f(t) - f(a+\lambda)][1-g(t)]p(t) dt + \int_{a+\lambda}^b [f(a+\lambda) - f(t)]g(t)p(t) dt \\ & = -\int_a^{a+\lambda} \left[\int_a^x (1-g(t))p(t) dt \right] df(x) - \int_{a+\lambda}^b \left[\int_x^b g(t)p(t) dt \right] df(x) \\ & = -\int_a^b G_1(x) df(x) = -\int_a^b G_1(x) f'(x) dx. \end{aligned}$$

Applying identity (1) to f' and replacing n with $n-1$ we have

$$f'(x) = \frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} + \frac{1}{(n-2)!} \int_a^b T_{n-1}(x, s) f^{(n)}(s) ds.$$

Now we obtain

$$\begin{aligned} & \int_a^b G_1(x) f'(x) dx = \int_a^b G_1(x) \left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right) dx \\ & + \frac{1}{(n-2)!} \int_a^b G_1(x) \left(\int_a^b T_{n-1}(x, s) f^{(n)}(s) ds \right) dx. \end{aligned} \quad (6)$$

After applying Fubini's theorem on the last term in (6) we obtain (5). \square

Theorem 2.2. Let $f : I \rightarrow \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a, b] \rightarrow \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let

$\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_2 be defined by

$$G_2(x) = \begin{cases} \int_a^x g(t)p(t)dt, & x \in [a, b - \lambda], \\ \int_x^b (1 - g(t))p(t)dt, & x \in [b - \lambda, b]. \end{cases} \tag{7}$$

Then

$$\begin{aligned} & \int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt \\ & + \int_a^b G_2(x) \left(\frac{f(b) - f(a)}{b - a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right) dx \\ & = -\frac{1}{(n-2)!} \int_a^b \left(\int_a^b G_2(x)T_{n-1}(x, s)dx \right) f^{(n)}(s)ds. \end{aligned} \tag{8}$$

Proof. Similarly as in the proof of Theorem 2.1, we use the identity

$$\int_{[a,b]} f(t)g(t)p(t)dt - \int_{(b-\lambda,b]} f(t)p(t)dt = \int_{[a,b-\lambda]} f(t)g(t)p(t)dt - \int_{(b-\lambda,b]} f(t)(1-g(t))p(t)dt. \quad \square$$

Now, using the above obtained identities we give generalization of Steffensen’s inequality for n -convex functions.

Theorem 2.3. Let $f : I \rightarrow \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I, a < b$. Let $g, p : [a, b] \rightarrow \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_1 be defined by (4). If f is n -convex and

$$\int_a^b G_1(x)T_{n-1}(x, s)dx \geq 0, \quad s \in [a, b], \tag{9}$$

then

$$\begin{aligned} & \int_a^b f(t)g(t)p(t)dt \geq \int_a^{a+\lambda} f(t)p(t)dt \\ & + \int_a^b G_1(x) \left(\frac{f(b) - f(a)}{b - a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right) dx. \end{aligned} \tag{10}$$

Proof. If the function f is n -convex, without loss of generality we can assume that f is n -times differentiable and $f^{(n)} \geq 0$ see [7, p. 16 and p. 293]. Now we can apply Theorem 2.1 to obtain (10). \square

Theorem 2.4. Let $f : I \rightarrow \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I, a < b$. Let $g, p : [a, b] \rightarrow \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_2 be defined by (7). If f is n -convex and

$$\int_a^b G_2(x)T_{n-1}(x, s)dx \geq 0, \quad s \in [a, b], \tag{11}$$

then

$$\begin{aligned} & \int_a^b f(t)g(t)p(t)dt \leq \int_{b-\lambda}^b f(t)p(t)dt \\ & - \int_a^b G_2(x) \left(\frac{f(b) - f(a)}{b - a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right) dx. \end{aligned} \tag{12}$$

Proof. Similar to the proof of Theorem 2.3. □

Remark 2.5. *If the integrals in (9) and (11) are nonpositive, then the reverse inequalities in (10) and (12) hold. Note that in this case for some odd $n \geq 3$, functions G_i , $i = 1, 2$ are nonnegative so integrals in (9) and (11) are nonpositive. Hence, inequalities (10) and (12) are reversed.*

3 Ostrowski-type inequalities

In this section we give the Ostrowski-type inequalities related to generalizations obtained in the previous section.

Here, the symbol $L_p[a, b]$ ($1 \leq p < \infty$) denotes the space of p -power integrable functions on the interval $[a, b]$ equipped with the norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}$$

and $L_\infty[a, b]$ denotes the space of essentially bounded functions on $[a, b]$ with the norm

$$\|f\|_\infty = \operatorname{ess\,sup}_{t \in [a, b]} |f(t)|.$$

Theorem 3.1. *Suppose that all assumptions of Theorem 2.1 hold. Assume also that (p, q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$ and $f^{(n)} \in L_p[a, b]$ for some $n \geq 2$. Then we have*

$$\begin{aligned} & \left| \int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \right. \\ & \left. + \int_a^b G_1(x) \left(\frac{f(b) - f(a)}{b - a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right) dx \right| \\ & \leq \frac{1}{(n-2)!} \|f^{(n)}\|_p \left\| \int_a^b G_1(x) T_{n-1}(x, \cdot) dx \right\|_q. \end{aligned} \quad (13)$$

The constant on the right-hand side of (13) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Let's denote

$$C(s) = \frac{1}{(n-2)!} \int_a^b G_1(x) T_{n-1}(x, s) dx.$$

By taking the modulus of (5) and applying Hölder's inequality we obtain

$$\begin{aligned} & \left| \int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \right. \\ & \left. + \int_a^b G_1(x) \left(\frac{f(b) - f(a)}{b - a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right) dx \right| \\ & = \left| \int_a^b C(s) f^{(n)}(s) ds \right| \leq \|f^{(n)}\|_p \|C\|_q. \end{aligned}$$

For the proof of the sharpness of the constant $\|C\|_q$ let us find a function f for which the equality in (13) is obtained.

For $1 < p < \infty$ take f to be such that

$$f^{(n)}(s) = \operatorname{sgn} C(s) |C(s)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $f^{(n)}(s) = \text{sgn } C(s)$.

For $p = 1$ we prove that

$$\left| \int_a^b C(s) f^{(n)}(s) dt \right| \leq \max_{t \in [a,b]} |C(t)| \left(\int_a^b |f^{(n)}(s)| ds \right) \tag{14}$$

is the best possible inequality. $C(\cdot)$ is a continuous function on $[a, b]$ and so is $|C(\cdot)|$. Suppose that $|C(\cdot)|$ attains its maximum at $s_0 \in [a, b]$. First we assume that $C(s_0) > 0$. For $\varepsilon > 0$ small enough we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} 0, & a \leq s \leq s_0, \\ \frac{1}{\varepsilon n!} (s - s_0)^n, & s_0 \leq s \leq s_0 + \varepsilon, \\ \frac{1}{n!} (s - s_0)^{n-1}, & s_0 + \varepsilon \leq s \leq b. \end{cases}$$

Then

$$\left| \int_a^b C(s) f_\varepsilon^{(n)}(s) ds \right| = \left| \int_{s_0}^{s_0+\varepsilon} C(s) \frac{1}{\varepsilon} ds \right| = \frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} C(s) ds.$$

Now from the inequality (14) we have

$$\frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} C(s) ds \leq \frac{1}{\varepsilon} C(s_0) \int_{s_0}^{s_0+\varepsilon} ds = C(s_0).$$

Since,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} C(s) ds = C(s_0)$$

the statement follows. In the case $C(s_0) < 0$, we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} \frac{1}{n!} (s - s_0 - \varepsilon)^{n-1}, & a \leq s \leq s_0, \\ -\frac{1}{\varepsilon n!} (s - s_0 - \varepsilon)^n, & s_0 \leq s \leq s_0 + \varepsilon, \\ 0, & s_0 + \varepsilon \leq s \leq b, \end{cases}$$

and the rest of the proof is the same as above. □

Theorem 3.2. *Suppose that all assumptions of Theorem 2.2 hold. Assume also (p, q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty, 1/p + 1/q = 1$. Let $f^{(n)} \in L_p[a, b]$ for some $n \geq 2$. Then we have*

$$\begin{aligned} & \left| \int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt \right. \\ & \left. + \int_a^b G_2(x) \left(\frac{f(b) - f(a)}{b - a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right) dx \right| \\ & \leq \frac{1}{(n-2)!} \|f^{(n)}\|_p \left\| \int_a^b G_2(x) T_{n-1}(x, \cdot) dx \right\|_q. \end{aligned} \tag{15}$$

The constant on the right-hand side of (15) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Similar to the proof of Theorem 3.1. □

4 Generalizations related to the bounds for the Chebyshev functional

Let $f, h : [a, b] \rightarrow \mathbb{R}$ be Lebesgue integrable functions. We define the Chebyshev functional $T(f, h)$ by

$$T(f, h) := \frac{1}{b-a} \int_a^b f(t)h(t)dt - \frac{1}{b-a} \int_a^b f(t)dt \cdot \frac{1}{b-a} \int_a^b h(t)dt.$$

In 1882 Chebyshev proved that

$$|T(f, h)| \leq \frac{1}{12} \|f'\|_{\infty} \|h'\|_{\infty} (b-a)^2,$$

provided that f', h' exist and are continuous on $[a, b]$ and $\|f'\|_{\infty} = \sup_{t \in [a, b]} |f'(t)|$. It also holds if $f, h : [a, b] \rightarrow \mathbb{R}$ are absolutely continuous and $f', g' \in L_{\infty}[a, b]$ while $\|f'\|_{\infty} = \text{ess sup}_{t \in [a, b]} |f'(t)|$.

In 1934, Grüss in his paper [10] proved that

$$|T(f, h)| \leq \frac{1}{4} (M-m)(N-n),$$

provided that there exist the real numbers m, M, n, N such that

$$m \leq f(t) \leq M, \quad n \leq h(t) \leq N$$

for a.e. $t \in [a, b]$. The constant $1/4$ is the best possible.

In [11] Cerone and Dragomir proved the following theorems:

Theorem 4.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a Lebesgue integrable function and $h : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function with $(\cdot - a)(b - \cdot)[h']^2 \in L_1[a, b]$. Then we have the inequality

$$|T(f, h)| \leq \frac{1}{\sqrt{2}} [T(f, f)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left(\int_a^b (x-a)(b-x)[h'(x)]^2 dx \right)^{\frac{1}{2}}. \quad (16)$$

The constant $\frac{1}{\sqrt{2}}$ in (16) is the best possible.

Theorem 4.2. Assume that $h : [a, b] \rightarrow \mathbb{R}$ is monotonic nondecreasing on $[a, b]$ and $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous with $f' \in L_{\infty}[a, b]$. Then we have the inequality

$$|T(f, h)| \leq \frac{1}{2(b-a)} \|f'\|_{\infty} \int_a^b (x-a)(b-x)dh(x). \quad (17)$$

The constant $\frac{1}{2}$ in (17) is the best possible.

In the sequel we use the above theorems to obtain some new bounds for integrals on the left hand side in the perturbed version of identities (5) and (8).

Firstly, let us denote

$$\Omega_i(s) = \int_a^b G_i(x)T_{n-1}(x, s)dx, \quad i = 1, 2. \quad (18)$$

Theorem 4.3. Let $f : I \rightarrow \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$ and $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L_1[a, b]$. Let $g, p : [a, b] \rightarrow \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the functions G_1 and Ω_1 be defined by (4) and

(18). Then

$$\begin{aligned} & \int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \\ & + \int_a^b G_1(x) \left(\frac{f(b) - f(a)}{b - a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right) dx \\ & + \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n-2)!} \int_a^b \Omega_1(s)ds = S_n^1(f; a, b), \end{aligned} \tag{19}$$

where the remainder $S_n^1(f; a, b)$ satisfies the estimation

$$\left| S_n^1(f; a, b) \right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!} [T(\Omega_1, \Omega_1)]^{\frac{1}{2}} \left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds \right)^{\frac{1}{2}}. \tag{20}$$

Proof. Applying Theorem 4.1 for $f \rightarrow \Omega_1$ and $h \rightarrow f^{(n)}$ we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \Omega_1(s)f^{(n)}(s)ds - \frac{1}{b-a} \int_a^b \Omega_1(s)ds \cdot \frac{1}{b-a} \int_a^b f^{(n)}(s)ds \right| \\ & \leq \frac{1}{\sqrt{2}} [T(\Omega_1, \Omega_1)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 dt \right)^{\frac{1}{2}}. \end{aligned} \tag{21}$$

Now if we add

$$\frac{1}{(b-a)(n-2)!} \int_a^b \Omega_1(s)ds \int_a^b f^{(n)}(s)ds = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n-2)!} \int_a^b \Omega_1(s)ds$$

to both sides of identity (5) and use inequality (21), we obtain representation (19) and bound (20). □

Similarly, using identity (8) we obtain the following result:

Theorem 4.4. Let $f : I \rightarrow \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$ and $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L_1[a, b]$. Let $g, p : [a, b] \rightarrow \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the functions G_2 and Ω_2 be defined by (7) and (18). Then

$$\begin{aligned} & \int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt \\ & + \int_a^b G_2(x) \left(\frac{f(b) - f(a)}{b - a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right) dx \\ & + \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n-2)!} \int_a^b \Omega_2(s)ds = S_n^2(f; a, b), \end{aligned} \tag{22}$$

where the remainder $S_n^2(f; a, b)$ satisfies the estimation

$$\left| S_n^2(f; a, b) \right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!} [T(\Omega_2, \Omega_2)]^{\frac{1}{2}} \left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds \right)^{\frac{1}{2}}.$$

Proof. Similar to the proof of Theorem 4.3. □

The following Grüss-type inequalities also hold.

Theorem 4.5. Let $f : I \rightarrow \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n \geq 2$ and $f^{(n+1)} \geq 0$ on $[a, b]$. Let functions Ω_i , $i = 1, 2$ be defined by (18).

(a) Let $\int_a^{a+\lambda} p(t) dt = \int_a^b g(t)p(t) dt$. Then we have representation (19) and the remainder $S_n^1(f; a, b)$ satisfies the bound

$$\left| S_n^1(f; a, b) \right| \leq \frac{b-a}{(n-2)!} \|\Omega_1'\|_\infty \left\{ \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b) - f^{(n-2)}(a)}{b-a} \right\}. \quad (23)$$

(b) Let $\int_{b-\lambda}^b p(t) dt = \int_a^b g(t)p(t) dt$. Then we have representation (22) and the remainder $S_n^2(f; a, b)$ satisfies the bound

$$\left| S_n^2(f; a, b) \right| \leq \frac{b-a}{(n-2)!} \|\Omega_2'\|_\infty \left\{ \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b) - f^{(n-2)}(a)}{b-a} \right\}.$$

Proof. (a) Applying Theorem 4.2 for $f \rightarrow \Omega_1$, $h \rightarrow f^{(n)}$ we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \Omega_1(s) f^{(n)}(s) ds - \frac{1}{b-a} \int_a^b \Omega_1(s) ds \cdot \frac{1}{b-a} \int_a^b f^{(n)}(s) ds \right| \\ & \leq \frac{1}{2(b-a)} \|\Omega_1'\|_\infty \int_a^b (s-a)(b-s) f^{(n+1)}(s) ds. \end{aligned} \quad (24)$$

Since

$$\begin{aligned} \int_a^b (s-a)(b-s) f^{(n+1)}(s) ds &= \int_a^b [2s - (a+b)] f^{(n)}(s) ds \\ &= (b-a) [f^{(n-1)}(b) + f^{(n-1)}(a)] - 2 (f^{(n-2)}(b) - f^{(n-2)}(a)). \end{aligned}$$

Using representation (5) and inequality (24) we deduce (23).

(b) Similar to the (a)-part. □

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