Open Mathematics

Research Article

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Disjointed sum of products by a novel technique of orthogonalizing ORing

Abstract: This work presents a novel combining method called 'orthogonalizing ORing $\varoslash$ ' which enables the building of the union of two conjunctions whereby the result consists of disjointed conjunctions. The advantage of this novel technique is that the results are already presented in an orthogonal form which has a significant advantage for further calculations as the Boolean Differential Calculus. By orthogonalizing ORing two calculation steps - building the disjunction and the subsequent orthogonalization of two conjunctions - are performed in one step. Postulates, axioms and rules for this linking technique are also defined which have to be considered getting correct results. Additionally, a novel equation, based on orthogonalizing ORing, is set up for orthogonalization of every Boolean function of disjunctive form. Thus, disjointed Sum of Products can be easily calculated in a mathematical way by this equation.

1 Introduction

A Boolean function or a switching function, respectively, is defined as a mapping $f(\underline{x}) : \mathbf{B}^n \rightarrow \mathbf{B}$ with $\mathbf{B} = \{0, 1\}$. It can be expressed by using Boolean variables $x_i = \{x_1, x_2, \ldots, x_n\}$ [1, 3, 10, 14, 20, 21]. As shown in Table 1, Boolean variables which are either direct x_n or negated $\bar{x}_n$ are connected by operations like conjunction ($\wedge$, or no operation sign), disjunction ($\vee$), antivalence ($\oplus$) and equivalence ($\odot$).

Table 1. Boolean operations of two variables

x_i	x_j	$\bar{x}_i$	$\bar{x}_j$	$x_i \wedge x_j$	$x_i \vee x_j$	$x_i \oplus x_j$	$x_i \odot x_j$
0	0	1	1	0	0	0	1
0	1	1	0	0	1	1	0
1	0	0	1	0	1	1	0
1	1	0	0	1	1	0	1

There are four standard forms $f^S(\underline{x})$ of switching function [1, 6, 19] which are either connected by conjunctions $c_i(\underline{x}) = \bigwedge_{j=1}^n x_j = x_1 \cdot \ldots \cdot x_{n-1} \cdot x_n$ or by disjunctions $d_i(\underline{x}) = \bigvee_{j=1}^n x_j = x_1 \vee \ldots \vee x_{n-1} \vee x_n$ [3]. Conjunctions are connected by disjunctions in the disjunctive form DF (1) or by antivalence-operations in the antivalence form AF (3) and disjunctions are connected by conjunctions in the conjunctive form CF (2) or by equivalence-

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operations in the equivalence form *EF* (4). The connection of canonical conjunctions or disjunctions is named as normal form: disjunctive normal form *DNF*, antivalence normal form *ANF*, conjunctive normal form *CNF* and equivalence normal form *ENF*. Therefore, the orthogonal form of a DF can also be named as disjointed Sum of Products (dSOP) which is the set of products terms (conjunctions) whose disjunction equals $f(\underline{x})$ in non-orthogonal form, such that no two product terms cover the same 1. Consequently, dSOP consist of non intersecting cubes. Deriving an orthogonal form of DF is a classical problem of Boolean theory. In this work, this problem is supported by a contribution of a novel solution.

Definition 1.1 (Standard Forms with $m \in \mathbb{N} \setminus \{1\}$)**.**

$$1.\ \text{Disjunctive Form:}\quad f_{DF}(\underline{x}) = \bigvee_{i=1}^{m} c_i(\underline{x}). \tag{1}$$

$$2.\ \text{Conjunctive Form:}\quad f_{CF}(\underline{x}) = \bigwedge_{i=1}^{m} d_i(\underline{x}). \tag{2}$$

$$3.\ \text{Antivalence Form:}\quad f_{AF}(\underline{x}) = \bigoplus_{i=1}^{m} c_i(\underline{x}). \tag{3}$$

$$4.\ \text{Equivalence Form:}\quad f_{EF}(\underline{x}) = \bigodot_{i=1}^{m} d_i(\underline{x}). \tag{4}$$

2 Characteristic of orthogonality

The characteristic of orthogonality is a special attribute of a switching function [1, 5, 6, 15–18]. The orthogonal form of a switching function is characterized by conjunctions or disjunctions which are disjointed to one another in pairs. This means, that for each pair of conjunctions, one of them contains a direct Boolean variable (x_i) and the other contains the negation $(\bar{x}_i)$ of the same Boolean variable. Consequently, the intersection of each pair of these conjunctions $(c_{i,j}(\underline{x}))$ results in 0, as shown in Eq. (5). In contrast, the disjunction of each pair of orthogonal disjunctions $(d_{i,j}(\underline{x}))$ results in 1, as shown in Eq. (6).

Definition 2.1 (Orthogonality of Conjunctions or Disjunctions)**.** *Two conjunctions $c_i(\underline{x})$ and $c_j(\underline{x})$ are orthogonal to each other if the following applies:*

$$c_i(\underline{x}) \wedge c_j(\underline{x}) = 0 \quad i \neq j. \tag{5}$$

Two disjunctions $d_i(\underline{x})$ and $d_j(\underline{x})$ are orthogonal to each other if the following applies:

$$d_i(\underline{x}) \vee d_j(\underline{x}) = 1 \quad i \neq j. \tag{6}$$

The orthogonal form of a $f^S(\underline{x})$ enables its transformation in another form, which will have equivalent function values. This means, that the native form and the transformed form have the same function values if the same input values are used in each case. Therefore, orthogonalization simplifies the handling for further calculation steps, especially in the application of electrical engineering, e.g. for further calculation step as the Boolean Differential Calculus (BDC) [2, 3] by which all possible test patterns for a combinational circuit can be determined. Test patterns are used to detect feasible faults in combinational circuits. Additionally, it facilitates the calculation of BDC particularly in Ternary-Vector-List (TVL) arithmetic [11, 12, 16–18]. TVL is a kind of matrix which simplifies the computational representation of Boolean functions and their computational handling of tasks in a facilitated way.

Theorem 2.2 (Orthogonality of DF and AF)**.** *If the intersection of every two conjunctions $c_{i,j}(\underline{x})$ of a given DF is 0 then $DF^{orth} = AF^{orth}$ applies. That means, an orthogonal disjunctive form DF^{orth} is equivalent to the orthogonal*

antivalence form AF^{orth} including the same conjunctions [1, 3, 10, 14, 15, 21]:

$$\bigvee_{k=1}^{m} c_k = \bigoplus_{k=1}^{m} c_k. \tag{7}$$

Proof of Theorem 2.2. By using (5) for orthogonal conjunctions $c_{i,j}(\underline{x})$ the relation in (7) applies. The respective proof is brought by the following relation. The disjunction of two conjunctions $c_{i,j}(\underline{x})$ on the right side is equivalent to the antivalence operation of the same two conjunctions on the right side. This is the procedure of reshaping of a disjunctive form in the antivalence form. For the case that both conjunctions are orthogonal the last term on right side results in 0. An antivalence-operation with 0 is to be neglected, as $x_i \oplus 0 = x_i$ applies. Consequently, this leads to the relation in (7).

$$c_i(\underline{x}) \vee c_j(\underline{x}) = c_i(\underline{x}) \oplus c_j(\underline{x}) \oplus \underbrace{(c_i(\underline{x}) \wedge c_j(\underline{x}))}_{=0}. \tag{8}$$

$\square$

Theorem 2.3 (Orthogonality of CF and EF). *If the union of every two disjunctions $d_{i,j}(\underline{x})$ of a given CF is 1 then $CF^{orth} = EF^{orth}$ applies. Thus, an orthogonal conjunctive form CF^{orth} is equivalent to the orthogonal equivalence form EF^{orth} including the same disjunctions [1, 3, 10, 14, 15, 21]:*

$$\bigwedge_{k=1}^{m} d_k = \bigodot_{k=1}^{m} d_k. \tag{9}$$

Proof of Theorem 2.3. A *CF* can be represented as an *EF* by using the following condition in (10). By using (6) for orthogonal disjunctions $d_{i,j}(\underline{x})$ in this case the relation in (9) applies. The conjunction of two disjunctions $d_{i,j}(\underline{x})$, that means a CF, on the left side is equivalent to the right side which illustrates the equivalence operation of the same two disjunctions. If these both disjunctions $d_{i,j}(\underline{x})$ are orthogonal then the union of them results in 1, as shown by the last term on the right side. An equivalence-operation with 1 is to be neglected, as $x_i \odot 1 = x_i$ applies. Consequently, the relation in (9) applies.

$$d_i(\underline{x}) \wedge d_j(\underline{x}) = d_i(\underline{x}) \odot d_j(\underline{x}) \odot \underbrace{(d_i(\underline{x}) \vee d_j(\underline{x}))}_{=1}. \tag{10}$$

$\square$

3 Elementary operations of two conjunctions

In this section the elementary operations (intersection, union, difference-building) of conjunctions, which are deduced out of the set theory due to the isomorphism, are defined for the switching algebra. These formulas for different operations of conjunctions specify the order in which the variables of the given conjunction have to be linked. That means, if a variable is displayed negated, the corresponding literal of the given conjunction must be negated at this point. The number of variables in their respective conjunction is defined by n or n'.

Theorem 3.1 (Intersection of two conjunctions). *The intersection of any two conjunctions $c_{i,j}(\underline{x})$ with $n, n' \in \mathbb{N}$ is calculated by:*

$$c_i(\underline{x}) \wedge c_j(\underline{x}) = \bigwedge_{i=1}^{n} x_i \wedge \bigwedge_{j=1}^{n'} x_j. \tag{11}$$

Theorem 3.2 (Union of two conjunctions). *The union of any two conjunctions $c_{i,j}(\underline{x})$ with $n, n' \in \mathbb{N}$ is given by:*

$$c_i(\underline{x}) \vee c_j(\underline{x}) = \bigwedge_{i=1}^{n} x_i \vee \bigwedge_{j=1}^{n'} x_j. \tag{12}$$

Theorem 3.3 (Difference-building of two conjunctions)**.** *The difference-building of a conjunction $c_m(\underline{x})$ as minuend and another conjunction $c_s(\underline{x})$ as subtrahend with $n, n' \in \mathbb{N}$ is calculated by following equation, which is deduced out of the set theory [4, 13]. That means, for the difference of two sets $M - S$ it applies $M \cap \bar{S}$ which is transferred to the switching algebra due to the isomorphism. Consequently, the difference-building of two conjunctions is the intersection of the minuend and the complement of the subtrahend. By building the difference several conjunctions arise which are not disjointed (orthogonal) to each other.*

$$c_m(\underline{x}) - c_s(\underline{x}) = \bigwedge_{m=1}^{n} x_m - \bigwedge_{s=1}^{n'} x_s = \bigwedge_{m=1}^{n} x_m \wedge \overline{\bigwedge_{s=1}^{n'} x_s} = \bigwedge_{m=1}^{n} x_m \wedge \bigvee_{s=1}^{n'} \bar{x}_s. \tag{13}$$

4 Orthogonalizing difference-building

The technique of orthogonalizing difference-building $\ominus$ is used to calculate the orthogonal difference of two conjunctions ($c_{i,j}(\underline{x})$) whereby the result is orthogonal. This method is generally valid and equivalent to the usual method of difference-building [6–9]. Orthogonalizing difference-building $\ominus$ is the composition of two calculation steps - the difference-building and the subsequent orthogonalization.

Definition 4.1 (Orthogonalizing difference-building)**.** *Orthogonalizing difference-building $\ominus$ corresponds to the removal of the intersection which is formed between the minuend conjunction $c_m(\underline{x})$ and the subtrahend conjunction $c_s(\underline{x})$ from the minuend $c_m(\underline{x})$, which means $c_m(\underline{x}) - (c_m(\underline{x}) \wedge c_s(\underline{x}))$; the result is orthogonal. Orthogonalizing difference-building of two conjunctions with $n, n' \in \mathbb{N}$ is defined as:*

$$c_m(\underline{x}) \ominus c_s(\underline{x}) = \bigwedge_{m=1}^{n} x_m \ominus \bigwedge_{s=1}^{n'} x_s := \bigwedge_{m=1}^{n} x_m \wedge \bigvee_{s=1}^{n'_j} \bar{x}_{s_j} = \tag{14}$$

$$= \left(x_1 \cdot \ldots \cdot x_{n-1} \cdot x_n \right)_m \wedge \left(\bar{x}_{1_j} \vee x_{1_j} \cdot \bar{x}_{2_j} \vee \ldots \vee x_{1_j} \cdot \ldots \cdot x_{(n-1)_j} \cdot \bar{x}_{n'_j} \right)_s.$$

In this case, the formula ($\bigvee_{i=1}^{n_j} \bar{x}_{i_j} = \bar{x}_{1_j} \vee x_{1_j} \bar{x}_{2_j} \vee \ldots \vee x_{1_j} x_{2_j} \cdot \ldots \cdot \bar{x}_{n_j}$) from [10] is used to describe the orthogonalizing difference-building in a mathematically easier way, where $x_{1_j}, x_{2_j}, \ldots, x_{n_j}$ are literals of the subtrahend conjunction $c_s(\underline{x})$.

This method is explained by the following example and description. Additionally, this example is also illustrated in a K-map (Figure 1).

Example 4.2. *A subtrahend ($c_s(\underline{x}) = x_1 x_2 x_4$) is subtracted from a minuend ($c_m(\underline{x}) = x_3$). It is a result of several conjunctions (1st cube, 2nd cube, 3rd cube) which are orthogonal to each other and cover all of the remaining 1s.*

$$\underbrace{x_3}_{Minuend} \ominus \underbrace{x_1 x_2 x_4}_{Subtrahend} = \underbrace{\bar{x}_1 x_3}_{1^{st} cube} \vee \underbrace{x_1 \bar{x}_2 x_3}_{2^{nd} cube} \vee \underbrace{x_1 x_2 x_3 \bar{x}_4}_{3^{rd} cube}.$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{orthogonal\ difference}$$

- The first literal of the subtrahend, here x_1, is complemented and builds the intersection with the minuend, here x_3. Consequently, the first conjunction of the difference is $\bar{x}_1 x_3$.
- Then the second literal, here x_2, is complemented and forms the intersection with the minuend, and the first literal x_1 of the subtrahend is built. Therefore, the second conjunction is $x_1 \bar{x}_2 x_3$.
- The next literal, here x_4, is complemented and forms the intersection with the minuend, and the first literal x_1 and second literal x_2 of the subtrahend is built. Thus, the third term of the difference is $x_1 x_2 x_3 \bar{x}_4$.
- This process is continued until all literals of the subtrahend are singly complemented and linked by building the intersection with the minuend in a separate conjunction.

Fig. 1. Example 4.2 in a K-map
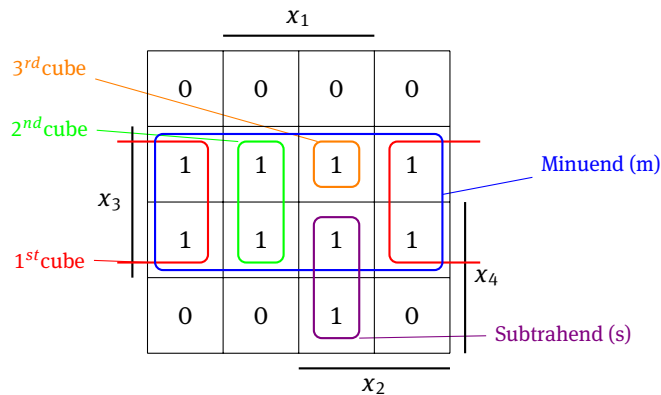
5 Orthogonalizing ORing

By a further novel technique 'orthogonalizing ORing $\varovee$', which is based on the orthogonalizing difference-building $\ominus$, the union of two conjunctions $(c_{i,j}(\underline{x}))$ can be calculated of two conjunctions $(c_{i,j}(\underline{x}))$ whereby the result is orthogonal.

Definition 5.1 (Orthogonalizing ORing). *The intersection of a conjunction $c_{s_1}(\underline{x})$, called as the first summand, and a second conjunction $c_{s_2}(\underline{x})$, called as the second summand, is removed by the method $\ominus$ from the first summand $c_{s_1}(\underline{x})$, and the second summand $c_{s_2}(\underline{x})$ is linked to that subtraction by a disjunction; the result is orthogonal. This procedure is labeled as orthogonalizing ORing and is defined with $n, n' \in \mathbb{N}$ as:*

$$c_{s_1}(\underline{x}) \varovee c_{s_2}(\underline{x}) = \bigwedge_{s_1=1}^{n} x_{s_1} \varovee \bigwedge_{s_2=1}^{n'} x_{s_2} := \tag{15}$$

$$= \left[\bigwedge_{s_1=1}^{n} x_{s_1} \ominus \bigwedge_{s_2=1}^{n'} x_{s_2} \right] \vee \bigwedge_{s_2=1}^{n'} x_{s_2} =$$

$$= \left[\bigwedge_{s_1=1}^{n} x_{s_1} \wedge \bigvee_{s_2=1}^{n'_j} \bar{x}_{s_{2_j}} \right] \vee \bigwedge_{s_2=1}^{n'} x_{s_2} =$$

$$= \left[\left(x_1 \cdot \ldots \cdot x_n \right)_{s_1} \wedge \left(\bar{x}_{1_j} \vee \ldots \vee x_{1_j} \cdot \ldots \cdot \bar{x}_{n'_j} \right)_{s_{2_j}} \right] \vee \left(x_1 \cdot \ldots \cdot x_{n'} \right)_{s_2}.$$

This method of orthogonalizing ORing is explained by the Example 5.2 which is also illustrated in a K-map (Fig. 2).

Example 5.2. *The orthogonalizing ORing is built of two conjunctions $c_{s_1}(\underline{x}) = \bar{x}_2$ and $c_{s_2}(\underline{x}) = x_1 x_3$ by using Eq. (15).*

$$\underbrace{\bar{x}_2}_{1^{st}sum.} \varovee \underbrace{x_1 x_3}_{2^{nd}sum.} = \underbrace{\bar{x}_1 \bar{x}_2}_{1^{st}cube} \vee \underbrace{x_1 \bar{x}_2 \bar{x}_3}_{2^{nd}cube} \vee \underbrace{x_1 x_3}_{2^{nd}sum.} .$$
$$\underbrace{\phantom{\bar{x}_1 \bar{x}_2 \vee x_1 \bar{x}_2 \bar{x}_3 \vee x_1 x_3}}_{disjointed\ conjunctions}$$

It is a result of several conjunctions or cubes which are disjointed to each other in pair (Fig. 2). The following points explain this unprecedented technique:

- The first literal of the second summand $c_{s_2}(\underline{x})$, here x_1, is complemented and ANDed to the first summand $c_{s_1}(\underline{x})$. Consequently, the first conjunction of the orthogonal result arises, $\bar{x}_1 \bar{x}_2$.

- Then the second literal of the second summand $c_{s_2}(\underline{x})$ is complemented, here x_3, and ANDed with the first literal of the second summand $c_{s_2}(\underline{x})$ to the first summand $c_{s_1}(\underline{x})$. Therefore, the second conjunction of the orthogonal result is developed, $x_1 \bar{x}_2 \bar{x}_3$.
- This process is continued until all literals of the second summand $c_{s_2}(\underline{x})$ are singly complemented and linked by ANDing to the first summand $c_{s_1}(\underline{x})$ in a separate term.
- At last the second summand $c_{s_2}(\underline{x})$ is added to the heretofore calculated conjunctions.

Fig. 2. Example 5.2 in a K-map
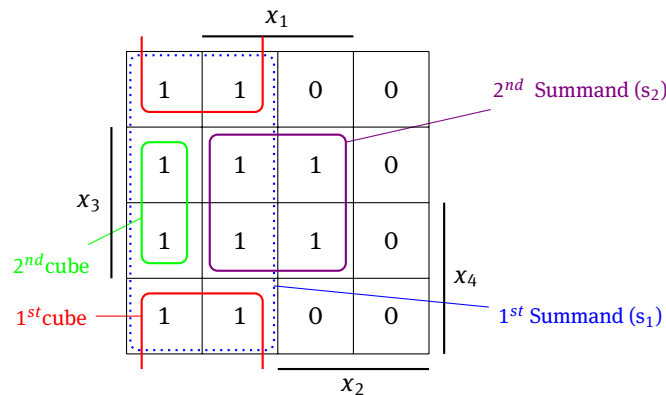
By swapping the position of the summands, the result changes as shown in the following:

$$x_1 x_3 \oslash \bar{x}_2 = x_1 x_2 x_3 \vee x_1 \bar{x}_2.$$

However, both solutions are equivalent because the same 1s are covered. They only differ in the form of their coverage. But in order to represent an orthogonal result with a fewer number of conjunctions, the conjunction with more literals has to be accepted as the first summand. That is possible because orthogonalizing ORing has the property of commutativity. By the mathematical induction the general validity of Eq. (15) is given in Proof 1.

The number of conjunctions in the result, called n_x, corresponds to the number of literals presented in the second summand $c_{s_2}(\underline{x})$ and not presented in the first summand $c_{s_1}(\underline{x})$ at the same time; in addition, a 1 is added to n_x, which stands for the second summand $c_{s_2}(\underline{x})$ as the last linked term. It applies:

$$n_x + 1 \quad \text{with} \quad n_x \in \mathbb{N}. \tag{16}$$

Furthermore, the number of possible results, which primarily depends on n_x, can be charged by:

$$n_x! \quad \text{for} \quad n_x > 0. \tag{17}$$

Depending on the starting literal the result may differ. There are many equivalent options which only differ in the form of their coverage. This novel technique contains the composition of two calculation procedures – the union '$\vee$' and the subsequent orthogonalization. The result out of orthogonalizing ORing is orthogonal in contrast to the result out of the usual method of union $\vee$. Both results are different in their representations but cover the same 1s. Hereinafter, the proof of this equivalence between orthogonalizing ORing and union is exemplified. On the left side it is denoted the orthogonalizing ORing of two sets S_1, S_2 and on the right side the union of the same sets. Due to the axiom of absorption, the equivalence between orthogonalizing ORing and ORing is verified. The right side is the orthogonal form of the left side, which are equivalent but different in their form of coverage.

$$S_1 \oslash S_2 = S_1 \vee S_2$$
$$S_1 \bar{S}_2 \vee S_2 = S_1 \vee S_2. \tag{18}$$

□

Proof. 1

General validity of Eq. (15):

$\forall(n, n') \in \mathbb{N} \setminus \{1\}, \quad (n, n') \geq n_0$ applies $A(n, n')$:

if $A(n_0) \wedge (\forall(n, n') \in \mathbb{N}, (n, n') \geq n_0 : A(n, n') \to A((n, n') + 1)) \Rrightarrow \forall(n, n') \in \mathbb{N}, (n, n') \geq n_0 : A(n, n')$

Basis $A(n_0)$: $n_0, n'_0 = 2$

$$\left[\bigwedge_{s_1=1}^{2} x_{s_1} \wedge \bigvee_{s_2=1}^{2_j} \bar{x}_{s_{2_j}}\right]_{s_{2_j}} \vee \bigwedge_{s_2=1}^{2} x_{s_2} = \left[(x_1 x_2)_{s_1} \wedge \left(\bar{x}_{1_j} \vee x_{1_j} \bar{x}_{2_j}\right)_{s_{2_j}}\right] \vee (x_1 x_2)_{s_2}$$

$$\left[(x_1 x_2)_{s_1} \wedge \left(\bar{x}_{1_j} \vee x_{1_j} \bar{x}_{2_j}\right)_{s_{2_j}}\right] \vee (x_1 x_2)_{s_2} = \left[(x_1 x_2)_{s_1} \wedge \left(\bar{x}_{1_j} \vee x_{1_j} \bar{x}_{2_j}\right)_{s_{2_j}}\right] \vee (x_1 x_2)_{s_2}$$

Inductive step $A(n, n') \to A(n+1, n'+1)$: $n \to n+1$ and $n' \to n'+1$

$$\left[\bigwedge_{s_1=1}^{n+1} x_{s_1} \wedge \bigvee_{s_2=1}^{(n'+1)_j} \bar{x}_{s_{2_j}}\right] \vee \bigwedge_{s_2=1}^{n'+1} x_{s_2} = \left[(x_1 \cdots \cdot x_n x_{n+1})_{s_1} \wedge \left(\bar{x}_{1_j} \vee \cdots \vee x_{1_j} \cdots \cdot \bar{x}_{n_j}' \vee x_{1_j} \cdots \cdots x_{n_j}' \bar{x}_{(n'+1)_j}\right)_{s_{2_j}}\right] \vee$$

$$\vee \underbrace{(x_1 \cdots \cdot x_{n_j}' x_{(n'+1)})}_{}_{s_2}$$

$$\left[\left(\bigwedge_{s_1=1}^{n} x_{s_1} \cdot x_{n+1}\right) \wedge \left(\bigvee_{s_2=1}^{n'_j} \bar{x}_{s_{2_j}} \vee x_{1_j} \cdots \cdots x_{n_j}' \bar{x}_{(n'+1)_j}\right)\right] \vee \bigwedge_{s_2=1}^{n'} x_{s_2} \cdot x_{(n'+1)} = \left[(x_1 \cdots \cdot x_n x_{n+1})_{s_1} \wedge \left(\bar{x}_{1_j} \vee \cdots \vee x_{1_j} \cdots \cdots \bar{x}_{n_j}' \vee x_{1_j} \cdots \cdots x_{n_j}' \bar{x}_{(n+1)_j}\right)_{s_{2_j}}\right] \vee$$

$$\vee \underbrace{(x_1 \cdots \cdot x_{n_j}' x_{(n'+1)})}_{}_{s_2}$$

$$\left[\left(\bigwedge_{s_1=1}^{n} x_{s_1} \cdot x_{n+1}\right) \wedge \left(\bigvee_{s_2=1}^{n'_j} \bar{x}_{s_{2_j}} \vee x_{1_j} \cdots \cdots x_{n_j}' \bar{x}_{(n'+1)_j}\right)\right] \vee \bigwedge_{s_2=1}^{n'} x_{s_2} \cdot x_{(n'+1)} = \left[\left(\bigwedge_{s_1=1}^{n} x_{s_1} \cdot x_{n+1}\right) \wedge \left(\bigvee_{s_2=1}^{n'_j} \bar{x}_{s_{2_j}} \vee x_{1_j} \cdots \cdots x_{n_j}' \bar{x}_{(n+1)_j}\right)\right] \vee \bigwedge_{s_2=1}^{n'} x_{s_2} \cdot x_{(n'+1)}$$

5.1 Orthogonalizing ORing between a DF and a conjunction

The orthogonalizing ORing of an orthogonal DF $f(\underline{x})^{orth}$ and a conjunction $c_{s_2}(\underline{x})$ can be reached by Eq. (19). With $l_{s_1} \in \mathbb{N}^+$ as the number of conjunctions in the given function the following applies:

$$f_{s_1}(\underline{x})^{orth} \varoslash c_{s_2}(\underline{x}) = \bigvee_{i=1}^{l_{s_1}} \bigwedge_{s_1=1}^{n} x_{i,s_1} \varoslash \bigwedge_{s_2=1}^{n'} x_{s_2} := \tag{19}$$

$$= \bigvee_{i=1}^{l_{s_1}} \left[\bigwedge_{s_1=1}^{n} x_{i,s_1} \ominus \bigwedge_{s_2=1}^{n'} x_{s_2} \right] \vee \bigwedge_{s_2=1}^{n'} x_{s_2} =$$

$$= \bigvee_{i=1}^{l_{s_1}} \left[\bigwedge_{s_1=1}^{n} x_{i,s_1} \wedge \bigvee_{s_2=1}^{n'_j} \bar{x}_{s_{2_j}} \right] \vee \bigwedge_{s_2=1}^{n'} x_{s_2}.$$

Since the general validity for orthogonalizing ORing is given, there is no need to prove the general validity for Eq. (19) in this case. As shown in Example 5.3 the use of (19) is illustrated.

Example 5.3. *Let DF $f_1(\underline{x}) = x_1 \vee x_2 x_3$ and a conjunction $c_1(\underline{x}) = \bar{x}_2 \bar{x}_3$. The orthogonalizing ORing between both has to be calculated.*

In this case the orthogonal form of $f_1(\underline{x})$ has to be calculated by Eq. (15) otherwise by Eq. (36):

$$x_1 \varoslash x_2 x_3 = x_1 \bar{x}_2 \vee x_1 x_2 \bar{x}_3 \vee x_2 x_3.$$

The orthogonal form is generated:

$$f_1(\underline{x})^{orth} = x_1 \bar{x}_2 \vee x_1 x_2 \bar{x}_3 \vee x_2 x_3.$$

Next step is the application of Eq. (19). Each conjunction of $f_1(\underline{x})^{orth}$ is combined by orthogonalizing ORing with $c_1(\underline{x})$. However, the adding of the second summand at last is fulfilled after each combining step.

$$f_1(\underline{x})^{orth} \varoslash c_1(\underline{x}) = (x_1 \bar{x}_2 \vee x_1 x_2 \bar{x}_3 \vee x_2 x_3) \varoslash \bar{x}_2 \bar{x}_3$$

$$= x_1 \bar{x}_2 x_3 \vee x_1 x_2 \bar{x}_3 \vee x_2 x_3 \vee \bar{x}_2 \bar{x}_3.$$

In the K-map on the left side the cubes of $f_1(\underline{x})$ and $c_1(\underline{x})$ are represented, and on the K-map on the right side the result after the procedure of orthogonalizing ORing is illustrated (Fig. 3).

Fig. 3. Before and after the process of orthogonalizing ORing
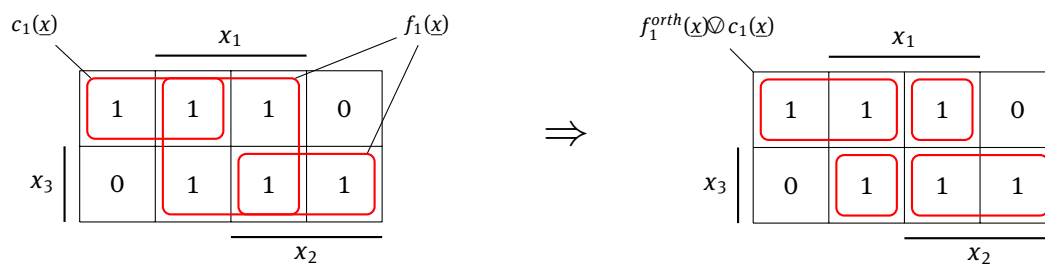
5.2 Axioms and Postulates

5.2.1 Postulates

The following postulates are necessary for getting correct results after each operation of orthogonalizing ORing.

– If two conjunctions are already orthogonal to each other ($c_{s_1}(\underline{x}) \perp c_{s_2}(\underline{x})$) the result corresponds to the disjunction of both conjunctions:

$$c_{s_1}(\underline{x}) \oslash c_{s_2}(\underline{x}) = c_{s_1}(\underline{x}) \lor c_{s_2}(\underline{x}). \tag{20}$$

– If the first conjunction is the subset of the second conjunction ($c_{s_1}(\underline{x}) \subset c_{s_2}(\underline{x})$) it follows:

$$c_{s_1}(\underline{x}) \oslash c_{s_2}(\underline{x}) = c_{s_2}(\underline{x}). \tag{21}$$

– and in the reverse case ($c_{s_2}(\underline{x}) \subset c_{s_1}(\underline{x})$) it follows:

$$c_{s_1}(\underline{x}) \oslash c_{s_2}(\underline{x}) = c_{s_1}(\underline{x}). \tag{22}$$

5.2.2 Axioms for variables

The following rules apply for the linking of variables and constants.

– The orthogonalizing ORing of a variable and constant 0 results in the variable itself (23) and the orthogonalizing ORing of a variable and constant 1 results in 1 (24).

$$x_i \oslash 0 = x_i, \tag{23}$$

$$x_i \oslash 1 = 1. \tag{24}$$

– Furthermore, the orthogonalizing ORing of a variable with the same variable leads to the variable itself (25) and the orthogonalizing ORing of a variable with its negated form leads to the union of both (26). Consequently, this results in 1 at last

$$x_i \oslash x_i = x_i, \tag{25}$$

$$x_i \oslash \bar{x}_i = x_i \lor \bar{x}_i \quad (= 1). \tag{26}$$

5.2.3 Axioms for conjunctions

Following axioms for conjunctions are deduced out of the axioms for variables.

– The neutral element of orthogonalizing ORing is 0:

$$c_i(\underline{x}) \oslash 0 = c_i(\underline{x}). \tag{27}$$

– The result of orthogonalizing ORing between 0 and a conjunction $c_i(\underline{x})$ is this conjunction $c_i(\underline{x})$ itself:

$$0 \oslash c_i(\underline{x}) = c_i(\underline{x}). \tag{28}$$

– The orthogonalizing ORing of a conjunction $c_i(\underline{x})$ with the unit-term 1 leads to 1:

$$c_i(\underline{x}) \oslash 1 = 1. \tag{29}$$

– The result of orthogonalizing ORing between 1 and a conjunction $c_i(\underline{x})$ is 1:

$$1 \oslash c_i(\underline{x}) = 1. \tag{30}$$

– The result of the orthogonalizing ORing between two the same conjunctions $c_i(\underline{x})$ results to this conjunction itself:

$$c_i(\underline{x}) \oslash c_i(\underline{x}) = c_i(\underline{x}). \tag{31}$$

– The orthogonalizing ORing of a conjunction $c_i(\underline{x})$ with its complement $\overline{c_i(\underline{x})}$ results in an unit-term 1:

$$c_i(\underline{x}) \oslash \overline{c_i(\underline{x})} = 1. \tag{32}$$

5.2.4 Commutativity

Commutativity is the property of operation which allows the changing of the terms in their position in such that the value of the expression will not change. As the novel method of orthogonalizing ORing is commutative, the sequence of its execution can be changed.

$$c_1(\underline{x}) \oslash c_2(\underline{x}) = c_2(\underline{x}) \oslash c_1(\underline{x}). \tag{33}$$

The value of both sides are equivalent and orthogonal. They can only differ in the form of coverage. The following Example 5.4 gives an overview about the commutative property.

Example 5.4.

$$x_1 \oslash x_2 \bar{x}_3 = x_2 \bar{x}_3 \oslash x_1$$
$$x_1 \bar{x}_2 \vee x_1 x_2 x_3 \vee x_2 \bar{x}_3 = \bar{x}_1 x_2 \bar{x}_3 \vee x_1.$$

The left side differs only in the form of coverage in contrast to the right side, which is shown in the corresponding K-maps (Fig. 4). Both sides consist of disjointed cubes.

Fig. 4. Left and right side of Ex. 5.4
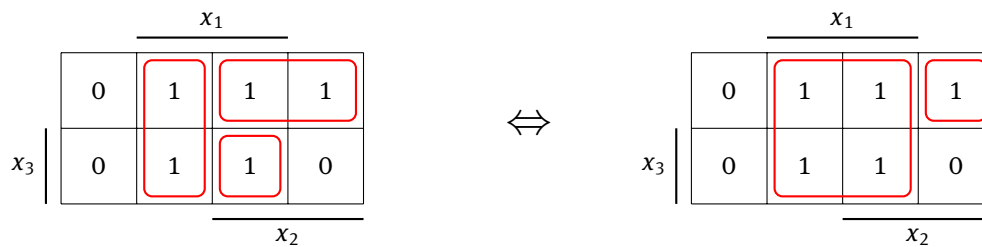
5.2.5 Associativity

Associativity is the property of an operation which allows the rearranging of the parentheses in such that the value of the expression will not change. As orthogonalizing ORing is associative, the position of the parentheses can be changed.

$$\left(c_1(\underline{x}) \oslash c_2(\underline{x})\right) \oslash c_3(\underline{x}) = c_1(\underline{x}) \oslash \left(c_2(\underline{x}) \oslash c_3(\underline{x})\right). \tag{34}$$

The value of both sides are equivalent and orthogonal. Only the form of their coverage can be different. Following Example 5.5 illustrates this characteristic of associativity.

Example 5.5.

$$(x_1 \oslash x_2 \bar{x}_3) \oslash \bar{x}_1 x_2 = x_1 \oslash (x_2 \bar{x}_3 \oslash \bar{x}_1 x_2)$$
$$(x_1 \bar{x}_2 \vee x_1 x_2 x_3 \vee x_2 \bar{x}_3) \oslash \bar{x}_1 x_2 = x_1 \oslash (x_1 x_2 \bar{x}_3 \vee \bar{x}_1 x_2)$$
$$x_1 \bar{x}_2 \vee x_1 x_2 x_3 \vee x_1 x_2 \bar{x}_3 \vee \bar{x}_1 x_2 = x_1 \vee x_1 \vee \bar{x}_1 x_2$$
$$x_1 \bar{x}_2 \vee x_1 x_2 x_3 \vee x_1 x_2 \bar{x}_3 \vee \bar{x}_1 x_2 = x_1 \vee \bar{x}_1 x_2.$$

Both sides are homogeneous and orthogonal as shown in the K-maps in Figure 5. They only differ in their form of coverage.

Fig. 5. Left and right side of Ex. 5.5
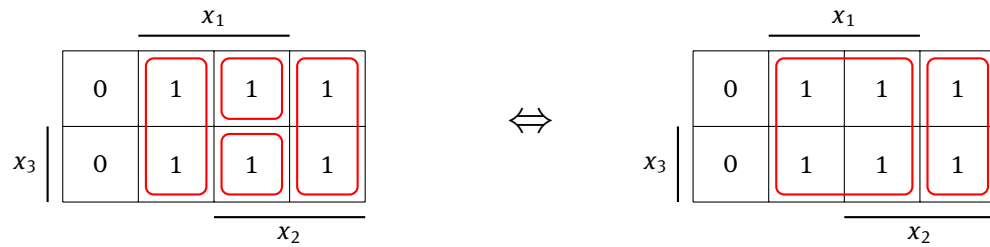
5.2.6 Distributivity

The distributive property of an operation allows the exclusion of the same term. That means, that a term can be factored out. Hereby, the orthogonality of both sides has to be insisted. In this case, the distributive law for ANDing out applies for left and right side.

$$c_1(\underline{x}) \cdot \left(c_2(\underline{x}) \oslash c_3(\underline{x})\right) = \left(c_1(\underline{x}) \cdot c_2(\underline{x})\right) \oslash \left(c_1(\underline{x}) \cdot c_3(\underline{x})\right). \tag{35}$$

The validity of the distributive property is given by the following proof:

$$c_1(\underline{x}) \cdot \left(c_2(\underline{x})\overline{c_3(\underline{x})} \vee c_3(\underline{x})\right) = c_1(\underline{x}) \cdot c_2(\underline{x}) \cdot \overline{c_3(\underline{x})} \vee c_1(\underline{x}) \cdot c_3(\underline{x})$$

$$c_1(\underline{x}) \cdot c_2(\underline{x}) \cdot \overline{c_3(\underline{x})} \vee c_1(\underline{x}) \cdot c_3(\underline{x}) = c_1(\underline{x}) \cdot c_2(\underline{x}) \cdot \overline{c_3(\underline{x})} \vee c_1(\underline{x}) \cdot c_3(\underline{x}).$$

Both sides are equivalent and orthogonal. They can only differ in the form of their coverage. This charasteristic of distributivity is demonstrated by the following Example 5.6 whereby both sides result to the same term.

Example 5.6.

$$x_1 \cdot (x_2\bar{x}_3 \oslash x_1x_2) = (x_1 \cdot x_2\bar{x}_3) \oslash (x_1 \cdot x_1x_2)$$

$$x_1 \cdot (\bar{x}_1x_2\bar{x}_3 \vee x_1x_2) = x_1x_2\bar{x}_3 \oslash x_1x_2$$

$$x_1x_2 = x_1x_2.$$

6 Disjointed sum of products

Based on this technique of orthogonalizing ORing a novel Equation (36) is formed which enables the orthogonalization of every disjunctive form DF. That means, this formula enables the transformation of a SOP in a homogeneous dSOP. With $m \in \mathbb{N}$ as the number of conjunctions $c_i(\underline{x})$ that are included in the given DF, which has to be orthogonalized, it follows that:

$$f_{DF}(\underline{x})^{orth} = \bigoplus_{i=1}^{m} c_i(\underline{x}) = c_1(\underline{x}) \oslash c_2(\underline{x}) \oslash \ldots \oslash c_{m-1}(\underline{x}) \oslash c_m(\underline{x}). \tag{36}$$

The explanation for Equation (36) is provided as follows:
– Orthogonalizing ORing is realized between the first and the second conjunction, $c_1(\underline{x}) \oslash c_2(\underline{x})$, by Eq. (15). After that, orthogonalizing ORing is calculated between the result of $\left(c_1(\underline{x}) \oslash c_2(\underline{x})\right)$ and the third conjunction, $\left(c_1(\underline{x}) \oslash c_2(\underline{x})\right) \oslash c_3(\underline{x})$, by Eq. (19).
– This procedure is continued until the last conjunction $c_m(\underline{x})$.

The general validity of (36) is proven by the following mathematical induction:

Proof. 2

- $\forall (m) \in \mathbb{N}, \quad (m) \geq m_0$ applies $A(m)$:

$$\overset{m}{\underset{i=1}{\bigcirc\!\!\!\vee}} c_i(\underline{x}) = c_1(\underline{x}) \varoslash c_2(\underline{x}) \varoslash \ldots \varoslash c_m(\underline{x}).$$

- if $A(m_0) \wedge \big(\forall(m) \in \mathbb{N}, \ m \geq m_0 : A(m) \to A(m+1)\big) \Rightarrow \forall(m) \in \mathbb{N}, \ m \geq m_0 : A(m)$

- Basis $A(m_0)$: $m_0 = 2$

$$\overset{2}{\underset{i=1}{\bigcirc\!\!\!\vee}} c_i(\underline{x}) = c_1(\underline{x}) \varoslash c_2(\underline{x})$$

$$c_1(\underline{x}) \varoslash c_2(\underline{x}) = c_1(\underline{x}) \varoslash c_2(\underline{x}).$$

- Inductive step $A(m) \to A(m+1)$: $m \to m+1$

$$\overset{m+1}{\underset{i=1}{\bigcirc\!\!\!\vee}} c_i(\underline{x}) = c_1(\underline{x}) \varoslash c_2(\underline{x}) \varoslash \ldots \varoslash c_m(\underline{x}) \varoslash c_{(m+1)}(\underline{x})$$

$$\overset{m}{\underset{i=1}{\bigcirc\!\!\!\vee}} c_i(\underline{x}) \varoslash c_{(m+1)}(\underline{x}) = \underbrace{c_1(\underline{x}) \varoslash c_2(\underline{x}) \varoslash \ldots \varoslash c_m(\underline{x})}_{\overset{m}{\underset{i=1}{\bigcirc\!\!\!\vee}} c_i(\underline{x})} \varoslash c_{(m+1)}(\underline{x}).$$

□

The orthogonal result may differ depending on the order of the conjunctions because of the property of commutativity. Thus, the conjunctions can be changed if necessary. However, all solutions are equivalent and orthogonal. In Example 6.1 it is given an overview of the use of Eq. (36)

Example 6.1. *Function $f_2(\underline{x}) = x_3 \vee x_1 \vee x_2 \bar{x}_3$ has to be orthogonalized by Eq.* (36).

$$f_2(\underline{x})^{orth} = x_3 \varoslash x_1 \varoslash x_2 \bar{x}_3 = (\bar{x}_1 x_3 \vee x_1) \varoslash x_2 \bar{x}_3 = \bar{x}_1 x_3 \vee x_1 \bar{x}_2 \vee x_1 x_2 x_3 \vee x_2 \bar{x}_3.$$

Function $f_2(\underline{x})^{orth}$ is the orthogonal form of function $f_2(\underline{x})$, illustrated in the K-maps (Fig. 6).

Fig. 6. $f_2(\underline{x})$, $f_2(\underline{x})^{orth}$ and $f_2(\underline{x})^{orth}_{sort}$ in K-maps
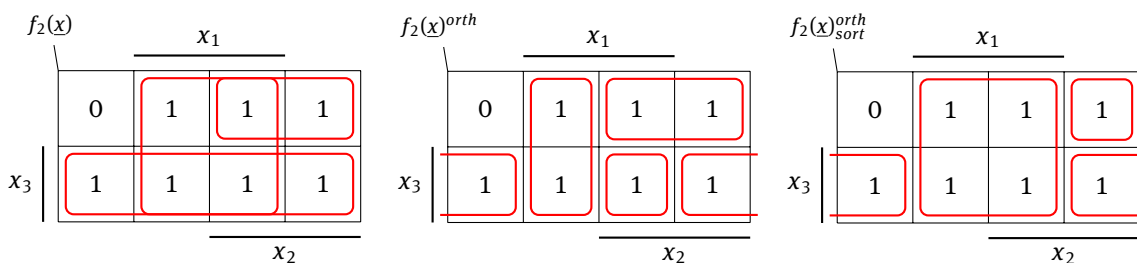

By rearrangement of the order of the consisting conjunctions of a DF we obtain fewer number of conjunctions in the derived orthogonal form. This procedure of sorting is carried out from large to small. That means, it takes place from conjunctions of higher number of variables to conjunctions of fewer number of variables. The following Example clarifies this advantage of sorting.

Example 6.2. *Function $f_2(\underline{x}) = x_3 \vee x_1 \vee x_2\bar{x}_3$ has to be orthogonalized after resorting.*

$$f_2(\underline{x})^{orth}_{sort} = x_2\bar{x}_3 \oslash x_3 \oslash x_1 = (x_2\bar{x}_3 \vee x_3)\oslash x_1 = \bar{x}_1x_2\bar{x}_3 \vee \bar{x}_1x_3 \vee x_1.$$

The orthogonalized form of the sorted DF contains fewer number of conjunctions (see Fig. 6).

Fig. 7. Average number of conjunctions in the orthogonal result
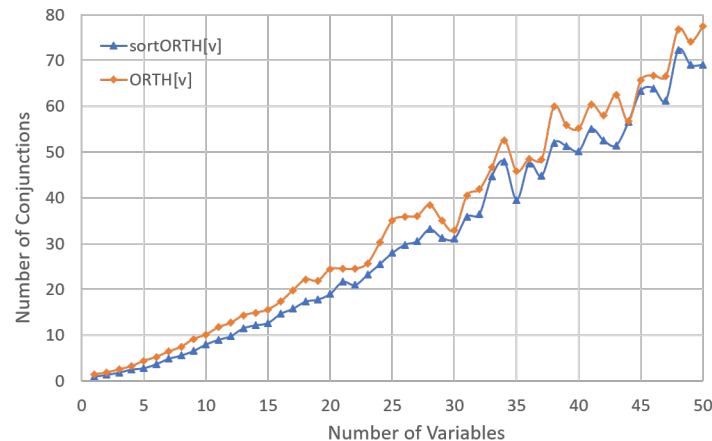
The analysis of a measurement, as shown in Fig. 7, gives an overview of the comparison of the orthogonalization process depending on sorting. The orthogonalization of unsorted DF called as $ORTH[\vee]$ and the sorted of the same DFs called as $sortORTH[\vee]$ are compared. Thereby, 50 different functions with five conjunctions were orthogonalized by the novel technique with regards to the tuple length (number of variables) which runs from 1 to 50. Out of these 50 calculations per number of variables the number of conjunctions were determined, from which an average value was calculated for each tuple length. These range of averages were subsequently plotted in the diagram to show the deviation of $sortORTH[\vee]$ from $ORTH[\vee]$. This comparison illustrates a reduction of conjunctions in the orthogonal form when the DF was sorted before. This reduction is approximately 19% on average (see Table 2). Additionally, this feature allows the reducing of operation for subsequent calculation steps. Hereby, the relation in (37) is confirmed by the comparison in Fig. 7. The number of conjunctions of an orthogonalized DF $\left(\mathrm{P}^s_{c(\underline{x})}(f(\underline{x})^{orth}_{sort})\right)$ which was sorted before ($sortORTH[\vee]$) is smaller than the number of conjunctions of the orthogonalized DF $\left(\mathrm{P}_{c(\underline{x})}(f(\underline{x})^{orth})\right)$ which was not sorted before ($ORTH[\vee]$):

$$\mathrm{P}^s_{c(\underline{x})}(f(\underline{x})^{orth}_{sort}) \leq \mathrm{P}_{c(\underline{x})}(f(\underline{x})^{orth}). \tag{37}$$

7 Conclusion

This work showed a novel technique for building a union of disjointed conjunctions which is called as orthogonalizing ORing. Its results are orthogonal. Orthogonalizing ORing is used to calculate the orthogonal form of building the union of two conjunctions. This linking technique replaces two calculation steps - building a union and the subsequent orthogonalization - by one step. Orthogonalizing ORing is valid in general, which was proven by the mathematical induction, and is also equivalent to the usual method of union $\vee$. Additionally, postulates related to commutativity, distributivity and associativity and axioms for this method are also defined. Furthermore, every Boolean function of disjunctive form or every Sum of Products, respectively, can easily be orthogonalized mathematically by a novel equation which is based on this linking technique of orthogonalizing ORing. By this orthogonalization a disjointed Sum of Products

can be reached in a simpler way. The general validity was also proven by the mathematical induction. An additional step of sorting before the step of orthogonalization achieves a reduction of approximately 19% of the number of conjunctions in the orthogonal result. This feature was illustrated by a measurement whereby the orthogonalization took place before and after sorting.

Table 2. Average number of conjunctions in the orthogonal result

x_n	sortORTH[$\vee$]	ORTH[$\vee$]	deviation in %
1	1	1.48	48.0
2	1.38	1.84	33.3
3	1.88	2.54	35.1
4	2.54	3.24	27.6
5	2.84	4.4	54.9
⋮	⋮	⋮	⋮
26	29.76	35.88	20.6
27	30.54	36.06	18.1
28	33.16	38.44	15.9
29	31.26	34.98	11.9
⋮	⋮	⋮	⋮
46	63.96	66.72	4.3
47	61.26	66.52	8.6
48	72.28	76.84	6.3
49	69.02	74.06	7.3
50	69.02	77.46	12.2
average deviation in %			19.3

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