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Singular integrals with variable kernel and fractional differentiation in homogeneous Morrey-Herz-type Hardy spaces with variable exponents

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Abstract: Let T be the singular integral operator with variable kernel defined by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n} f(y) dy$$

and D^γ ($0 \leq \gamma \leq 1$) be the fractional differentiation operator. Let T^* and $T^\#$ be the adjoint of T and the pseudo-adjoint of T , respectively. The aim of this paper is to establish some boundedness for $TD^\gamma - D^\gamma T$ and $(T^* - T^\#)D^\gamma$ on the homogeneous Morrey-Herz-type Hardy spaces with variable exponents $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n)$ via the convolution operator $T_{m,j}$ and Calderón-Zygmund operator, and then establish their boundedness on these spaces. The boundedness on $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n)$ is shown to hold for $TD^\gamma - D^\gamma T$ and $(T^* - T^\#)D^\gamma$. Moreover, the authors also establish various norm characterizations for the product $T_1 T_2$ and the pseudo-product $T_1 \circ T_2$.

Keywords: Variable kernel, Fractional differentiation, Sobolev spaces $I_\gamma(BMO)$, Morrey-Herz-type Hardy space with variable exponents

MSC: 42B20, 42B25, 42B35

1 Introduction and main results

Let $\Omega(x, z) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function and satisfy the following conditions:

$$\Omega(x, \lambda z) = \Omega(x, z), \text{ for any } x, z \in \mathbb{R}^n \text{ and } \lambda > 0, \quad (1.1)$$

$$\int_{S^{n-1}} \Omega(x, z') d\sigma(z') = 0, \text{ for any } x \in \mathbb{R}^n, \quad (1.2)$$

where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) with normalized Lebesgue measure $d\sigma$. Then the singular integral operator with variable kernel $\Omega(x, z)$ is defined by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n} f(y) dy. \quad (1.3)$$

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Boundedness properties of the above operator in a variety of functional spaces have been extensively studied. In particular, Calderón and Zygmund proved that T is bounded on the $L^2(\mathbb{R}^n)$ in the Mihlin conditions (see [1]). Other references with results of this sort include [2-5] and the references within. On the other hand, these estimates played an important role in the theory of non-divergent elliptic equations with discontinuous coefficients (see [6,7]).

Let $0 \leq \gamma \leq 1$. For tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ ($n = 1, 2, \dots$), the fractional differentiation operators D^γ of order γ are defined by $\widehat{D^\gamma f}(\xi) = |\xi|^\gamma \widehat{f}(\xi)$, i.e., $D^\gamma f(x) = (|\xi|^\gamma \widehat{f}(\xi))^\vee(x)$. We will denote by I_γ the Riesz potential operator of order γ that is defined on the space of tempered distributions modulo polynomials by setting $\widehat{I_\gamma f}(\xi) = |\xi|^{-\gamma} \widehat{f}(\xi)$. It is easy to see that a locally integrable function b belongs to $I_\gamma(\text{BMO})(\mathbb{R}^n)$ if and only if $D^\gamma b \in \text{BMO}(\mathbb{R}^n)$. Strichartz (see [8]) showed that, for $\gamma \in (0, 1)$, $I_\gamma(\text{BMO})(\mathbb{R}^n)$ is a space of functions modulo constants which is properly contained in $\text{Lip}_\gamma(\mathbb{R}^n)$.

We make some conventions. In what follows, χ_E denotes the characteristic function for a μ -measurable set E . We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \leq CB$. For any index $p \in (1, \infty)$, we denote by p' its conjugate index, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

Denote T^* and $T^\sharp$ to be the adjoint of T and the pseudo-adjoint of T respectively (see (3.2) and (3.3) below). Let T_1 and T_2 be the operators defined in (1.1) which are differentiated by its kernel $\Omega_1(x, y)$ and $\Omega_2(x, y)$. Let $T_1 T_2$, $T_1 \circ T_2$ denote the product and pseudo-product of T_1 and T_2 , respectively. In 1957, Calderón and Zygmund found that these operators are closely related to the second order linear elliptic equations with variable coefficients and established the following results of the operators T_1^* , $T_1^\sharp$, $T_1 T_2$, $T_1 \circ T_2$ and D on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) (see [1]).

Theorem A ([1]). Let $1 < p < \infty$, $\Omega_1(x, y)$, $\Omega_2(x, y) \in C^\beta(C^\infty)$, $\beta > 1$ satisfy (1.1) and (1.2). Then there is a constant C such that

- (1) $\|(T_1 D - D T_1)f\|_{L^p} \lesssim \|f\|_{L^p}$;
- (2) $\|(T_1^* - T_1^\sharp)Df\|_{L^p} \lesssim \|f\|_{L^p}$;
- (3) $\|(T_1 \circ T_2 - T_1 T_2)Df\|_{L^p} \lesssim \|f\|_{L^p}$.

In 2015, Chen and Zhu proved that Theorem A was also true on Weighted Lebesgue space and Morrey space (see [9]). In 2016, Tao and Yang obtained the boundedness of those operators on the weighted Morrey-Herz spaces (see [5]). Later, the boundedness of those operators on the Lebesgue spaces with variable exponents were obtained [10]. Inspired by the ideas mentioned previously, the aim of this paper is to deal with the boundedness of the singular integrals with variable kernel and fractional differentiations in the setting of the Morrey-Herz-type Hardy Spaces with variable exponents (which will be defined in the next section).

The main theorems are presented in this section. The definitions of the Morrey-Herz spaces with variable exponents, the Morrey-Herz-type Hardy spaces with variable exponents and the preliminary lemmas are presented in Section 2. In Section 3, we will introduce the spherical harmonical expansions and give the boundedness of $T_{m,j}$. The proofs of Theorems are given in Section 4.

Theorem 1.1. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$, and $0 \leq \lambda < \infty$. If $\alpha(\cdot)$ is a bounded and log-Hölder continuous both at the origin and infinity such that $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0)$, $\alpha_\infty \leq n\delta_2 + \delta$ with some $\delta > \max\{\alpha(0) - \delta_2, \alpha_\infty - n\delta_2\}$ and δ_2 as in Lemma 2.6. Assume that T is defined by (1.3) and $\Omega(x, y)$, which satisfies (1.1), (1.2), meet a condition

$$\max_{|j| \leq 2n} \|D_x^\gamma (\partial^j / \partial y^j) \Omega(x, y)\|_{L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})} < \infty, \quad (1.4)$$

then we have

- (1) $\|(TD^\gamma - D^\gamma T)f\|_{\dot{M}_{p(\cdot), \lambda}^{\alpha(\cdot), q}} \lesssim \|f\|_{\dot{H}M_{p(\cdot), \lambda}^{\alpha(\cdot), q}}$;
- (2) $\|(T^* - T^\sharp)D^\gamma f\|_{\dot{M}_{p(\cdot), \lambda}^{\alpha(\cdot), q}} \lesssim \|f\|_{\dot{H}M_{p(\cdot), \lambda}^{\alpha(\cdot), q}}$.

Theorem 1.2. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$, and $0 \leq \lambda < \infty$. If $\alpha(\cdot)$ is a bounded and log-Hölder continuous both at the origin and infinity such that $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0)$, $\alpha_\infty \leq n\delta_2 + \delta$ with some $\delta > \max\{\alpha(0) - \delta_2, \alpha_\infty - n\delta_2\}$ and δ_2 as in Lemma 2.6. Suppose that $\Omega_1(x, y)$ and $\Omega_2(x, y)$ satisfy (1.1) and (1.2). If $\Omega_2(x, y)$ satisfies (1.4) and

$\Omega_1(x, y)$ satisfies

$$\max_{|j| \leq 2n} \|(\partial^j / \partial y^j) \Omega_1(x, y)\|_{L^\infty(\mathbb{R}^n \times S^{n-1})} < \infty, \quad (1.5)$$

then we have

$$\|(T_1 \circ T_2 - T_1 T_2) D^\gamma f\|_{M\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}} \lesssim \|f\|_{H\dot{M}\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}}.$$

Theorem 1.3. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$, and $0 \leq \lambda < \infty$. If $\alpha(\cdot)$ is a bounded and log-Hölder continuous both at the origin and infinity such that $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0)$, $\alpha_\infty \leq n\delta_2 + \delta$ with some $\delta > \max\{\alpha(0) - \delta_2, \alpha_\infty - n\delta_2\}$ and δ_2 as in Lemma 2.6. Suppose that $\Omega_i(x, y)$ ($i = 1, 2$) satisfies (1.1), (1.2) and (1.5), then we have

- (1) $\|(T_1 \mathcal{I} - \mathcal{I} T_1) f\|_{M\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}} \lesssim \|f\|_{H\dot{M}\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}};$
- (2) $\|(T_1^* - T_1^\sharp) \mathcal{I} f\|_{M\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}} \lesssim \|f\|_{H\dot{M}\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}};$
- (3) $\|(T_1 \circ T_2 - T_1 T_2) \mathcal{I} f\|_{M\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}} \lesssim \|f\|_{H\dot{M}\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}}.$

Theorem 1.4. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$, and $0 \leq \lambda < \infty$. If $\alpha(\cdot)$ is a bounded and log-Hölder continuous both at the origin and infinity such that $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0)$, $\alpha_\infty \leq n\delta_2 + \delta$ with some $\delta > \max\{\alpha(0) - \delta_2, \alpha_\infty - n\delta_2\}$ and δ_2 as in Lemma 2.6. Suppose that $\Omega(x, y)$ satisfies (1.1), (1.2) and

$$\max_{|j| \leq 2n} \|\nabla_x (\partial^j / \partial y^j) \Omega(x, y)\|_{L^\infty(\mathbb{R}^n \times S^{n-1})} < \infty, \quad (1.6)$$

then we have

- (1) $\|(TD - DT) f\|_{M\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}} \lesssim \|f\|_{H\dot{M}\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}};$
- (2) $\|(T^* - T^\sharp) Df\|_{M\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}} \lesssim \|f\|_{H\dot{M}\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}}.$

Theorem 1.5. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$, and $0 \leq \lambda < \infty$. If $\alpha(\cdot)$ is a bounded and log-Hölder continuous both at the origin and infinity such that $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0)$, $\alpha_\infty \leq n\delta_2 + \delta$ with some $\delta > \max\{\alpha(0) - \delta_2, \alpha_\infty - n\delta_2\}$ and δ_2 as in Lemma 2.6. Suppose that $\Omega_1(x, y)$ and $\Omega_2(x, y)$ satisfies (1.1), (1.2). If $\Omega_1(x, y)$ satisfies (1.5) and $\Omega_2(x, y)$ satisfies (1.6), then we have

$$\|(T_1 \circ T_2 - T_1 T_2) Df\|_{M\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}} \lesssim \|f\|_{H\dot{M}\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}}.$$

2 Definitions and preliminaries

In this section, the Morrey-Herz spaces with variable exponents $M\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}$ and the Morrey-Herz-type Hardy spaces with variable exponents $H\dot{M}\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q}$ will be introduced. Some preliminary lemmas will be given as well.

Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ become one of important function spaces due to the fundamental paper [11] by Kováčik Rákosník. In the past 20 years, the theory of these spaces have made progress rapidly. On the other hand, the function spaces with variable exponent have been applied in fluid dynamics, elasticity dynamics, calculus of variations and differential equations with non-standard growth conditions (see [12-16]). In [17], authors proved the extrapolation theorem which leads to the boundedness of some classical operators including the commutators on $L^{p(\cdot)}(\mathbb{R}^n)$. Karlovich and Lerner also obtained the boundedness of the singular integral commutators in [18]. The boundedness of some typical operators has been studied with keen interest (see [18-25]). Recently, Xu and Yang have introduced the Morrey-Herz-type Hardy spaces with variable exponents and established the boundedness of singular integral operators on these spaces in [26].

Definition 2.1 ([21]). Let $\alpha(\cdot)$ be a real-valued function on $\mathbb{R}^n$. If there exist $C > 0$ such that for any $x, y \in \mathbb{R}^n$, $|x - y| < 1/2$,

$$|\alpha(x) - \alpha(y)| \leq \frac{C}{-\log(|x - y|)},$$

then $\alpha(\cdot)$ is said to be local log-Hölder continuous on $\mathbb{R}^n$.

If there exist $C > 0$ such that for all $x \in \mathbb{R}^n$,

$$|\alpha(x) - \alpha(0)| \leq \frac{C}{\log(e + 1/|x|)},$$

then $\alpha(\cdot)$ is said to be log-Hölder continuous at origin.

If there exist $\alpha_\infty \in \mathbb{R}$ and a constant $C > 0$ such that all $x \in \mathbb{R}^n$,

$$|\alpha(x) - \alpha_\infty| \leq \frac{C}{\log(e + |x|)},$$

then $\alpha(\cdot)$ is said to be log-Hölder continuous at infinity.

Definition 2.2 ([11]). Let $p : \mathbb{R}^n \rightarrow [1, \infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f \text{ is measurable} : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx < \infty \text{ for some constant } \eta > 0 \right\}.$$

Equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \eta > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx \leq 1 \right\}$$

we denote

$$p_- = \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^n\}, \quad p_+ = \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^n\}.$$

Then $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Let M be the Hardy-Littlewood maximal operator. We denote $\mathcal{B}(\mathbb{R}^n)$ to be the set of all functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

We now make some conventions. Throughout this paper, let $k \in \mathbb{Z}$, $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$, $L \in \mathbb{Z}$ and $\chi_k = \chi_{C_k}$.

Definition 2.3 ([26]). Let $0 < q \leq \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $0 \leq \lambda < \infty$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous Morrey-Herz space $\dot{M}_{p(\cdot), \lambda}^{\alpha(\cdot), q}$ is defined by

$$\dot{M}_{p(\cdot), \lambda}^{\alpha(\cdot), q}(\mathbb{R}^n) = \{f \in L_{\operatorname{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{M}_{p(\cdot), \lambda}^{\alpha(\cdot), q}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{M}_{p(\cdot), \lambda}^{\alpha(\cdot), q}(\mathbb{R}^n)} = \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L \|2^{\alpha(\cdot)k} f \chi_k\|_{L^{p(\cdot)}}^q \right\}^{1/q},$$

with the corresponding modification for $q = \infty$.

Next let us recall the definition of Morrey-Herz-type Hardy spaces with variable exponents $HM_{p(\cdot), \lambda}^{\alpha(\cdot), q}$, which was firstly introduced by Xu and Yang in [26]. To do this, we need some notations. $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz spaces of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) = \sup_{\phi \in \mathcal{A}_N} |\phi_{\nabla}^*(f)(x)|, \quad x \in \mathbb{R}^n,$$

where $\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\nu|, |\beta| \leq N, \forall x \in \mathbb{R}^n} |x^\nu D^\beta \phi(x)| \leq 1\}$ and $N > n + 1$ and ϕ_{∇}^* is the nontangential maximal operator defined by $\phi_{\nabla}^*(f)(x) = \sup_{|y-x| < t} |\phi_t * f(y)|$, where $\phi_t(x) = t^{-n} \phi(\frac{x}{t})$ for any $x \in \mathbb{R}^n$ and $t > 0$.

The grand maximal G_N was firstly introduced by Fefferman and Stein in [27] to study classical Hardy spaces. We refer the reader to [28-30] for details on the classical Hardy spaces. The variable exponent case is shown in [22] by Nakai and Sawano.

Definition 2.4 ([26]). Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $0 < q \leq \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \leq \lambda < \infty$, and $N > n + 1$. The homogeneous Morrey-Herz-type Hardy space with variable exponents $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ is defined by

$$HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} = \|G_N f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} < \infty \right\}.$$

Obviously, if $\alpha(\cdot) = \alpha$ and $\lambda = 0$, these spaces were considered by Wang and Liu in [23]. If $p(\cdot)$ and $\alpha(\cdot)$ are constant and $\lambda = 0$, these are the classical Herz type Hardy spaces (see [31]).

Definition 2.5 ([26]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and infinity, and nonnegative integer $s \geq [\alpha_r - n\delta_2]$, here $\alpha_r = \alpha(0)$, if $r < 1$, and $\alpha_r = \alpha_\infty$, if $r \geq 1$, $n\delta_2 \leq \alpha_r < \infty$ and δ_2 as in Lemma 2.6.

(1) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$ -atom, if it satisfies (1) $\text{supp } a \subset B(0, r)$; (2) $\|a\|_{L^{p(\cdot)}} \leq |B(0, r)|^{-\alpha_r/n}$; (3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$, $|\beta| \leq s$.

(2) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$ -atom of restricted type, if it satisfies (1) $\text{supp } a \subset B(0, r)$, $r \geq 1$; (2) $\|a\|_{L^{p(\cdot)}} \leq |B(0, r)|^{-\alpha_r/n}$; (3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$, $|\beta| \leq s$.

Lemma 2.6 ([18]). If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exist constants $\delta_1, \delta_2 > 0$, such that for all balls $B \subset \mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim \frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

Lemma 2.7 ([21, Theorem 13]). Let $0 < q < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 \leq \lambda < \infty$, and $\alpha(\cdot) \in L^\infty$ be log-Hölder continuous both at the origin and infinity, $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0)$, $\alpha_\infty < \infty$ with δ_2 as in Lemma 2.5, then $f \in HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ if and only if $f = \sum_{k=-\infty}^\infty \lambda_k a_k$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where each a_k is a central $(\alpha(\cdot), p(\cdot))$ -atom with support contained in B_k and $\sup_{L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=-\infty}^L |\lambda_k|^q < \infty$. Moreover,

$$\|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \approx \inf \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^L |\lambda_k|^q \right)^{1/q},$$

where the infimum is taken over all above decompositions of f .

Lemma 2.8 ([11]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}$ and $g \in L^{p'(\cdot)}$, then fg is integrable on $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq r_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where $r_p = 1 + 1/p^- - 1/p^+$.

Lemma 2.9 ([26]). Let $\alpha(\cdot)$ be a bounded and log-Hölder continuous both at the origin and infinity such that $n\delta_2 \leq \alpha(0)$, $\alpha_\infty < n\delta_2 + \delta$ with some $\delta > \max\{\alpha(0) - n\delta_2, \alpha_\infty - n\delta_2\}$ and δ_2 as in Lemma 2.6. Suppose T is a Calderón-Zygmund operator associated to a standard kernel K . If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < \lambda < \infty$ and $0 \leq q < \infty$, then we have

$$\|Tf\|_{M\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}} \lesssim \|f\|_{HM\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}}.$$

Lemma 2.10 ([26]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q \in (0, \infty]$, and $\lambda \in (0, \infty]$. If $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, then

$$\|f\|_{M\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}} \approx \max \left\{ \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^L 2^{kq\alpha(0)} \|f\chi_k\|_{L^{p(\cdot)}}^q \right)^{1/q}, \right. \\ \left. \sup_{L > 0, L \in \mathbb{Z}} \left[2^{-L\lambda} \left(\sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|f\chi_k\|_{L^{p(\cdot)}}^q \right)^{1/q} + 2^{-L\lambda} \left(\sum_{k=0}^L 2^{kq\alpha_\infty} \|f\chi_k\|_{L^{p(\cdot)}}^q \right)^{1/q} \right] \right\}.$$

3 Spherical harmonics and boundedness of $T_{m,l}$

In this section, we will recall the spherical harmonical expansion and give the boundedness of $T_{m,l}$, which are very vital in our proofs of Theorems.

We let $\mathcal{H}_m$ denote the space of spherical harmonics homogeneous polynomials of degree m . Let $\dim \mathcal{H}_m = d_m$ and $\{Y_{m,j}\}_{j=1}^{d_m}$ be an orthonormal system of $\mathcal{H}_m$. It is showed that $\{Y_{m,j}\}_{j=1}^{d_m}$, $m = 0, 1, \dots$, is a complete orthonormal system in $L^2(S^{n-1})$ (see [32]). We can expand the kernel $\Omega(x, z')$ in spherical harmonics as

$$\Omega(x, z') = \sum_{m \geq 0} \sum_{j=1}^{d_m} a_{m,j}(x) Y_{m,j}(z'),$$

where

$$a_{m,j}(x) = \int_{S^{n-1}} \Omega(x, z') \overline{Y_{m,j}(z')} d\sigma(z').$$

If $\int_{S^{n-1}} \Omega(x, z') d\sigma(z') = 0$, then $a_{0,j} = 0$ for any $x \in \mathbb{R}^n$. Let

$$T_{m,j}f(x) = \left(\frac{Y_{m,j}}{|\cdot|^n} * f \right)(x). \quad (3.1)$$

Then T , defined in (1.3), can be written as

$$Tf(x) = \sum_{m \geq 1} \sum_{j=1}^{d_m} a_{m,j}(x) T_{m,j}f(x).$$

Let T^* and $T^\#$ be the adjoint of T and the pseudo-adjoint of T respectively, defined by

$$T^*f(x) = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} (-1)^m T_{m,j}(\overline{a_{m,j}}f)(x) \quad (3.2)$$

and

$$T^\#f(x) = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} (-1)^m \overline{a_{m,j}}(x) T_{m,j}f(x). \quad (3.3)$$

Lemma 3.1. Let $\alpha(\cdot) \in L^\infty$ be log-Hölder continuous both at the origin and infinity, $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0)$, $\alpha_\infty < n\delta_2 + \delta$ with some $\delta > \max\{\alpha(0) - n\delta_2, \alpha_\infty - n\delta_2\}$ and δ_2 as in Lemma 2.6. The $T_{m,j}$ defined by (3.1) is bounded from $HM\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}$ to $M\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}$, i.e.

$$\|T_{m,j}f\|_{M\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}} \lesssim \|f\|_{HM\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}}.$$

Proof. Suppose $f \in M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$. By Lemma 2.7, $f = \sum_{j=-\infty}^{\infty} \lambda_j b_j$ converges in $S'(\mathbb{R}^n)$, where each b_j is a central $(\alpha(\cdot), q(\cdot))$ -atom with support contained in B_j and

$$\|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \approx \inf_{L \in \mathbb{Z}} \sup 2^{-L\lambda} \left(\sum_{j=-\infty}^L |\lambda_j|^q \right)^{\frac{1}{q}}.$$

For simplicity, we denote $\Phi = \sum_{L \in \mathbb{Z}} 2^{-L\lambda} \sum_{j=-\infty}^L |\lambda_j|^q$. By Lemma 2.10, we have

$$\begin{aligned} \|T_{m,j}f\|_{M\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}}^q &\approx \max\left\{ \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \|(T_{m,j}f)\chi_k\|_{L^{p(\cdot)}}^q, \right. \\ &\quad \left. \sup_{L > 0, L \in \mathbb{Z}} 2^{-L\lambda q} \left(\sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|(T_{m,j}f)\chi_k\|_{L^{p(\cdot)}}^q + \sum_{k=0}^L 2^{kq\alpha(\infty)} \|(T_{m,j}f)\chi_k\|_{L^{p(\cdot)}}^q \right) \right\} \\ &=: \max\{I, II + III\}, \end{aligned}$$

where

$$I = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \|(T_m, lf)\chi_k\|_{L^p(\cdot)}^q, II = \sup_{L > 0, L \in \mathbb{Z}} \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|(T_m, lf)\chi_k\|_{L^p(\cdot)}^q,$$

$$III = \sup_{L > 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha(\infty)} \|(T_m, lf)\chi_k\|_{L^p(\cdot)}^q.$$

To complete our proof, we only need to show that $I, II, III \lesssim m^{nq/2}\Phi$. First, we estimate I :

$$I = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \|(T_m, lf)\chi_k\|_{L^p(\cdot)}^q$$

$$\lesssim \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| \| [T_m, l b_j] \chi_k \|_{L^p(\cdot)} \right)^q$$

$$+ \lesssim \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \| [T_m, l b_j] \chi_k \|_{L^p(\cdot)} \right)^q$$

$$=: I_1 + I_2.$$

By the result that $T_{m,l}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (see [10]), we have

$$\|[(T_m, l b_l)\chi_k]\|_{L^p(\cdot)} \lesssim m^{n/2} \|b_j\|_{L^p(\cdot)} \lesssim m^{n/2} |B_j|^{-\alpha_j/n} = m^{n/2} 2^{-\alpha_j j}.$$

Therefore, when we get $0 < q \leq 1$, we get

$$I_1 \lesssim \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| m^{n/2} 2^{-\alpha_j j} \right)^q$$

$$\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \times \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{j=k}^{-1} |\lambda_j|^q 2^{-\alpha(0)jq} + \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq} \right)$$

$$\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q}$$

$$+ m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq}$$

$$\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q}$$

$$+ m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-j\lambda q} |\lambda_j|^q 2^{(\lambda-\alpha_{\infty})jq} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq}$$

$$\leq m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^L |\lambda_j|^q + m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q}$$

$$+ m^{nq/2} \Phi \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{(\lambda-\alpha_{\infty})jq} \sum_{k=-\infty}^L 2^{\alpha(0)kq-L\lambda q}$$

$$\lesssim m^{nq/2} \Phi + m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{-1} 2^{-j\lambda q} |\lambda_j|^q 2^{(j-L)\lambda q} \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q} + m^{nq/2} \Phi$$

$$\lesssim m^{nq/2} \Phi + m^{nq/2} \Phi \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{-1} 2^{(j-L)\lambda q} \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q}$$

$$\lesssim m^{nq/2} \Phi.$$

As $1 < q < \infty$, we can obtain

$$I_1 \lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| 2^{-\alpha_j j} \right)^q$$

$$\begin{aligned}
&\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{j=k}^{-1} |\lambda_j| 2^{\alpha(0)(k-j)} \right)^q \\
&+ m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=0}^{\infty} |\lambda_j| 2^{-\alpha_{\infty} j} \right)^q \\
&\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q/2} \right) \times \left(\sum_{j=k}^{-1} 2^{\alpha(0)(k-j)q'/2} \right)^{q/q'} \\
&+ m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty} j} \right) \times \left(\sum_{j=0}^{\infty} 2^{-\alpha_{\infty} j} \right)^{q/q'} \\
&\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q/2} \\
&+ m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty} j} \\
&\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q/2} \\
&+ m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{-j\lambda q} |\lambda_j|^q 2^{(\lambda - \alpha_{\infty}/2)jq} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \\
&\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^L |\lambda_j|^q + m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q/2} \\
&+ m^{nq/2} \Phi \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{(\lambda - \alpha_{\infty}/2)jq} \sum_{k=-\infty}^L 2^{\alpha(0)kq - L\lambda q} \\
&\lesssim m^{nq/2} 2^{-L\lambda q} \sum_{j=-\infty}^L |\lambda_j|^q + m^{nq/2} 2^{-L\lambda q} \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q/2} \\
&+ m^{nq/2} \Phi \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{(\lambda - \alpha_{\infty}/2)jq} \sum_{k=-\infty}^L 2^{\alpha(0)kq - L\lambda q} \\
&\lesssim m^{nq/2} \Phi + m^{nq/2} \Phi \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{-1} 2^{-j\lambda q} |\lambda_j|^q 2^{(j-L)\lambda q} \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q/2} + m^{nq/2} \Phi \\
&\lesssim m^{nq/2} \Phi + m^{nq/2} \Phi \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{-1} 2^{(j-L)\lambda q} \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q/2} \\
&\lesssim m^{nq/2} \Phi.
\end{aligned}$$

Hence, we have $I_1 \lesssim m^{nq/2} \Phi$.

Secondly, we estimate I_2 . A simple computation shows that there exists a constant $\delta > 0$ such that $T_{m,l}$ satisfies the following size condition

$$|T_{m,l}f| \lesssim m^{n/2} (\text{diam}(\text{supp}f))^{\delta} |x|^{-(n+\delta)} \|f\|_1, \text{ when } \text{dist}(x, \text{supp}f) \geq \frac{|x|}{2},$$

and with the help of Lemma 2.8, we get

$$\begin{aligned}
|T_{m,l}b_j(x)| &\lesssim m^{n/2} |x|^{-(n+\delta)} 2^{j\delta} \int_{B_j} |b_j(y)| dy \\
&\lesssim m^{n/2} 2^{-k(n+\delta)} 2^{j\delta} \|b_j\|_{L^p(\cdot)} \|\chi_{B_j}\|_{L^{p'}(\cdot)} \\
&\lesssim m^{n/2} 2^{j(\delta - \alpha_j) - k(\delta + n)} \|\chi_{B_j}\|_{L^{p'}(\cdot)}.
\end{aligned}$$

So by Lemma 2.6 and 2.8, we have

$$\|(T_{m,l}b_j)\chi_k\|_{L^p(\cdot)} \lesssim m^{n/2} 2^{j(\delta - \alpha_j) - k(\delta + n)} \|\chi_{B_j}\|_{L^{p'}(\cdot)} \|\chi_{B_k}\|_{L^p(\cdot)}$$

$$\begin{aligned}
&\lesssim m^{n/2} 2^{j(\delta-\alpha_j)-k\delta} 2^{-kn} \left(\|B_k\| \|\chi_{B_k}\|_{L^{p'(\cdot)}}^{-1} \right) \|\chi_{B_j}\|_{L^{p'(\cdot)}} \\
&\lesssim m^{n/2} 2^{j(\delta-\alpha_j)-k\delta} \frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}}} \\
&\lesssim m^{n/2} 2^{(\delta+n\delta_2)(j-k)-j\alpha_j}.
\end{aligned}$$

Therefore, when $0 < q \leq 1$, by $n\delta_2 \leq \alpha(0) < \delta + n\delta_2$ we get

$$\begin{aligned}
I_2 &= \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \| (T_{m,l} b_j) \chi_k \|_{L^{p(\cdot)}} \right)^q \\
&\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{[(\delta+n\delta_2)(j-k)-j\alpha(0)]q} \right) \\
&\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^L |\lambda_j|^q \sum_{k=j+1}^{-1} 2^{(j-k)(\delta+n\delta_2-\alpha(0))q} \\
&\lesssim m^{nq/2} \Phi.
\end{aligned}$$

When $1 < q < \infty$, let $1/q + 1/q' = 1$. Since $n\delta_2 \leq \alpha(0) \leq \delta + n\delta_2$, by Hölder's inequality, we have

$$\begin{aligned}
I_2 &\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| 2^{(\delta+n\delta_2)(j-k)-j\alpha(0)} \right)^q \\
&\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{(j-k)(\delta+n\delta_2-\alpha(0))q/2} \right) \\
&\quad \times \left(\sum_{j=-\infty}^{k-1} 2^{(j-k)(\delta+n\delta_2-\alpha(0))q'/2} \right)^{q/q'} \\
&\lesssim m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{(j-k)(\delta+n\delta_2-\alpha(0))q/2} \right) \\
&= m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^L |\lambda_j|^q \sum_{k=j+1}^{-1} 2^{(j-k)(\delta+n\delta_2-\alpha(0))q/2} \\
&\lesssim m^{nq/2} \Phi.
\end{aligned}$$

Hence, we have $I \lesssim m^{nq/2} \Phi$.

Thirdly, we estimate II . Consider

$$\begin{aligned}
II &\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| \| (T_{m,l} b_j) \chi_k \|_{L^{p(\cdot)}} \right)^q + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \| (T_{m,l} b_j) \chi_k \|_{L^{p(\cdot)}} \right)^q \\
&=: II_1 + II_2.
\end{aligned}$$

When $0 < q \leq 1$, we get

$$\begin{aligned}
II_1 &\lesssim m^{nq/2} \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| 2^{-\alpha_j j} \right)^q \\
&\lesssim m^{nq/2} \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{-1} |\lambda_j|^q 2^{-\alpha(0)jq} + \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq} \right) \\
&\lesssim m^{nq/2} \sum_{k=-\infty}^{-1} \sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q} + m^{nq/2} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq} \\
&\lesssim m^{nq/2} \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q} + m^{nq/2} \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq}
\end{aligned}$$

$$\begin{aligned}
&\lesssim m^{nq/2} \sum_{j=-\infty}^{-1} |\lambda_j|^q + m^{nq/2} \sum_{j=0}^{\infty} 2^{-j\lambda q} |\lambda_j|^q 2^{-\alpha_{\infty} j q} \sum_{k=-\infty}^{-1} 2^{\alpha(0) k q} \\
&\lesssim m^{nq/2} \Phi + m^{nq/2} \Phi \sum_{i=-\infty}^j |\lambda_i|^q \sum_{j=0}^{\infty} 2^{(\lambda - \alpha_{\infty}) j q} \sum_{k=-\infty}^j 2^{\alpha(0) k q} \\
&\lesssim m^{nq/2} \Phi.
\end{aligned}$$

As $1 < q < \infty$, we obtain

$$\begin{aligned}
II_1 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| \| (T_{m,l} b_j) \chi_k \|_{L^{p(\cdot)}} \right)^q \\
&\lesssim m^{nq/2} \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| 2^{-\alpha_{\infty} j} \right)^q \\
&\lesssim m^{nq/2} \sum_{k=-\infty}^{-1} \left(\sum_{j=k}^{-1} |\lambda_j| 2^{\alpha(0)(k-j)} \right)^q + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=0}^{\infty} |\lambda_j| 2^{-\alpha_{\infty} j} \right)^q \\
&\lesssim m^{nq/2} \sum_{k=-\infty}^{-1} \left(\sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q/2} \right) \left(\sum_{j=k}^{-1} 2^{\alpha(0)(k-j)q'/2} \right)^{q/q'} \\
&\quad + m^{nq/2} 2^{\alpha(0)kq} \left(\sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty} j q/2} \right) \left(\sum_{j=0}^{\infty} 2^{-\alpha_{\infty} j q'/2} \right)^{q/q'} \\
&\lesssim m^{nq/2} \sum_{k=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q/2} + m^{nq/2} \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty} j q/2} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\lesssim m^{nq/2} \sum_{k=-\infty}^{-1} |\lambda_j|^q + m^{nq/2} \sum_{j=0}^{\infty} 2^{(\lambda - \alpha_{\infty}/2) j q} 2^{-j\lambda q} \sum_{i=-\infty}^j |\lambda_i|^q \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\lesssim m^{nq/2} \Phi + m^{nq/2} \Phi \sum_{j=0}^{\infty} 2^{(\lambda - \alpha_{\infty}/2) j q} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\lesssim m^{nq/2} \Phi.
\end{aligned}$$

For II_2 , as $0 < q \leq 1$, noting that $n\delta_2 \leq \alpha(0) < \delta + n\delta_2$, we have

$$\begin{aligned}
II_2 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \| (T_{m,l} b_j) \chi_k \|_{L^{p(\cdot)}} \right)^q \\
&\lesssim m^{nq/2} \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{[(\delta + n\delta_2)(j-k) - j\alpha(0)]q} \right) \\
&= m^{nq/2} \sum_{k=-\infty}^{-1} |\lambda_j|^q \sum_{k=j+1}^{-1} 2^{(j-k)(\delta + n\delta_2 - \alpha(0))q} \lesssim m^{nq/2} \sum_{k=-\infty}^{-1} |\lambda_j|^q \lesssim m^{nq/2} \Phi.
\end{aligned}$$

As $1 < q < \infty$ and $n\delta_2 \leq \alpha(0) < \delta + n\delta_2$, by Hölder's inequality, one has

$$\begin{aligned}
II_2 &\lesssim m^{nq/2} \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| 2^{(\delta + n\delta_2)(j-k) - j\alpha(0)} \right)^q \\
&\lesssim m^{nq/2} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{(j-k)(\delta + n\delta_2 - \alpha(0))q/2} \right) \\
&\quad \times \left(\sum_{j=-\infty}^{k-1} 2^{(j-k)(\delta + n\delta_2 - \alpha(0))q'/2} \right)^{q/q'} \\
&\lesssim m^{nq/2} \sum_{j=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{(j-k)(\delta + n\delta_2 - \alpha(0))q/2} \right)
\end{aligned}$$

$$\begin{aligned}
&= m^{nq/2} \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=j+1}^{-1} 2^{(j-k)(\delta+n\delta_2-\alpha(0))q/2} \\
&\lesssim m^{nq/2} \Phi.
\end{aligned}$$

So, we have $II \lesssim m^{nq/2} \Phi$.

Finally, we estimate *III*. Write

$$\begin{aligned}
III &\lesssim \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_\infty} \left(\sum_{j=k}^{\infty} |\lambda_j| \| (T_{m,l} b_j) \chi_k \|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_\infty} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \| (T_{m,l} b_j) \chi_k \|_{L^{p(\cdot)}} \right)^q \\
&=: III_1 + III_2.
\end{aligned}$$

When $0 < q \leq 1$, by the boundedness of $T_{m,l}$ in $L^{p(\cdot)}(\mathbb{R}^n)$ (see [10]), we obtain

$$\begin{aligned}
III_1 &\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_\infty} \sum_{j=k}^{\infty} |\lambda_j|^q \| (T_{m,l} b_j) \chi_k \|_{L^{p(\cdot)}}^q \\
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_\infty} \sum_{j=k}^{\infty} |\lambda_j|^q 2^{-\alpha_j j q} \\
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_\infty} \sum_{j=k}^{\infty} |\lambda_j|^q 2^{-\alpha_\infty j q} \\
&= m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^L |\lambda_j|^q \sum_{k=0}^j 2^{(k-j)\alpha_\infty q} + m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{\infty} |\lambda_j|^q \sum_{k=0}^L 2^{(k-j)\alpha_\infty q} \\
&\leq m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^L |\lambda_j|^q + m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j\lambda q - L\lambda q)} 2^{-j\lambda q} \sum_{i=-\infty}^j |\lambda_i|^q \sum_{k=0}^L 2^{(k-j)\alpha_\infty q} \\
&\lesssim m^{nq/2} \Phi + m^{nq/2} \Phi \sup_{L>0, L \in \mathbb{Z}} \sum_{j=L}^{\infty} 2^{(L-j)\alpha_\infty q} 2^{(j-L)\lambda q} \\
&\lesssim m^{nq/2} \Phi + m^{nq/2} \Phi \sup_{L>0, L \in \mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j-L)q(\lambda - \alpha_\infty)} \\
&\lesssim m^{nq/2} \Phi.
\end{aligned}$$

As $1 < q < \infty$, by the boundedness of $T_{m,l}$ in $L^{p(\cdot)}$ (see [10]) and Hölder's inequality, we have

$$\begin{aligned}
III_1 &\lesssim \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \left(\sum_{j=k}^{\infty} |\lambda_j|^q \| (T_{m,l} b_j) \chi_k \|_{L^{p(\cdot)}}^{q/2} \right) \times \left(\sum_{j=k}^{\infty} \| (T_{m,l} b_j) \chi_k \|_{L^{p(\cdot)}}^{q'/2} \right)^{q/q'} \\
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} \sum_{k=0}^L 2^{-L\lambda q} \left(\sum_{j=k}^{\infty} |\lambda_j|^q \| b_j \|_{L^{p(\cdot)}}^{q/2} \right) \times \left(\sum_{j=k}^{\infty} \| b_j \|_{L^{p(\cdot)}}^{q'/2} \right)^{q/q'} \\
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} \sum_{k=0}^L 2^{-L\lambda q} \left(\sum_{j=k}^{\infty} |\lambda_j|^q |B_j|^{-\alpha_j q/(2n)} \right) \times \left(\sum_{j=k}^{\infty} |B_j|^{-\alpha_j q'/(2n)} \right)^{q/q'} \\
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} \sum_{k=0}^L 2^{-L\lambda q} \left(\sum_{j=k}^{\infty} |\lambda_j|^q |B_j|^{-\alpha_j q/(2n)} \right) \\
&= m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^L |\lambda_j|^q \sum_{k=0}^L 2^{(k-j)\alpha_\infty q/2} \\
&\quad + m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{\infty} |\lambda_j|^q \sum_{k=0}^L 2^{(k-j)\alpha_\infty q/2}
\end{aligned}$$

$$\begin{aligned}
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^L |\lambda_j|^q \\
&\quad + m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{\infty} 2^{(j\lambda q - L\lambda q)} 2^{-j\lambda q} \sum_{i=-\infty}^j |\lambda_i|^q \sum_{k=0}^L 2^{(k-j)\alpha_{\infty} q/2} \\
&\lesssim m^{nq/2} \Phi + m^{nq/2} \Phi \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j-L)\lambda q} 2^{(L-j)\alpha_{\infty} q/2} \\
&\lesssim m^{nq/2} \Phi + m^{nq/2} \Phi \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j-L)q(\lambda - \alpha_{\infty}/2)} \\
&\lesssim m^{nq/2} \Phi.
\end{aligned}$$

For III_2 , as $0 < q \leq 1$ and $n\delta_2 \leq \alpha(0)$, $\alpha_{\infty} < \delta + n\delta_2$, we get

$$\begin{aligned}
III_2 &\lesssim \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} \left(\sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{[(\delta+n\delta_2)(j-k)-j\alpha_j]q} \right) \\
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} \left(\sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{[(\delta+n\delta_2)(j-k)-j\alpha(0)]q} \right) \\
&\quad + m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} \left(\sum_{j=0}^{k-1} |\lambda_j|^q 2^{[(\delta+n\delta_2)(j-k)-j\alpha_{\infty}]q} \right) \\
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{[\alpha_{\infty} - (\delta+n\delta_2)]kq} \times \sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{(\delta+n\delta_2 - \alpha(0))jq} \\
&\quad + m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L |\lambda_j|^q \sum_{k=j+1}^{\infty} 2^{(j-k)(\delta+n\delta_2 - \alpha_{\infty})q} \\
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q + m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L-1} |\lambda_j|^q \\
&\lesssim m^{nq/2} \Phi.
\end{aligned}$$

As $1 < q < \infty$ and $n\delta_2 \leq \alpha(0)$, $\alpha_{\infty} \leq \delta + n\delta_2$, by Hölder's inequality, we have

$$\begin{aligned}
III_2 &\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| 2^{[(\delta+n\delta_2)(j-k)-j\alpha_j]} \right)^q \\
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} \left(\sum_{j=-\infty}^{-1} |\lambda_j| 2^{[(\delta+n\delta_2)(j-k)-j\alpha(0)]} \right)^q \\
&\quad + m^{nq/2} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} \left(\sum_{j=0}^{k-1} |\lambda_j| 2^{[(\delta+n\delta_2)(j-k)-j\alpha_{\infty}]} \right)^q \\
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{[\alpha_{\infty} - (\delta+n\delta_2)]kq} \times \left(\sum_{j=-\infty}^{-1} 2^{(\delta+n\delta_2 - \alpha(0))j} \right)^q \\
&\quad + m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \left(\sum_{j=0}^{k-1} |\lambda_j| 2^{(j-k)(\delta+n\delta_2 - \alpha_{\infty})} \right)^q \\
&\lesssim m^{nq/2} \left(\sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{(\delta+n\delta_2 - \alpha(0))jq/2} \right) \left(\sum_{j=-\infty}^{-1} 2^{(\delta+n\delta_2 - \alpha(0))jq'/2} \right)^{q/q'} \\
&\quad + m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \left(\sum_{j=0}^{k-1} |\lambda_j|^q 2^{(j-k)(\delta+n\delta_2 - \alpha_{\infty})q/2} \right) \left(\sum_{j=0}^{k-1} 2^{(j-k)(\delta+n\delta_2 - \alpha_{\infty})q'/2} \right)^{q/q'} \\
&\lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{(\delta+n\delta_2 - \alpha(0))jq/2}
\end{aligned}$$

$$\begin{aligned}
& + m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \sum_{j=0}^{k-1} |\lambda_j|^q 2^{(j-k)(\delta+n\delta_2-\alpha_\infty)q/2} \\
& \lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q + m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L-1} |\lambda_j|^q \sum_{k=j+1}^L 2^{(j-k)(\delta+n\delta_2-\alpha_\infty)q/2} \\
& \lesssim m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q + m^{nq/2} \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L-1} |\lambda_j|^q \\
& \lesssim m^{nq/2} \Phi.
\end{aligned}$$

Thus, we have $III \lesssim m^{nq/2} \Phi$.

Combining the estimates *I*, *II*, *III*, we complete the proof of Lemma 3.1. $\square$

Lemma 3.2. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$, $0 \leq \lambda < \infty$ and $t(x)$ be a homogeneous of degree $-n-1$ and locally integrable in $|x| > 0$. Let $b \in Lip(\mathbb{R}^n)$ and K is defined by

$$Kf(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} t(x)(b(x) - b(y))f(y)dy.$$

Suppose that $t(x) \in C^1(S^{n-1})$, $\int_{S^{n-1}} t(x)x_j d\sigma(x) = 0$ ($j = 1, \dots, n$), and $\alpha(\cdot)$ is a bounded and log-Hölder continuous both at the origin and infinity such that $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0)$, $\alpha_\infty \leq n\delta_2 + \delta$ with some $\delta > \max\{\alpha(0) - \delta_2, \alpha_\infty - n\delta_2\}$ and δ_2 as in Lemma 2.6, then we have

$$\|Kf\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \lesssim (\|\nabla t\|_{L^\infty(S^{n-1})} + \|t\|_{L^\infty(S^{n-1})}) \|\nabla b\|_{L^\infty} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}.$$

Proof. Let $k(x, y) = t(x-y)(b(x) - b(y))$. For all $x, x_0, y \in \mathbb{R}^n$ with $|x - x_0| \leq 1/2|y - x|$, then k satisfies the following inequalities

$$|k(x, y) - k(x_0, y)| \lesssim \|\nabla s\|_{L^\infty(S^{n-1})} \|\nabla b\|_{L^\infty} |x - x_0| |y - x|^{-n-1} \quad (3.4)$$

and

$$|k(x, y)| \lesssim \|t\|_{L^\infty(S^{n-1})} \|\nabla b\|_{L^\infty} |y - x|^{-n}. \quad (3.5)$$

This, together with the boundedness of K on $L^2(\mathbb{R}^n)$ (see [33]), tells us K is a generalized Calderón-Zygmund operator (see [34]). Thus, by applying Lemma 2.9, we see that K is bounded from $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ to $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ with bound $(\|\nabla s\|_{L^\infty(S^{n-1})} + \|s\|_{L^\infty(S^{n-1})}) \|\nabla b\|_{L^\infty}$, i.e.

$$\|Kf\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \lesssim (\|\nabla s\|_{L^\infty(S^{n-1})} + \|s\|_{L^\infty(S^{n-1})}) \|\nabla b\|_{L^\infty} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}.$$

Therefore, the proof of Lemma 3.2 is finished. $\square$

Lemma 3.3. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$, $0 \leq \lambda < \infty$, $b \in Lip(\mathbb{R}^n)$, and T be a singular operator which is defined by

$$Tf(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} K(x-y)f(y)dy,$$

where $K(x) \in C^3(S^{n-1})$ satisfies $\int_{S^{n-1}} K(x)d\sigma(x) = 0$ and $K(\lambda x) = \lambda^{-n}K(x)$ for $x \in \mathbb{R}^n \setminus \{0\}$. If $\alpha(\cdot)$ is a bounded and log-Hölder continuous both at the origin and infinity such that $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0)$, $\alpha_\infty \leq n\delta_2 + \delta$ with some $\delta > \max\{\alpha(0) - \delta_2, \alpha_\infty - n\delta_2\}$ and δ_2 as in Lemma 2.6, then, for $f \in C_0^\infty(\mathbb{R}^n)$,

$$\| [b, T] \frac{\partial f}{\partial x_j} \|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \lesssim \max_{|\beta| \leq 2} \|\partial^\beta K\|_{L^\infty(S^{n-1})} \|\nabla b\|_{L^\infty} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}.$$

Proof. With an argument similar to that used in the proof of Lemma 5.2 in [9], together with Lemma 2.9 and Lemma 3.1, it is not difficult to obtain Lemma 3.3. Thus, we omit the details here. $\square$

4 Proofs of Theorems 1.1-1.5

Proof of Theorem 1.1. Let

$$\Omega(x, y) = \sum_{m \geq 1} \sum_{j=1}^{d_m} a_{m,j}(x) Y_{m,j}(y).$$

From [8], for any x , we can write the coefficients $a_{m,j}$ as

$$a_{m,j}(x) = (-1)^n m^{-n} (m+n-2)^{-n} \int_{S^{n-1}} L_{y'}^n(\Omega(x, y')) Y_{m,j}(y') d\sigma(y'), \quad m \geq 1, \quad (4.1)$$

where $L(F) = |x|^2 \Delta F(x)$.

We will firstly prove the conclusion (1). Write

$$\begin{aligned} (TD^\gamma - D^\gamma T)f &= \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} (a_{m,j} T_{m,j} D^\gamma - D^\gamma a_{m,j} T_{m,j})f \\ &= \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} (a_{m,j} D^\gamma T_{m,j} - D^\gamma a_{m,j} T_{m,j})f \\ &= \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} [a_{m,j}, D^\gamma] T_{m,j} f. \end{aligned}$$

By condition (4.1), it follows that

$$D^\gamma a_{m,j}(x) = (-1)^n m^{-n} (m+n-2)^{-n} \int_{S^{n-1}} D_x^\gamma L_{y'}^n(\Omega(x, y')) Y_{m,j}(y') d\sigma(y'), \quad m \geq 1.$$

Further, by applying the condition (1.4), we have

$$\|D^\gamma a_{m,j}\|_{L^\infty} \lesssim m^{-2n}. \quad (4.2)$$

Moreover, $[b, D^\gamma]$ is a generalized Calderón-Zygmund operator (see [35]), which is defined by

$$[b, D^\gamma]f(x) = C(\gamma) \int_{\mathbb{R}^n} \frac{(b(x) - b(y))}{|x - y|^{n+\gamma}} f(y) dy.$$

Thus, we see that $[b, D^\gamma]f(x)$ is bounded from $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ to $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ by applying Lemma 2.9. Namely

$$\|[b, D^\gamma]f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \lesssim \|D^\gamma b\|_{BMO} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \quad (4.3)$$

Then by $d_m \simeq m^{n-2}$ (see [7]), (4.2), (4.3) and Lemma 3.1, we have

$$\begin{aligned} \|(TD^\gamma - D^\gamma T)f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \|[a_{m,j}, D^\gamma] T_{m,j} f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \|D^\gamma a_{m,j}\|_{BMO} \|T_{m,j} f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} m^{\frac{n}{2}} \|D^\gamma a_{m,j}\|_{L^\infty} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} m^{n-2} m^{\frac{n}{2}} m^{-2n} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \end{aligned}$$

Now let us turn to estimate (2). By applying the definition of $T^\sharp$ and T^* we can deduce that

$$(T^\sharp - T^*)D^\gamma f = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} (-1)^m [\bar{a}_{m,j}, T_{m,j}] D^\gamma f. \quad (4.4)$$

To estimate $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ norm of $(T^* - T^\sharp)D^\gamma$, we first consider $[b, T_{m,j}]D^\gamma$ for any fixed $b \in I_\gamma(\text{BMO})$. Noting that $b(x) - b(y) = (b(x) - b(z)) - (b(y) - b(z))$, for any $x, y, z \in \mathbb{R}^n$, then we have

$$[b, T_{m,j}]D^\gamma f = [b, D^\gamma T_{m,j}]f - T_{m,j}[b, D^\gamma]f.$$

Thus, we get by (4.3) and Lemma 3.1

$$\|T_{m,j}[b, D^\gamma]f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \lesssim m^{\frac{n}{2}} \|D^\gamma b\|_{\text{BMO}} \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \quad (4.5)$$

Further, we estimate the $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ norm of $[b, D^\gamma T_{m,j}]f$. From the fact that $[b, D^\gamma T_{m,j}]$ is a generalized Calderón-Zygmund operator with kernel (see [9])

$$|k_{m,j}(x, y)| \lesssim m^{\frac{n}{2}-1+\gamma} \|D^\gamma b\|_{\text{BMO}} \frac{1}{|x-y|^n},$$

then we get by Lemma 2.9

$$\|[b, D^\gamma T_{m,j}]f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \lesssim m^{\frac{n}{2}+\gamma} \|D^\gamma b\|_{\text{BMO}} \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \quad (4.6)$$

Then, combining (4.5) with (4.6), we have

$$\begin{aligned} \|[b, T_{m,j}]D^\gamma f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\lesssim m^{\frac{n}{2}+\gamma} \|D^\gamma b\|_{\text{BMO}} \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} + m^{\frac{n}{2}} \|D^\gamma b\|_{\text{BMO}} \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim m^{\frac{n}{2}+\gamma} \|D^\gamma b\|_{\text{BMO}} \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \end{aligned} \quad (4.7)$$

By condition (4.2), (4.4) and (4.7), we get

$$\begin{aligned} \|(T^\sharp - T^*)D^\gamma f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \|[\bar{a}_{m,j}, T_{m,j}]D^\gamma f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} m^{\frac{n}{2}+\gamma} \|D^\gamma \bar{a}_{m,j}\|_{\text{BMO}} \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} m^{\frac{n}{2}+\gamma} \|D^\gamma \bar{a}_{m,j}\|_{L^\infty} \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} m^{n-2} m^{\frac{n}{2}+\gamma} m^{-2n} \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \end{aligned}$$

Thus we finish the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let

$$T_1 f(x) = \int_{\mathbb{R}^n} \frac{\Omega_1(x, x-y)}{|x-y|^n} f(y) dy \quad \text{and} \quad T_2 f(x) = \int_{\mathbb{R}^n} \frac{\Omega_2(x, x-y)}{|x-y|^n} f(y) dy.$$

Write

$$\Omega_1(x, y) = \sum_{m \geq 1} \sum_{j=1}^{d_m} a_{m,j}(x) Y_{m,j}(y) \quad \text{and} \quad \Omega_2(x, y) = \sum_{\lambda \geq 1} \sum_{\mu=1}^{d_\lambda} b_{\lambda,\mu}(x) Y_{\lambda,\mu}(y),$$

where

$$a_{m,j}(x) = \int_{S^{n-1}} \Omega_1(x, z') \overline{Y_{m,j}(z')} d\sigma(z') \quad \text{and} \quad b_{\lambda,\mu}(x) = \int_{S^{n-1}} \Omega_2(x, z') \overline{Y_{\lambda,\mu}(z')} d\sigma(z').$$

For any $x \in \mathbb{R}^n$, with a similar argument used in the proof of Theorem 1.1 in terms of (1.4) and (1.5), we can obtain that

$$\|a_{m,j}\|_{L^\infty} \lesssim m^{-2n}. \quad (4.8)$$

$$\|D^\gamma b_{\lambda,\mu}\|_{L^\infty} \lesssim m^{-2n}. \quad (4.9)$$

Let

$$T_{m,j}f(x) = \frac{Y_{m,j}}{|\cdot|^n} * f(x) \quad \text{and} \quad T_{\lambda,\mu}f(x) = \frac{Y_{\lambda,\mu}}{|\cdot|^n} * f(x).$$

Since $\Omega_1(x, y)$ and $\Omega_2(x, y)$ satisfy (1.2), then we get

$$T_1f(x) = \sum_{m \geq 1} \sum_{j=1}^{d_m} a_{m,j}(x) T_{m,j}f(x) \quad \text{and} \quad T_2f(x) = \sum_{\lambda \geq 1} \sum_{\mu=1}^{d_\lambda} b_{\lambda,\mu}(x) T_{\lambda,\mu}f(x).$$

Write (see [9])

$$(T_1 \circ T_2)f(x) = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} a_{m,j}(x) b_{\lambda,\mu}(x) (T_{m,j} T_{\lambda,\mu}f)(x),$$

$$(T_1 T_2)f(x) = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} a_{m,j} T_{m,j}(b_{\lambda,\mu} T_{\lambda,\mu}f)(x).$$

Then

$$\begin{aligned} (T_1 \circ T_2 - T_1 T_2)D^\gamma f &= \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} a_{m,j}(b_{\lambda,\mu}(x) T_{m,j} - T_{m,j} b_{\lambda,\mu}(x)) T_{\lambda,\mu} D^\gamma f \\ &= \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} a_{m,j}(b_{\lambda,\mu}(x) T_{m,j} - T_{m,j} b_{\lambda,\mu}(x)) D^\gamma T_{\lambda,\mu} f \\ &= \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} a_{m,j} [b_{\lambda,\mu}, T_{m,j}] D^\gamma T_{\lambda,\mu} f. \end{aligned}$$

Therefore, together with (4.7), (4.8), (4.9) and Lemma 3.1, we obtain

$$\begin{aligned} &\|(T_1 \circ T_2 - T_1 T_2)D^\gamma f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} \|a_{m,j}\|_{L^\infty} \|[b_{\lambda,\mu}, T_{m,j}] D^\gamma T_{\lambda,\mu} f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} \|a_{m,j}\|_{L^\infty} \|D^\gamma b_{\lambda,\mu}\|_{BMO} m^{\frac{n}{2}+\gamma} \|T_{\lambda,\mu} f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} \|a_{m,j}\|_{L^\infty} \|D^\gamma b_{\lambda,\mu}\|_{L^\infty} m^{\frac{n}{2}+\gamma} \lambda^{\frac{n}{2}} \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} m^{n-2} m^{-2n} m^{\frac{n}{2}+\gamma} \sum_{\lambda=1}^{\infty} \lambda^{n-2} \lambda^{-2n} \lambda^{\frac{n}{2}} \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \|f\|_{H\dot{M}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \end{aligned}$$

This finishes the proof of Theorem 1.2. $\square$

Proof of Theorem 1.3. We estimate that term exactly as we did for the corresponding boundedness in Theorem 1.1 in the above arguments. Without loss of generality, we only have to prove (2) and (3) of Theorem 1.3. By using the fact that $\Omega_1(x, y)$ and $\Omega_2(x, y)$ satisfy (1.5), we have shown that

$$\|a_{m,j}\|_{L^\infty} \lesssim m^{-2n}, \quad \|b_{\lambda,\mu}\|_{L^\infty} \lesssim \lambda^{-2n}. \quad (4.10)$$

Firstly, let's prove (2). As in the proof of Theorem 1.1, we can get

$$(T_1^\sharp - T_1^*)\mathcal{I}f = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} (-1)^m [\bar{a}_{m,j}, T_{m,j}] \mathcal{I}f.$$

We showed that $[b, T_{m,j}]$ is a special Calderón-Zygmund operator, so it is a bounded operator from $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ to $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ by applying Lemma 2.9. Thus we have

$$\|[b, T_{m,j}]f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \lesssim m^{\frac{n}{2}} \|b\|_{L^\infty} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \quad (4.11)$$

Then by (4.10), we get

$$\begin{aligned} \|(T_1^\sharp - T_1^*)\mathcal{I}f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\lesssim \sum_{m=1}^{\infty} m^{n-2} m^{-3n/2} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \end{aligned}$$

Thus the conclusion (2) is proved. We now estimate (3). Write

$$(T_1 \circ T_2 - T_1 T_2)\mathcal{I}f = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} [b_{\lambda,\mu}, T_{m,j}] T_{\lambda,\mu} \mathcal{I}f.$$

Therefore, by (4.10), (4.11) and Lemma 3.1, we get

$$\begin{aligned} \|(T_1 \circ T_2 - T_1 T_2)\mathcal{I}f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} \|a_{m,j}\|_{L^\infty} \|b_{\lambda,\mu}\|_{L^\infty} m^{\frac{n}{2}} \|T_{\lambda,\mu} \mathcal{I}f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} m^{n-2} m^{-2n} m^{n/2} \sum_{\lambda=1}^{\infty} \lambda^{n-2} \lambda^{-2n} \lambda^{n/2} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \end{aligned}$$

Thus the conclusion (3) is also proved. Hence the proof of Theorem 1.3 is finished. $\square$

Proof of Theorem 1.4. In the first place, we will prove the conclusion (1). Write $D = \sum_{k=1}^n \mathcal{R}_k \frac{\partial}{\partial x_k}$, where $\mathcal{R}_k$ denotes the Riesz transform. As in the proof of Theorem 1.1, we have

$$\begin{aligned} (TD - DT)f(x) &= \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} [a_{m,j}, D] T_{m,j} f(x) \\ &= \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{k=1}^n \mathcal{R}_k [a_{m,j}, \frac{\partial}{\partial x_k}] T_{m,j} f(x) + \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{k=1}^n [a_{m,j}, \mathcal{R}_k] \frac{\partial}{\partial x_k} (T_{m,j} f)(x) \\ &=: J_1 + J_2. \end{aligned}$$

We have by the Leibniz's rules that

$$J_1 = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{k=1}^n \mathcal{R}_k \left(\frac{\partial}{\partial x_k} (a_{m,j}) T_{m,j} f \right).$$

Thus we deduce from (4.1) that

$$\frac{a_{m,j}}{\partial x_k}(x) = (-1)^n m^{-n} (m+n-2)^{-n} \int_{S^{n-1}} \partial_{x_k} L_{y'}^n(\Omega(x, y')) Y_{m,j}(y') d\sigma(y'), m \geq 1.$$

From this and (1.6), we get for $k = 1, \dots, n$,

$$\left\| \frac{\partial a_{m,j}}{\partial x_k} \right\|_{L^\infty} \lesssim m^{-2n}. \quad (4.12)$$

By using the fact that $\|\mathcal{R}_k g\|_{WM\dot{K}_{p,1}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} \lesssim \|g\|_{M\dot{K}_{p,1}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}$, $d_m \simeq m^{n-2}$ and Lemma 3.2, then we have

$$\begin{aligned} \|J_1\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{k=1}^n \|\mathcal{R}_k(\frac{\partial}{\partial x_k}(a_{m,j})T_{m,j}f)\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} m^{-2n} m^{n/2} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \sum_{m=1}^{\infty} m^{n-2} m^{-2n} m^{n/2} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \end{aligned}$$

By Lemma 3.3 and (4.12), a trivial computation shows that for I_2 ,

$$\begin{aligned} \|J_2\|_{WM\dot{K}_{p,1}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{k=1}^n \|\nabla a_{m,j}\|_{L^\infty} \|T_{m,j}f\|_{WM\dot{K}_{p,1}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} \\ &\lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} m^{-2n} m^{n/2} \|f\|_{M\dot{K}_{p,1}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} \\ &\lesssim \sum_{m=1}^{\infty} m^{n-2} m^{-2n} m^{n/2} \|f\|_{M\dot{K}_{p,1}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{M\dot{K}_{p,1}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}. \end{aligned}$$

Combining the estimates above, we arrive at the desired boundedness

$$\|(TD - DT)f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \lesssim \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}.$$

We posterior prove the conclusion (2). Write $D = \sum_{k=1}^n \mathcal{R}_k \frac{\partial}{\partial x_k}$, we have

$$(T^\sharp - T^*)Df(x) = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} (-1)^m [\bar{a}_{m,j}, T_{m,j}] Df(x) = \sum_{k=1}^n \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} (-1)^m [\bar{a}_{m,j}, T_{m,j}] \frac{\partial}{\partial x_k} (\mathcal{R}_k f)(x). \quad (4.13)$$

We now turn to estimate the $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ norm of $[\bar{a}_{m,j}, T_{m,j}] \frac{\partial}{\partial x_k} (\mathcal{R}_k f)$. Applying (4.12), Lemma 3.3 and the fact that for any multi-index β and $x \in \mathbb{R}^n \setminus \{0\}$, $m = 1, 2, \dots$ (see [1]),

$$|\partial^\beta (|x|^m) Y_{m,j}| \leq C(n) |x|^{m-|\beta|} m^{|\beta|+(n-2)/2}. \quad (4.14)$$

Hence, we get

$$\begin{aligned} \left\| [\bar{a}_{m,j}, T_{m,j}] \frac{\partial}{\partial x_k} (\mathcal{R}_k f) \right\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\lesssim \|\nabla \bar{a}_{m,j}\|_{L^\infty} \max_{|\beta| \leq 2} \|\partial^\beta Y_{m,j}\|_{L^\infty(S^{n-1})} \|\mathcal{R}_k f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim m^{-2n} m^{n/2+1} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ &\lesssim m^{-3n/2+1} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \end{aligned} \quad (4.15)$$

Combining the estimates of (4.13) with (4.15), we have

$$\|(T^\sharp - T^*)Df\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \lesssim \sum_{m=1}^{\infty} m^{n-2} m^{-3n/2+1} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \lesssim \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}.$$

Consequently, the proof of Theorem 1.4 is completed. $\square$

Proof of Theorem 1.5. Similarly to the proof of Theorem 1.2, we easily see that

$$(T_1 \circ T_2 - T_1 T_2)Df = \sum_{m=1}^{\infty} \sum_{d=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} a_{m,j} [b_{\lambda,\mu}, T_{m,j}] DT_{\lambda,\mu} f,$$

where $a_{m,j}$ and $b_{\lambda,\mu}$ are the same as in the proof of Theorem 1.2. By (1.5) and (1.6), we have

$$\|a_{m,j}\|_{L^\infty} \lesssim m^{-2n}. \quad (4.16)$$

$$\|\nabla b_{\lambda,\mu}\|_{L^\infty} \lesssim \lambda^{-2n}. \quad (4.17)$$

Write $D = \sum_{k=1}^n \frac{\partial}{\partial x_k} \mathcal{R}_k$, it then follows that

$$\begin{aligned} & \| (T_1 \circ T_2 - T_1 T_2) Df \|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ & \lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} \|a_{m,j}\|_{L^\infty} \left\| [b_{\lambda,\mu}, T_{m,j}] \left(\sum_{k=1}^n \frac{\partial}{\partial x_k} \mathcal{R}_k T_{\lambda,\mu} f \right) \right\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ & \lesssim \sum_{k=1}^n \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} \|a_{m,j}\|_{L^\infty} \left\| [b_{\lambda,\mu}, T_{m,j}] \left(\frac{\partial}{\partial x_k} \mathcal{R}_k T_{\lambda,\mu} f \right) \right\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \end{aligned}$$

The above estimate, via Lemma 3.1, leads to

$$\begin{aligned} & \| (T_1 \circ T_2 - T_1 T_2) Df \|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ & \lesssim \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} \|a_{m,j}\|_{L^\infty} \|\nabla b_{\lambda,\mu}\|_{L^\infty} \max_{|\beta| \leq 2} \|\partial^\beta Y_{m,j}\|_{L^\infty(S^{n-1})} \|T_{\lambda,\mu} \mathcal{R}_k f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \end{aligned}$$

We thus obtain from (4.14), (4.16), (4.17) and Lemma 3.1 that

$$\begin{aligned} & \| (T_1 \circ T_2 - T_1 T_2) Df \|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ & \lesssim \sum_{m=1}^{\infty} m^{n/2-1} m^{-2n} m^{n/2+1} \sum_{\lambda=1}^{\infty} \lambda^{n/2-1} \lambda^{-2n} \lambda^{n/2} \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\ & \lesssim \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}. \end{aligned}$$

Consequently, the proof of Theorem 1.5 is finished. $\square$

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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