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Boundedness control sets for linear systems on Lie groups

Abstract: Let Σ be a linear system on a connected Lie group G and assume that the reachable set $\mathcal{A}$ from the identity element $e \in G$ is open. In this paper, we give an algebraic condition to warrant the boundedness of the existent control set with a nonempty interior containing e. We concentrate the search for the class of decomposable groups which includes any solvable group and also every compact semisimple group.

Keywords: Decomposable group, Linear system, Control set, Boundedness

MSC: 16W25, 93B05, 93C05

1 Introduction

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. In [1] the authors introduce the notion of a linear system Σ on G which is determined by a family of differential equations

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t) X^j(g(t)),$$

where $\mathcal{X}$ is a linear vector field, $X^j \in \mathfrak{g}$ considered as right invariant vector fields and $u \in \mathcal{U} \subset L^{\infty}(\mathbb{R}, \Omega \subset \mathbb{R}^m)$ is the class of admissible controls. We deal with Ω as a subset of $\mathbb{R}^m$ with $0 \in \text{int } \Omega$. Furthermore, Σ is called *restricted* if Ω is compact and *unrestricted* if $\Omega = \mathbb{R}^m$.

We denote by $\phi_{t,u}(g) = \phi(t, g, u)$ the solution of Σ with control u, initial condition g at time t.

The controllability property of any system is a relevant issue in system theory. It gives you the possibility to connect any two arbitrary states of the manifold through a Σ-solution in positive time. For instance, when G is the Euclidean space $\mathbb{R}^n$ an unrestricted linear system is controllable if and only if it satisfies de Kalman rank condition [1], which is nothing more than the ad-rank condition, see Remark 2.2 in chapter two. However, controllability is rare in the literature, especially for Σ. Assume G is nilpotent and the accessibility set $\mathcal{A}$ from the identity element $e \in G$ provided by

$$\mathcal{A} = \{x \in G : \exists u \in \mathcal{U} \text{ and } t \geq 0 \text{ with } x = \phi_{t,u}(e)\}$$

is an open set. It turns out that

$$\Sigma \text{ is controllable on } G \Leftrightarrow \text{Spec}_{Ly}(\mathcal{D}) \cap \mathbb{R} = \{0\}.$$

Here, $\mathcal{D} \in \partial \mathfrak{g}$ is a $\mathfrak{g}$-derivation associated to $\mathcal{X}$ and the Lyapunov spectrum $\text{Spec}_{Ly}(\mathcal{D})$ consists of the real parts of the $\mathcal{D}$-eigenvalues.

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Recently, the authors in [2] proved that the requirement $\text{Spec}_{Ly}(\mathcal{D}) \cap \mathbb{R} = \{0\}$ implies controllability for any Lie group with finite semisimple center, that is, for any Lie group which admits a maximal semisimple Lie subgroup with finite center. Certainly, the condition on the Lyapunov spectrum of $\mathcal{D}$ is very strong. Actually, each $\mathcal{D}$-eigenvalue must live on the axis $i\mathbb{R}$.

For restricted system there exists the notion of control set introduced in [3], basically, a subset $\mathcal{C}$ where controllability holds on $int(\mathcal{C})$. For a locally controllable system, it is shown in [4] the shape of the control sets with nonempty interior. Under our assumptions, the control set containing the identity element $e \in G$ reads as

$$\mathcal{C}_e = \text{cl}(\mathcal{A}) \cap \mathcal{A}^*$$

where $\mathcal{A}^*$ is the reachable set of Σ^-, i.e., when time in Σ is reversed.

In this paper, we are interested in research on algebraic condition to ensure the boundedness of $\mathcal{C}_e$. We concentrate the study on solvable Lie groups because in this case the space state is firstly decomposable. This means that G can be written as a direct product of the closed subgroups G^+, G^0 and G^- with Lie algebras $\mathfrak{g}^+$, $\mathfrak{g}^0$, and $\mathfrak{g}^-$ induced by the $\mathfrak{g}$-derivation $\mathcal{D}$ which determines the drift vector field $\mathcal{X}$. Secondly, any solvable Lie group has trivially the finite semisimple center property. Hence, we can apply any result about control sets from [5]. In particular, denote by $\mathcal{A}_{G^-} = \mathcal{A} \cap G^-$ and $\mathcal{A}_{G^+}^* = \mathcal{A}^* \cap G^+$. The authors show that for semisimple or nilpotent Lie groups the compactness of $\mathcal{A}_{G^-}$, $\mathcal{A}_{G^+}^*$ and G^0 together is a sufficient condition for the boundedness of $\mathcal{C}_e$. Furthermore, for the class of nilpotent simply connected Lie groups these conditions are also necessary. However, to compute effectively these three sets is a very hard task. Hence, our main aims in this paper are to search for algebraic computable conditions to get the boundedness of $\mathcal{C}_e$. Next, we resume the chapters.

In Section 2, we review some of the standard facts on linear systems. In particular, we summarize without proof the primary relevant material on the dynamic structure, the reachable sets and the existence and uniqueness of control sets with a nonempty interior of Σ. We also mention the $\mathcal{D}$-decomposition of the Lie algebra $\mathfrak{g}$ and the corresponding Lie groups induced by $\mathcal{X}$. In Section 3, our main result is stated and proved. A sufficient algebraic condition for the boundedness of $\mathcal{C}_e$ is given. In Section 4, we remark some possible extensions. And finally, Section 5 contains a couple of examples in low dimensional Lie groups.

2 Preliminaries

In what follows Σ will denote a linear system on a connected Lie group G. In this section, we establish the basic definitions and the main results about the topological and dynamic structure of Σ. In particular, we list some properties of the reachable sets of Σ and we mention the Lie algebra decomposition induces by the drift vector field $\mathcal{X}$ on $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$ and, its dynamics consequences on the corresponding closed subgroups G^+, G^0, and G^- of G.

2.1 The dynamic structure of Σ

As we mention, a linear system Σ is furnished by

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t) X^j(g(t)), \ u \in \mathcal{U}.$$

Essentially, its dynamic behavior is determined by two different classes of vector fields. First, the uncontrolled differential equation $\dot{g}(t) = \mathcal{X}(g(t))$. Denote by $(\varphi_t)_{t \in \mathbb{R}}$ the flow of $\mathcal{X}$. By definition, $\mathcal{X}$ is an infinitesimal automorphism, which means

$$\{\varphi_t : t \in \mathbb{R}\} \text{ is a subgroup of } Aut(G),$$

where $Aut(G)$ is the Lie group of all G-automorphisms. Associated with $\mathcal{X}$ there exists a derivation $\mathcal{D}$ of $\mathfrak{g}$ supplied by

$$\mathcal{D}Y = -[\mathcal{X}, Y](e), \text{ for all } Y \in \mathfrak{g}.$$

The relationship between φ_t and $\mathcal{D}$ is given by the formulas, [6],

$$(d\varphi_t)_e = e^{t\mathcal{D}} \text{ and } \varphi_t(\exp Y) = \exp(e^{t\mathcal{D}}Y), \ t \in \mathbb{R}, \ Y \in \mathfrak{g}.$$

On the other hand, the family of vector field $X^u = \sum_{j=1}^m u_j X^j$ depends on m fixed right invariant vector fields $X^j \in \mathfrak{g}$ and the family of admissible control $u = (u_1, ..., u_m) \in \mathcal{U}$ which has the mission to redirect $\mathcal{X}$ to reach the desired goal.

2.2 Reachable sets

For a state $g \in G$, the reachable set from g up to the time t is defined by

$$\mathcal{A}_{\leq t}(g) = \{h \in G : \exists u \in \mathcal{U} \text{ and } \tau \in [0, t] \text{ with } h = \phi_{\tau, u}(g)\}.$$

and $\mathcal{A}(g) = \bigcup_{t > 0} \mathcal{A}_{\leq t}(g)$ is the reachable set from g. We denote $\mathcal{A}(e)$ by $\mathcal{A}$.

Next, we collect the main properties of the reachable sets, see [7] and [8].

Proposition 2.1. *For a linear system Σ on the connected Lie group G it holds*
1. $0 \leq t_1 \leq t_2$ *implies* $\mathcal{A}_{t_1} \subset \mathcal{A}_{t_2}$
2. *for all* $g \in G$, $\mathcal{A}_t(g) = \mathcal{A}_t \varphi_t(g)$
3. *for all* $u \in \mathcal{U}$, $g \in G$ *and* $t \geq 0$ *it follows* $\phi_{t,u}(\mathcal{A}(g)) \subset \mathcal{A}(g)$
4. $e \in \text{int } \mathcal{A}$ *if and only if* $\mathcal{A}$ *is open*

The controllable set to g up to the time t is defined by

$$\mathcal{A}^*_{\leq t}(g) = \{h \in M : \exists u \in \mathcal{U} \text{ and } \tau \in [0, t] \text{ and } \phi_{\tau, u}(h) = g\}.$$

The controlled set to g is $\mathcal{A}^*(g) = \bigcup_{t > 0} \mathcal{A}^*_{\leq t}(g)$. We denote $\mathcal{A}^*(e)$ by $\mathcal{A}^*$.

Remark 2.2. *We assume from the start that $\mathcal{A}$ is an open set and it happens for instance, when the system satisfies the ad-rank condition, i.e.,*

$$Span\left\{\mathcal{D}^i(Y^j) : \text{where } \mathcal{D}^0 = Id, \ j = 1, ..., m \text{ and } i = 0, 1, ...\right\} = \mathfrak{g}.$$

The system is said to be locally accessible at g if $int(\mathcal{A}_{\leq t}(g))$ and $int(\mathcal{A}^*_{\leq t}(g))$ is nonempty for any $t \geq 0$, and controllable from g if $\mathcal{A}(g) = G$.

2.3 $\mathcal{D}$-Decomposable Lie groups

In this section, we look more closely at the Lie algebra decomposition induced by the derivation $\mathcal{D}$ associated with the drift vector field $\mathcal{X}$. We address the generalized eigenspaces of $\mathcal{D}$ provided by

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} : (\mathcal{D} - \alpha)^n Y = 0 \text{ for some } n \geq 1\}.$$

Here, α runs over the spectrum $Spec(\mathcal{D})$. It turns out that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in Spec(\mathcal{D})$ and 0 otherwise. Of course, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-, \text{ where}$$

$$\mathfrak{g}^+ = \bigoplus_{\alpha \, : \, Re(\alpha) > 0} \mathfrak{g}_\alpha \, , \ \mathfrak{g}^0 = \bigoplus_{\alpha \, : \, Re(\alpha) = 0} \mathfrak{g}_\alpha \text{ and } \mathfrak{g}^- = \bigoplus_{\alpha \, : \, Re(\alpha) < 0} \mathfrak{g}_\alpha.$$

It follows that $\mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^-$ are Lie subalgebras and $\mathfrak{g}^+, \mathfrak{g}^-$ are nilpotent, see Proposition 3.1 in [6].

Let us denote by G^+, G^-, G^0, $G^{+,0}$, and $G^{-,0}$ the connected Lie subgroups of G with Lie algebras $\mathfrak{g}^+$, $\mathfrak{g}^-$, $\mathfrak{g}^0$, $\mathfrak{g}^{+,0} = \mathfrak{g}^+ \oplus \mathfrak{g}^0$ and $\mathfrak{g}^{-,0} = \mathfrak{g}^- \oplus \mathfrak{g}^0$, respectively.

Definition 2.3. *Let $\mathcal{D}$ be a $\mathfrak{g}$-derivation. The Lie algebra $\mathfrak{g}$ is said to be $\mathcal{D}$-decomposable if $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$.*

We collect some basic properties of these subgroups, Proposition 2.9 in [7].

Proposition 2.4. *Let $\mathcal{D}$ be a $\mathfrak{g}$-derivation. It holds,*
1. $G^{+,0} = G^+ G^0 = G^0 G^+$ *and* $G^{-,0} = G^- G^0 = G^0 G^-$
2. $G^+ \cap G^- = G^{+,0} \cap G^- = G^{-,0} \cap G^+ = \{e\}$
3. $G^{+,0} \cap G^{-,0} = G^0$
4. $G^+, G^0, G^-, G^{+,0}$ *and* $G^{-,0}$ *are closed*
5. *If G is solvable then G is decomposable*

2.4 Control sets

A more realistic approach to the controllability property of a system comes from the following notion. A nonempty set $\mathcal{C} \subset G$ is called a control set, [3] if
i)   for every $g \in G$ there exists $u \in \mathcal{U}$ such that $\phi(t, g, u) \subset \mathcal{C}, t \geq 0$
ii) $\mathcal{C} \subset \mathrm{cl}(\mathcal{A}(g))$ for every $g \in \mathcal{C}$
iii) $\mathcal{C}$ is maximal with properties (i) and (ii).

In [4] the authors prove general results about the shape of an existent control set, that we specialize in our particular class of linear systems, as follows:

Lemma 2.5. *Let $\mathcal{C}$ be a control set of Σ. If the system is locally accessible at any point of $\mathrm{int}(\mathcal{C})$ then for any $y \in \mathrm{int}\,\mathcal{C}$*

$$\mathcal{C} = \mathrm{cl}(\mathcal{A}(y)) \cap \mathcal{A}^*(y).$$

In particular, the system is controllable on $\mathrm{int}\,\mathcal{C}$.

Instead to study the strong (global) controllability property of Σ we are looking for a weak conditions to obtain regions where controllability holds.

3 Main result

In this section, our main results are stated and proved. For that, we apply several results appearing in [5]. From now we assume that G is decomposable. It turns out that $\mathcal{A}$ and $\mathcal{A}^*$ are also decomposable. Denote by $\mathcal{A}_{G^-} = \mathcal{A} \cap G^-$ and $\mathcal{A}_{G^+}^* = \mathcal{A}^* \cap G^+$ then

$$\mathcal{A} = \mathcal{A}_{G^-} G^{+,0}, \;\; \mathcal{A}^* = \mathcal{A}_{G^+}^* G^{-,0}.$$

Furthermore, $\mathcal{A}_{G^-}$, $\mathcal{A}_{G^+}^*$ and G^0 are contained in $\mathcal{A} \cap \mathcal{A}^*$.

     We assume that the reachable set $\mathcal{A}$ is open then the system is locally accessible in a neighborhood of e. From Lemma 2.5, Σ has a control set $\mathcal{C}_e = \mathrm{cl}(\mathcal{A}) \cap \mathcal{A}^*$. On the other hand, by hypothesis $\mathfrak{g}$ is $\mathcal{D}$-decomposable hence $\mathcal{C}_e$ is the only control sets with nonempty interior.

     It is clear that

$$\mathcal{C}_e = \mathrm{cl}(\mathcal{A}) \cap \mathcal{A}^* \text{ bounded} \Rightarrow \mathrm{cl}(\mathcal{A}_{G^-}), \; \mathrm{cl}(\mathcal{A}_{G^+}^*) \text{ and } G^0 \text{ bounded}.$$

In the sequel, we analyze a kind of converse. Actually, in some special cases, the boundedness of these three sets imply the boundedness of $\mathcal{C}_e$. The following two results were proved in [5].

Theorem 3.1. *Let us assume that G is semisimple or nilpotent. If $\mathrm{cl}(\mathcal{A}_{G^-})$, $\mathrm{cl}(\mathcal{A}_{G^+}^*)$ and G^0 are compact subsets of G then $\mathcal{C}_e$ is bounded.*

Recall that a linear transformation L is said to be hyperbolic if L has just eigenvalues with nonzero real parts, in other words

$$\mathrm{Spec}_{Ly}(\mathcal{D}) \cap \mathbb{R} = \{0\}.$$

Theorem 3.2. *Let G be a nilpotent simply connected Lie group. Then,*

$$\mathcal{C} \text{ is bounded } \Leftrightarrow \mathrm{cl}(\mathcal{A}_{G^-}), \ \mathrm{cl}(\mathcal{A}^*_{G^+}) \text{ are compacts and } \mathcal{D} \text{ is hyperbolic.}$$

Remark 3.3. *The main aim of the paper is to find algebraic conditions to decide wether* $\mathrm{cl}(\mathcal{A}_{G^-}), \ \mathrm{cl}(\mathcal{A}^*_{G^+})$ *are bounded sets. With that and the hyperbolic notion we can ensure the boundedness of the control set $\mathcal{C}_e$.*

Let Σ be a linear system on a connected Lie group G. If $\mathcal{D}$ is a stable matrix the reachable set on G^- is bounded. More precisely:

Proposition 3.4. *Let $\mathcal{D}$ be the derivation associated with $\mathcal{X}$. If $Spec_{Ly}(\mathcal{D}) \subset \mathbb{R}^-$ then the reachable set $\mathcal{A} = \mathcal{A}^-$ is bounded*

Proof. Let us denote by ρ the left invariant metric of G, [9]. There exist $c > 1$ and $\lambda > 0$ such that

$$\rho(\varphi_t(g), e) \leq c^{-1} e^{-\lambda t} \rho(g, e), \text{ for any } t \geq 0, \quad (*)$$

In fact, consider a curve $\gamma : [0, 1] \to G$ with $\gamma(0) = e$ and $\gamma(1) = g$. Thus, $\varphi_t \circ \gamma$ is a curve connecting e to $\varphi_t(g)$ and

$$\rho(\varphi_t(g), e) \leq \int_0^1 \left\| (d(\varphi_t)_{\gamma(s)}(\dot{\gamma}(s)) \right\| ds.$$

Now, any G-homomorphism ϕ satisfies the formula

$$\phi \circ L_g = L_{\phi(g)} \circ \phi.$$

Subsequently,

$$(d\phi)_g = (dL_{\phi(g)})_e \circ (d\phi)_e \circ (dL_{g^{-1}})_g.$$

The homomorphism φ_t belongs to $Aut(G)$ for any $t \in \mathbb{R}$ and $\mathcal{D} = d(\varphi_t)_e$. Since the metric ρ is left invariant, we get

$$\| d(\varphi_t)_g \| = \left\| e^{t\mathcal{D}} \right\|.$$

By hypothesis $\mathcal{D}$ is a stable matrix, then $(*)$ follows.

Take $t > 0$ such that $\mathcal{A}_t \subset B(e, 1)$ the open ball with center e and radius 1. Just observe that for every positive number τ

$$\varphi_\tau(B(e, 1)) \subset B(e, c^{-1} e^{-\lambda \tau}).$$

By using the same argument we obtain

$$\mathcal{A}_{(n+1)t} = \mathcal{A}_t \, \varphi_t(\mathcal{A}_t) \, \varphi_{2t}(\mathcal{A}_t) ... \varphi_{nt}(\mathcal{A}_t) \subset B(e, 1) B(e, c^{-1} e^{-\lambda t}) B(e, c^{-1} e^{-2\lambda t}) ... B(e, c^{-1} e^{-n\lambda t}).$$

Now, any $g \in G$ can be decomposed as

$$g = g_0 g_1 g_2 ... g_n \text{ with } g_i \in B(e, c^{-1} e^{-it\lambda}), i = 0, 1, ..., n.$$

Since the metric is left invariant, the following inequalities are true

$$\begin{aligned}
\rho(g, e) &= \rho(g_0 g_1 g_2 ... g_n, e) \\
&\leq \rho(g_0 g_1 g_2 ... g_n, g_0) + \rho(g_0, e) \\
&\leq \rho(g_1 g_2 ... g_n, e) + \rho(g_0, e) \leq \rho(g_1 g_2 ... g_n, g_1) + \rho(g_1, e) + \rho(g_0, e) \\
&\leq \rho(g_2 ... g_n, e) + \rho(g_1, e) + \rho(g_0, e)
\end{aligned}$$

$$\cdots$$
$$\le \Sigma_{i=0}^{n}\rho(g_i, e) < \Sigma_{i=0}^{n} c^{-1} e^{-i\lambda t} < c^{-1}\Sigma_{i=0}^{\infty}(e^{-\lambda t})^i < \infty.$$

Hence, there exists a radius R such that

$$R = c^{-1}\Sigma_{i=0}^{\infty}(e^{-\lambda t})^i > 0 \;\Rightarrow\; \mathcal{A}_{nt} \subset B(e, R), \text{ for any } n \in \mathbb{N}.$$

This ends the proof, actually

$$\mathcal{A} = \cup_{n\in\mathbb{N}}\mathcal{A}_{nt} \subset B(e, R). \hspace{3cm} \square$$

Now, we are able to prove our main result.

Theorem 3.5. *Let Σ be a linear system on a decomposable connected Lie group G. Assume that $\mathfrak{g}^{+,0}$ is an ideal of $\mathfrak{g}$ then $\mathrm{cl}(\mathcal{A}_{G^-})$ is bounded.*

Proof. According to our hypothesis the group is decomposable, thus

$$G = G^- G^0 G^+.$$

Since $\mathfrak{g}^{+,0}$ is an ideal the Lie subgroup $G^{+,0}$ is normal. In particular, the homogeneous space $G/G^{+,0}$ is a Lie group isomorphic to G^-. Let us consider the canonical projection $\pi : G \to G/G^{+,0}$. It turns out that $\pi(\mathcal{A}) = \mathcal{A}^-$. Furthermore, on $G/G^{+,0}$ the derivation $\mathcal{D}$ associated to the drift vector field $\mathcal{X}$ of Σ has just eigenvalues with negative real parts. In other words, $\mathcal{D}^-$ is the corresponding derivation associated with the system Σ^- in G^-. In fact, the Lie algebra of $G/G^{+,0}$ is isomorphic to $\mathfrak{g}^{+,0} = \mathfrak{g}^+ \oplus \mathfrak{g}^0$ which is isomorphic to $\mathfrak{g}^-$.

Therefore, Proposition 3.4 implies that the reachable set $\mathcal{A}^-$ of Σ^- is bounded in G^-. In the sequel, we prove that this condition is enough to show that the reachable set $\mathcal{A}$ is bounded in G^-. However, we first need to show that

$$\pi_{G^-} : G^- \to G/G^{+,0} \text{ is a homeomorphism.}$$

Actually, since any element in G has a unique decomposition in $G^- G^0 G^+$ the application is bijective. By the own definition of the quotient topology on $G/G^{+,0}$ the projection π restricted to G^- is continuous. Next, we prove that π_{G^-} is an open map. First, there are neigborhoods $V \subset G^-$ and $W \subset G^{+,0}$ of the identity $e \in G$ such that the product VW is also a neigborhood of e. In particular, $\pi_{G^-}(V) = \pi(V) = \pi(VW)$ is an open set in $G/G^{+,0}$. If $g \in G$ we consider the translations $L_g(V) = gV$ and $L_g(W) = gW$. Since Lg is a homeomorphism, the proof is done and π_{G^-} is a homeomorphism.

Once again, the group G is decomposable, thus $\pi(G^-) = G/G^{+,0}$ and it is possible to cover $\mathcal{A}$ with the projection of a compact subset of $\mathcal{A}$. In fact, for any compact K containing $\mathcal{A}^-$ define the compact set $K^- = (\pi_{G^-})^{-1}(K) \subset G^-$ such that $\mathcal{A}^- \subset \pi(K^-)$. From that, we obtain

$$\pi(\mathcal{A}) = \mathcal{A}^- \Rightarrow \mathcal{A} \subset K^- G^{+,0} \Rightarrow \mathcal{A}_{G^-} = \mathcal{A}\cap G^- \subset K^-.$$

Since $\mathcal{A}^-$ is bounded, it follows that $\mathrm{cl}(\mathcal{A}_{G^-})$ is also bounded as we claim. $\hspace{1cm}\square$

Corollary 3.6. *Let Σ be a linear system on a decomposable connected Lie group G. Assume that $\mathfrak{g}^{-,0}$ is an ideal of $\mathfrak{g}$ then $\mathrm{cl}(\mathcal{A}_{G^+}^*)$ is bounded.*

Proof. The proof is completely analogous to that of Theorem 3.5. $\hspace{1cm}\square$

Every nilpotent Lie group as a solvable group is decomposable, see [5].

Theorem 3.7. *Let Σ be a linear system on a nilpotent simply connected Lie group G. Assume that $\mathfrak{g}^{+,0}$ and $\mathfrak{g}^{-,0}$ are ideals of $\mathfrak{g}$. Then,*
1. *$\mathcal{D}$ hyperbolic $\Leftrightarrow$ the control set $\mathcal{C}_e = \mathrm{cl}(\mathcal{A})\cap \mathcal{A}^*$ is bounded*
2. *$G = G^- \Rightarrow \mathcal{A}^* = G$ and $\mathcal{C}_e = \mathrm{cl}(\mathcal{A})$ is compact*
3. *$G = G^+ \Rightarrow \mathcal{A} = G$ and $\mathcal{C}_e = \mathcal{A}^*$ is open*

Proof. 1.   We have,

$$\mathcal{D} \text{ hyperbolic} \Leftrightarrow \mathfrak{g}^0 = \{0\} \Leftrightarrow \mathfrak{g}^0 \text{ is compact} \Leftrightarrow G^0 \text{ is compact.}$$

The last equivalence depends strongly on the fact that in this particular case the exponential map is a global diffeomorphism. Just observe that in general this is not true. For instance, $\exp(\mathbb{R}) = S^1$, however, the 1-dimensional sphere is not simply connected. Now, our hypothesis and Theorem 3.5 implies that $\text{cl}(\mathcal{A}_{G^-})$ and $\text{cl}(\mathcal{A}_{G^+}^*)$ are bounded. Thus, Theorem 3.1 shows that $\mathcal{C}_e$ is bounded. On the other hand, if $\mathcal{C}_e$ is bounded it follows that $G^0 \subset \text{cl}(\mathcal{A})$ is compact and ending the proof.

2.   To prove the second item we observe that under the hypothesis $G \subset \mathcal{A}^*$. So, $\mathcal{C}_e = \text{cl}(\mathcal{A})$ is trivially closed and bounded by Theorem 3.1.

3.   If $G = G^+$ we get $\mathcal{A} = G$. Thus, $\mathcal{C}_e$ coincides with the open set $\mathcal{A}^*$. $\square$

Remark 3.8. *We observe that item third of Theorem 3.7 shows that Σ is controllable from the identity, i.e., for any arbitrary $g \in G$ there exists a control u and a positive time t such that $\phi_{t,u}(e) = g$. For other results in the same spirit, we invite the readers to take a look at the following references,* $[2, 7, 8, 10, 11]$. *Furthermore, in* $[12]$ *the author shows that the class of linear control systems is important in a theoretical way. He proves an equivalent theorem which involves a class of nonlinear control systems on general manifolds.*

A sufficient condition for the simultaneous boundedness of $\text{cl}(\mathcal{A}_{G^-})$ and $\text{cl}(\mathcal{A}_{G^+}^*)$ is to assume that both $\mathfrak{g}^{+,0}$ and $\mathfrak{g}^{-,0}$ are ideals of $\mathfrak{g}$. An equivalent condition is given by the next proposition.

Proposition 3.9. *Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{D} \in \partial\mathfrak{g}$. It turns out*

$$\mathfrak{g}^{+,0} \text{ and } \mathfrak{g}^{-,0} \text{ are ideals of } \mathfrak{g} \Leftrightarrow \left[\mathfrak{g}^0, \mathfrak{g}^+\right] = 0 \text{ and } \left[\mathfrak{g}^+, \mathfrak{g}^-\right] \subset \mathfrak{g}^0.$$

4 Extensions

In this paper, we concentrate the study on decomposable Lie groups. However, one might be tempted to try to extend the result to semisimple groups. Let us consider an unrestricted linear system Σ on a connected semisimple Lie group G. In this case, we have two possibilities

1.   The compact case

In $[10]$ the authors prove the following result:

Theorem 4.1. *If G is a connected and compact semisimple Lie group, a linear system Σ is controllable on G if and only if the system is transitive, i.e., satisfies the Lie algebra rank condition, (LARC), provided by*

$$Span_{LA} \left\{ \mathcal{D}^i(Y^j) : where \ \mathcal{D}^0 = Id, \ j = 1, \dots, m \ and \ i = 0, 1, \dots \right\} = \mathfrak{g}.$$

The LARC condition is weaker than the ad-rank condition. Actually, in the first case you are allowed to compute the Lie brackets $\left[\mathcal{D}^{i_1}(Y^{j_1}), \mathcal{D}^{i_2}(Y^{j_2})\right]$, which is forbidden in the other case. Therefore,

$$\mathcal{C}_e = G \text{ for any transitive linear system on } G.$$

2.   The noncompact case

Here, we just comment that except the case $G = G^0$, the space state cannot be decomposable. In fact, in $[5]$ we show that the set $G^- G^0 G^+ \subsetneq G$ is just an open Bruhat cell which is dense in the group all. In particular, our results can not be extended in this direction.

5 Examples

In this section, we give some examples of boundedness and unboundedness control sets on some decomposable Lie groups. But first, we explain how to find the face of the drift $\mathcal{X}$ when it is induced by an inner derivation.

Remark 5.1. *A particular class of linear vector fields is easy computed through a* 1*-parameter of inner G-automorphisms. Take $X \in \mathfrak{g}$ a right invariant vector field and consider the solution $X_t(g)$ with initial condition $g \in G$. By the right invariance, the solution through the initial condition g is provided by the right translation by g of the solution $X_t(e) = \exp_G(tX)$ through the identity element. In order words*

$$X_t(g) = \exp_G(tX)g.$$

Here, $\exp_G : \mathfrak{g} \to G$ is the usual exponential map. Hence, X defines by conjugation a 1-parameter group of inner automorphism as follows

$$\varphi_t(g) = X_t(e)\, g\, X_{-t}(e), g \in G, \text{ and } \varphi_t \in \text{Aut}(G) \text{ for any } t \in \mathbb{R}.$$

Therefore, it is possible to compute the linear vector field as

$$\mathcal{X}(g) = (\frac{d}{dt})_{t=0}\, \varphi_t(g).$$

The associated derivation $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$ is $\mathcal{D}(Y) = -[X, Y], Y \in \mathfrak{g}$.

    Recall that any derivation on a semisimple Lie group is inner. This property has interest for us in the compact case. On the other hand, in [13] we built an algorithm which provides an effective means to compute the Lie algebra $\partial\mathfrak{g}$ that we use in this section.

Example 5.2. *Consider the solvable affine group*

$$\left\{ G = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0 \text{ and } y \in \mathbb{R} \right\}$$

with Lie algebra $\mathfrak{g} = Span\{X, Y\}$ and $[X, Y] = Y$. An easy computation shows that $\partial\mathfrak{g}$ is given just by inner derivation with the shape

$$\partial\mathfrak{g} = \left\{ \mathcal{D} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

From Remark 5.1 the linear vector field $\mathcal{X}$ associated to $\mathcal{D}$ is given by

$$\mathcal{X}_{(x,y)} = \begin{pmatrix} 0 & a(x-1) + by \\ 0 & 0 \end{pmatrix}.$$

Let Σ be the transitive linear system on G defined by

$$\dot{g}(t) = \mathcal{X}(g(t)) + u(t)X(g(t)), \ u \in \mathcal{U}$$

where, $\mathcal{D} = ad(Y)$ comes from $a = -1$ and $b = 0$. Since $ad(Y)X = -Y$ then $Span\{X, \mathcal{D}X\} = \mathfrak{g}$. So, Σ satisfies the ad-rank condition, $\mathcal{A}$ is open and of course, Σ satisfies also LARC. Moreover, G is solvable thus the control set $\mathcal{C}_e$ is the only one with nonempty interior. It turns out that,

$$\mathfrak{g}^+ = \mathfrak{g}^- = 0 \ \Rightarrow \ \mathfrak{g}^0 = \mathfrak{g}.$$

Thus,

$$G^{+,0} = G \subset \mathcal{A} \text{ and } G^{-,0} = G \subset \mathcal{A}^* \Rightarrow \mathcal{C}_e = G.$$

To conclude, the system is controllable from the identity. This fact is completely concordant with Theorem 3 in [11]. Actually, it is shown there that a transitive system in a canonical form, like Σ, is controllable if and only if $b = 0$.

Example 5.3. *Let* $\mathfrak{g} = \mathbb{R}X + \mathbb{R}Y + \mathbb{R}Z$ *be the Lie algebra of the connected and simply connected Heisenberg Lie group G*

$$G = \left\{ g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : (x, y, z) \in \mathbb{R}^3 \right\}$$

of dimension three. The generators of $\mathfrak{g}$ *are provided by*

$$X = \frac{\partial}{\partial x}, \ Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \text{ and } Z = \frac{\partial}{\partial z}.$$

The only one non-vanishing Lie bracket is $[X, Y] = Z$. Any derivation $\mathcal{D}$ is represented by a 6 real parameters matrix in the basis $\{X, Y, Z\}$ as follow

$$\partial\mathfrak{g} = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a+d \end{pmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\}.$$

Consider the linear system Σ with derivation $\mathcal{D}$ determined by its coefficients $a = d = -1, b = 1, c = -1, e = f = 0$ and the control vectors X and Z,

$$\dot{g}(t) = \mathcal{X}(g(t)) + u_1(t)X(g(t)) + u_2(t)Z(g(t)), \ u \in \mathcal{U}, \text{ with } \Omega = [-1, 1].$$

We have, $Spec_{Ly}(\mathcal{D}) = \{-1, -2\}$. So, $\mathfrak{g}^{-,0} = \mathfrak{g}^- = \mathfrak{g}$ and $\mathfrak{g}^{+,0} = 0$ are both ideals of $\mathfrak{g}$. On the other hand,

$$Span\{X, Z, \mathcal{D}(X) = X - Y\} = \mathfrak{g}.$$

Since $\mathcal{D}$ is a hyperbolic derivation, Theorem 3.7 shows that the existent control set $\mathcal{C}_e$ is bounded.

Example 5.4. *On the rotational group* $S0(3, \mathbb{R})$ *with Lie algebra* $\mathfrak{so}(3, \mathbb{R})$ *the skew-symmetric real matrix of order three*

$$\mathfrak{g} = Span\{X, Y, Z\}$$

consider the system

$$\dot{g}(t) = \mathcal{X}(g(t)) + u_1(t)X(g(t)) + u_2(t)Y(g(t)), \ u \in \mathcal{U}, \text{ with } \Omega = \mathbb{R},$$

where $\mathcal{X} = ad(X)$*. Since* Σ *satisfies LARC, the control set is bounded and coincides with the group. The system is controllable from the identity.*

Example 5.5. *Take the linear system* Σ *on the Heisenberg group G like in Example 2, but with different dynamic behavior*

$$\dot{g}(t) = \mathcal{X}(g(t)) + u_1(t)(X - Y)(g(t)) + u_2(t)(X + Y + Z)(g(t)), \ u \in \mathcal{U}, \text{ with } \Omega = [-1, 1]$$

where the derivation $\mathcal{D}$ *is furnished by* $a = 1, d = -1$ *and* $b = c = e = f = 0$*. Hence,* $\mathcal{D}(X - Y) = X + Y$*. Thus,*

$$Span\{X - Y, Z, X + Y\} = \mathfrak{g}$$

an $\mathcal{A}$ *is an open set.*

If we restrict Σ to the plane $\mathbb{R}^2 = Span\{X, Y\}$ we get a classical linear system on the vector space $\Sigma_{\mathbb{R}^2}$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u : u \in \mathcal{U} \text{ with } \Omega = [-1, 1]$$

which satisfies Lemma 2.5. Moreover,

$$\mathcal{A} = \mathbb{R} \times (-1, 1) \text{ and } \mathcal{A}^* = (-1, 1) \times \mathbb{R}.$$

Thus, the control set $\mathcal{C}_e$ restricted to the plane is bounded and reads

$$(\mathcal{C}_e)_{\mathbb{R}^2} = (-1, 1) \times [-1, 1], \text{ see } [4].$$

However, $\mathcal{C}_e$ can not be bounded. Despite the fact that $\mathfrak{g}^{+,0} = Span\{X, Z\}$ and $\mathfrak{g}^{-,0} = Span\{Y, Z\}$ are ideals, the derivation $\mathcal{D} = diag(1, -1, 0)$ is just hyperbolic on the plane not on G. Actually,

$$\mathcal{C}_e = (-1, 1) \times [-1, 1] \times (1, 1, 1)\mathbb{R}.$$

Example 5.6. *Let us consider the nilpotent Lie group G with Lie algebra*

$$\mathfrak{g} = \mathbb{R}X_1 + \mathbb{R}X_2 + \mathbb{R}X_3 + \mathbb{R}X_4, \text{ and the rules}$$

$$[X_4, X_2] = X_4, \ [X_3, X_2] = X_4 + X_2, \ [X_1, X_2] = X_3 \text{ and } [X_1, X_3] = X_4.$$

Let Σ be a linear system with an arbitrary derivation $\mathcal{D} \in \partial\mathfrak{g}$ such that the reachable set $\mathcal{A}$ of Σ is open. Hence, the control set $\mathcal{C}_e$ is unbounded. In fact, a straightforward computation shows that the Lie algebra of $\mathfrak{g}$-derivations is five dimensional and reads as

$$\partial\mathfrak{g} = \left\{ \begin{pmatrix} a & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b & c & a+b & 0 \\ d & e & b+c+d & 2a+b \end{pmatrix} : a, b, c, d, e \in \mathbb{R} \right\}.$$

Since the underlying topological space of G is the connected and simply connected manifold $\mathbb{R}^4$, Theorem 3.5 applies. However, $0 \in Spec_{Ly}(\mathcal{D})$ for any $\mathcal{D} \in \partial\mathfrak{g}$. Thus, no hyperbolic derivation exits, ending the proof.

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