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Chen's inequalities for submanifolds in (κ, μ)-contact space form with a semi-symmetric metric connection

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Abstract: In this paper, we obtain Chen's inequalities for submanifolds in (κ, μ)-contact space form endowed with a semi-symmetric metric connection.

Keywords: (κ, μ)-contact space form, Semi-symmetric metric connection, Chen's inequalities

MSC: 53C40, 53B05

1 Introduction

Since Chen [1, 2] proposed the so-called Chen invariants, and chen's inequalities, the study of Chen invariants and inequalities has been an active field over the past two decades. There are many works studying Chen's inequalities for different submanifolds in various ambient spaces (one can see [3–12] for details).

In 1924, Friedmann and Schoutenn [13] first introduced the notion of a semi-symmetric linear connection on a differentiable manifold. Later many scholars studied the geometric and analytic problems on a manifold associated with a semi-symmetric metric connection. For instance, H. A. Hayden [14] studied the geometry of subspaces with a semi-symmetric metric connection. K. Yano [15] studied some geometric properties of a Riemannian manifold endowed with a semi-symmetric metric connection. In [16], Z. Nakao studied submanifolds of a Riemannian manifold with a semi-symmetric metric connection.

Recently, T. Koufogiorgos [17] introduced the notation of (κ, μ)-contact space form, in particular, when $\kappa = 1$, (κ, μ)-contact space form is the well known class of Sasakian space form, and A. Mihai and C. Özgür [18, 19] arrived at Chen's inequalities for submanifolds of Sasakian space forms, real(complex) space forms with a semi-symmetric metric connection. On the other hand, P. Zhang et al [20] also established Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection.

Motivated by the celebrated works above, a natural question is: can we establish the Chen's inequalities for submanifolds in a (κ, μ)-contact space form endowed with a semi-symmetric metric connection?

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In this paper, we will consider and derive Chen's first inequality and Chen-Ricci inequalities for submanifolds of a (κ, μ)-contact space form with a semi-symmetric metric connection.

2 Preliminaries

Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold with a Riemannian metric g and $\overline{\nabla}$ be a linear connection on N^{n+p}. If the torsion tensor $\overline{T}$ of $\overline{\nabla}$ satisfies

$$\overline{T}(\overline{X}, \overline{Y}) = \phi(\overline{Y})\overline{X} - \phi(\overline{X})\overline{Y}$$

for a 1-form ϕ, then the connection $\overline{\nabla}$ is called a semi-symmetric connection. Furthermore, if $\overline{\nabla}$ satisfies $\overline{\nabla} g = 0$, then $\overline{\nabla}$ is called a semi-symmetric metric connection.

Let $\overline{\nabla}'$ denote the Levi-Civita connection associated with the Riemannian metric g. In [15] K.Yano obtained a relation between the semi-symmetric metric connection $\overline{\nabla}$ and the Levi-Civita connection $\overline{\nabla}'$ which is given by

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla}'_{\overline{X}}\overline{Y} + \phi(\overline{Y})\overline{X} - g(\overline{X}, \overline{Y})U, \quad \forall \overline{X}, \overline{Y} \in \mathcal{X}(N^{n+p})$$

where U is a vector field given by $g(U, \overline{X}) = \phi(\overline{X})$.

Let M^n be an n-dimensional submanifold of N^{n+p} with the semi-symmetric metric connection $\overline{\nabla}$ and the Levi-Civita connection $\overline{\nabla}'$. On M^n we consider the induced semi-symmetric metric connection denoted by ∇ and the induced Levi-Civita connection denoted by ∇'. The Gauss formula with respect to ∇ and ∇' can be written as

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \overline{\nabla}'_X Y = \nabla'_X Y + \sigma'(X, Y), \quad \forall X, Y \in \mathcal{X}(M^n),$$

where σ' is the second fundamental form of M^n and σ is a $(0, 2)$-tensor on M^n. According to [16] σ is also symmetric.

Let $\overline{R}$ and $\overline{R}'$ denote the curvature tensors with respect to $\overline{\nabla}$ and $\overline{\nabla}'$ respectively. We also denote the curvature tensor associated with ∇ and ∇' by R and R', repectively. From [16], we know

$$\begin{aligned}\overline{R}(X, Y, Z, W) =& \overline{R}'(X, Y, Z, W) - \alpha(Y, Z)g(X, W) + \alpha(X, Z)g(Y, W) \\ &- \alpha(X, W)g(Y, Z) + \alpha(Y, W)g(X, Z),\end{aligned} \tag{1}$$

for all $X, Y, Z, W \in \mathcal{X}(M^n)$, where α is a $(0, 2)$-tensor field defined by

$$\alpha(X, Y) = (\overline{\nabla}'_X \phi)Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(U)g(X, Y).$$

Denote by λ the trace of α.

In [16] the Gauss equation for M^n with respect to the semi-symmetric metric connection is given by

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(\sigma(X, Z), \sigma(Y, W)) - g(\sigma(X, W), \sigma(Y, Z)) \tag{2}$$

for all $X, Y, Z, W \in \mathcal{X}(M^n)$.

In N^{n+p} we can choose a local orthonormal frame $\{e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}\}$ such that $\{e_1, e_2, \ldots, e_n\}$ are tangent to M^n. Setting $\sigma^r_{ij} = g(\sigma(e_i, e_j), e_r)$, then the squared length of σ is given by

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)) = \sum_{r=n}^{n+p}\sum_{i,j=1}^{n} (\sigma^r_{ij})^2.$$

The mean curvature vector of M^n associated to $\overline{\nabla}'$ is $H' = \frac{1}{n}\sum_{i=1}^n \sigma'(e_i, e_i)$. The mean curvature vector of M^n associated to $\overline{\nabla}$ is defined by $H = \frac{1}{n}\sum_{i=1}^n \sigma(e_i, e_i)$.

According to the formula (7) in [16], we have the following result.

Lemma 2.1 ([16]). *If U is a tangent vector field on M^n, then $H = H'$ and $\sigma = \sigma'$.*

Let $\pi \subset T_pM^n$ be a 2-plane section for any $p \in M^n$ and $K(\pi)$ be the sectional curvature of M^n associated to the semi-symmetric metric connection $\triangledown$. The scalar curvature τ associated to the semi-symmetric metric connection $\triangledown$ at p is defined by

$$\tau(p) = \sum_{1\leqslant i<j\leqslant n} K(e_i \wedge e_j). \tag{3}$$

Let L_k be a k-plane section of T_pM^n and $\{e_1, e_2, ..., e_k\}$ be any orthonormal basis of L_k. The scalar curvature $\tau(L_k)$ of L_k associated to the semi-symmetric metric connection $\triangledown$ is given by

$$\tau(L_k) = \sum_{1\leqslant i<j\leqslant k} K(e_i \wedge e_j). \tag{4}$$

We denote by $(\inf K)(p) = \inf\{K(\pi)|\pi \subset T_pM^n, dim\pi = 2\}$. In [1] B.-Y. Chen introduced the first Chen invariant $\delta_M(p) = \tau(p) - (\inf K)(p)$, which is certainly an intrinsic character of M^n .

Suppose L is a k-plane section of T_pM and X is a unit vector in L, we choose an orthonormal basis $\{e_1, e_2, ..., e_k\}$ of L, such that $e_1 = X$. The Ricci curvature Ric_p of L at X is given by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k} \tag{5}$$

where $K_{ij} = K(e_i \wedge e_j)$. We call $Ric_L(X)$ a k-Ricci curvature.

For each integer k, $2 \leqslant k \leqslant n$, the Riemannian invariant Θ_k [2] on M^n is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X}\{Ric_L(X)\}, p \in M^n \tag{6}$$

where L is a k-plane section in T_pM^n and X is a unit vector in L.

Recently, T. Koufogiorgos [17] introduced the notion of (κ, μ)-contact space form, which contains the well known class of Sasakian space forms for $\kappa = 1$. Thus it is worthwhile to study relationships between intrinsic and extrinsic invariants of submanifolds in a (κ, μ) -contact space form with a semi-symmetric metric connection $\overline{\triangledown}$.

A $(2m + 1)$-dimensional differentiable manifold $\hat{M}$ is called an almost contact metric manifold if there is an almost contact metric structure (φ, ξ, η, g) consisting of a $(1, 1)$ tensor field φ, a vector field ξ, a 1-form η and a compatible Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \qquad g(X, \xi) = \eta(X), \quad \forall X, Y \in \mathcal{X}(\hat{M}). \tag{7}$$

An almost contact metric structure becomes a contact metric structure if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \varphi Y)$ is the fundamental 2-form of $\hat{M}$.

In a contact metric manifold $\hat{M}$, the $(1, 1)$-tensor field h defined by $2h = \mathcal{L}_\xi\varphi$ is symmetric and satisfies

$$h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \overline{\triangledown}'\xi = -\varphi - \varphi h, \quad trace(h) = trace(\varphi h) = 0$$

where $\overline{\triangledown}'$ is a Levi-civita connection associated with the Riemannian metric g.

The (κ, μ)-nullity distribution of a contact metric manifold $\hat{M}$ is a distribution

$$N(\kappa, \mu) : p \to N_p(\kappa, \mu) = \{Z \in T_p\hat{M}|\hat{R}(X, Y)Z = \kappa[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY]\}$$

where κ and μ are constants. If $\xi \in N(\kappa, \mu)$, $\hat{M}$ is called a (κ, μ)-contact metric manifold. Since in a (κ, μ)-contact metric manifold one has $h^2 = (\kappa - 1)\varphi^2$, therefore $\kappa \leqslant 1$ and if $\kappa = 1$ then the structure is Sasakian.

The sectional curvature $\hat{K}(X, \varphi X)$ of a plane section spanned by a unit vector orthogonal to ξ is called a φ-sectional curvature. If the (κ, μ)-contact metric manifold $\hat{M}$ has constant φ-sectional curvature C, then it is called a (κ, μ)-contact space form and it is denoted by $\hat{M}(C)$. The curvature tensor of $\hat{M}(C)$ is given by [17].

$$\begin{aligned}\overline{R}'(X,Y)Z =& \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+3-4\kappa}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi)\} + \frac{c-1}{4}\{2g(X,\varphi Y)\varphi Z + g(X,\varphi Z)\varphi Y \\ &- g(Y,\varphi Z)\varphi X\} + \frac{1}{2}\{g(hY,Z)hX - g(hX,Z)hY + g(\varphi hX,Z)\varphi hY \\ &- g(\varphi hY,Z)\varphi hX\} - g(X,Z)hY + g(Y,Z)hX + \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX \\ &- g(hX,Z)Y + g(hY,Z)X - g(hY,Z)\eta(X)\xi + g(hX,Z)\eta(Y)\xi \\ &+ \mu\{\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY + g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi\}\end{aligned} \tag{8}$$

for all $X, Y, Z \in \mathcal{X}(\hat{M})$, where $c + 2\kappa = -1 = \kappa - \mu$ if $\kappa < 1$.

For a vector field X on a submanifold M of a (κ, μ)-contact form $\hat{M}(c)$, let PX be the tangential part of φX. Thus, P is an endomorphism of the tangent bundle of M and satisfies $g(X, PY) = -g(PX, Y)$ for all $X, Y \in \mathcal{X}(M)$. $(\varphi h)^T X$ and $h^T X$ are the tangential parts of φhX and hX respectively. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of T_pM. We set

$$\|\vartheta\|^2 = \sum_{i,j=1}^{n} g(e_i, \vartheta e_j)^2, \quad \vartheta \in \{P, (\varphi h)^T, h^T\}.$$

Let $\pi \subset T_pM$ be a 2-plane section spanned by an orthonormal basis $\{e_1, e_2\}$. Then, $\beta(\pi)$ is given by

$$\beta(\pi) = \langle e_1, Pe_2\rangle^2$$

is a real number in $[0, 1]$, which is independent of the choice of orthonormal basis $\{e_1, e_2\}$. Put

$$\gamma(\pi) = (\eta(e_1))^2 + (\eta(e_2))^2,$$

$$\theta(\pi) = \eta(e_1)^2 g(h^T e_2, e_2) + \eta(e_2)^2 g(h^T e_1, e_1) - 2\eta(e_1)\eta(e_2)g(h^T e_1, e_2).$$

Then $\gamma(\pi)$ and $\theta(\pi)$ are also real numbers and do not depend on the choice of orthonormal basis $\{e_1, e_2\}$, of course, $\gamma(\pi) \in [0, 1]$

3 Chen first inequality

For submanifolds of a (κ, μ) -contact space form endowed with a semi-symmetric metric connection $\overline{\nabla}$, we establish the following optimal inequality relating the scalar curvature and the squared mean curvature associated with the semi-symmetric metric connection, which is called Chen first inequality [1]. At first we recall the following lemma.

Lemma 3.1 ([20]). *Let $f(x_1, x_2, \ldots, x_n)$ $(n \geq 3)$ be a function in R^n defined by*

$$f(x_1, x_2, \ldots, x_n) = (x_1 + x_2)\sum_{i=3}^{n} x_i + \sum_{3 \leqslant i < j \leqslant n} x_i x_j.$$

If $x_1 + x_2 + \cdots + x_n = (n-1)\varepsilon$, then we have

$$f(x_1, x_2, \ldots, x_n) \leqslant \frac{(n-1)(n-2)}{2}\varepsilon^2$$

with the equality holding if and only if $x_1 + x_2 = x_3 = \cdots = x_n = \varepsilon$.

Theorem 3.2. *Let M ba an n-dimensional* $(n \geq 3)$ *submanifold of a* $(2m+1)$*-dimensional* (κ, μ)*-contact form* $\hat{M}(c)$ *endowed with a semi-symmetric metric connection* $\overline{\nabla}$ *such that* $\xi \in TM$. *For each* 2*-plane section* $\pi \subset T_pM$, *we have*

$$\begin{aligned}\tau(p) - K(\pi) \leqslant& \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{8}n(n-3)(c+3) + (n-1)\kappa + \frac{3(c-1)}{8}\{\|P\|^2 - 2\beta(\pi)\} \\ &+ \frac{1}{4}(c+3-4\kappa)\gamma(\pi) - (\mu-1)\theta(\pi) - \frac{1}{2}\{2trace(h|_\pi) + det(h|_\pi) - det((\varphi h)|_\pi)\} \\ &+ (\mu+n-2)trace(h^T) + \frac{1}{4}\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - (trace(\varphi h)^T)^2 + (trace(h^T))^2\} \\ &- (n-1)\lambda + trace(\alpha|_\pi).\end{aligned} \tag{9}$$

If U is a tangent vector field to M, then the equality in (9) *holds at* $p \in M$ *if and only if there exists an orthonormal basis* $\{e_1, e_2, \dots, e_n\}$ *of* T_pM *and an orthonormal basis* $\{e_{n+1}, \dots, e_{2m+1}\}$ *of* $T_p^{\perp}M$ *such that*

$$\pi = Span\{e_1, e_2\}$$

and the forms of shape operators $A_r \equiv A_{e_r}$ $(r = n+1, \dots, 2m+1)$ *become*

$$A_{n+1} = \begin{pmatrix} \sigma_{11}^{n+1} & 0 & 0 \\ 0 & \sigma_{22}^{n+1} & 0 \\ 0 & 0 & (\sigma_{11}^{n+1} + \sigma_{22}^{n+1})I_{n-2} \end{pmatrix},$$

$$A_r = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 \\ \sigma_{12}^r & -\sigma_{11}^r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \quad r = n+2, \cdots, 2m+1.$$

Proof. Let $\pi \subset T_pM$ be a 2-plane section. We choose an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ for T_pM and $\{e_{n+1}, \dots, e_{2m+1}\}$ for $T_p^{\perp}M$ such that $\pi = Span\{e_1, e_2\}$.

Setting $X = W = e_i$, $Y = Z = e_j$, $i \neq j$, $i, j = 1, \dots, n$, and using (1), (2), (8), we have

$$\begin{aligned}R_{ijji} =& \frac{c+3}{4} + \frac{c+3-4\kappa}{4}\{-\eta(e_i)^2 - \eta(e_j)^2\} + \frac{c-1}{4}\{3g(e_i, \varphi e_j)^2\} + \frac{1}{2}\{g(e_i, \varphi he_j)^2 \\ &- g(e_i, he_j)^2 + g(e_i, he_i)g(e_j, he_j) - g(e_i, \varphi he_i)g(e_j, \varphi he_j)\} + g(e_i, he_i) \\ &+ 2\eta(e_i)\eta(e_j)g(e_i, he_j) - g(he_i, e_i)\eta(e_j)^2 - g(he_j, e_j)\eta(e_i)^2 + g(he_j, e_j) \\ &+ \mu\{g(he_i, e_i)\eta(e_j)^2 + g(he_j, e_j)\eta(e_i)^2 - 2\eta(e_i)\eta(e_j)g(e_i, he_j)\} \\ &+ g(\sigma(e_i, e_i), \sigma(e_j, e_j)) - g(\sigma(e_i, e_j), \sigma(e_i, e_j)) - \alpha(e_i, e_i) - \alpha(e_j, e_j).\end{aligned} \tag{10}$$

From (10) we get

$$\begin{aligned}\tau =& \frac{1}{8}\{n(n-1)(c+3) + 3(c-1)\|P\|^2 - 2n(n-1)(c+3-4\kappa)\} + \frac{1}{4}\{\|(\varphi h)^T\|^2 - \|h^T\|^2 \\ &- (trace(\varphi h^T))^2 + (trace(h^T))^2\} + (\mu+n-2)trace(h^T) - (n-1)\lambda \\ &+ \sum_{r=1}^{2m+1}\sum_{1\leqslant i<j\leqslant n}[\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2].\end{aligned} \tag{11}$$

On the other hand, using (10) we have

$$\begin{aligned}R_{1221} =& \frac{1}{4}\{c+3+3(c-1)\beta(\pi) - (c+3-4\kappa)\gamma(\pi) + 4(\mu-1)\theta(\pi)\} \\ &+ \frac{1}{2}\{det(h|_\pi) - det((\varphi h)|_\pi) + 2trace(h|_\pi)\} - trace(\alpha|_\pi) \\ &+ \sum_{r=n+1}^{2m+1}[\sigma_{11}^r\sigma_{22}^r - (\sigma_{12}^r)^2].\end{aligned} \tag{12}$$

From (11) and (12) it follows that

$$\begin{aligned}\tau - K(\pi) =& \frac{1}{8}n(n-3)(c+3) + (n-1)\kappa + \frac{3(c-1)}{8}[\|P\|^2 - 2\beta(\pi)] + \frac{1}{4}(c+3-4\kappa)\gamma(\pi)\\ &- (\mu-1)\theta(\pi) - \frac{1}{2}\{2trace(h|_\pi) + det(h|_\pi) - det(\varphi h|_\pi)\} + (\mu+n-2)trace(h^T)\\ &+ \frac{1}{4}\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - (trace(\varphi h)^T)^2 + (trace(h^T))^2\} - (n-1)\lambda + trace(\alpha|_\pi)\\ &+ \sum_{r=n+1}^{2m+1}[(\sigma_{11}^r + \sigma_{22}^r)\sum_{3\leqslant i\leqslant n}\sigma_{ii}^r + \sum_{3\leqslant i<j\leqslant n}\sigma_{ii}^r\sigma_{jj}^r - \sum_{3\leqslant j\leqslant n}(\sigma_{1j}^r)^2 - \sum_{2\leqslant i<j\leqslant n}(\sigma_{ij}^r)^2)]\\ \leqslant& \frac{1}{8}n(n-3)(c+3) + (n-1)\kappa + \frac{3(c-1)}{8}[\|p\|^2 - 2\beta(\pi)] + \frac{1}{4}(c+3-4\kappa)\gamma(\pi)\\ &- (\mu-1)\theta(\pi) - \frac{1}{2}\{2trace(h|_\pi) + det(h|_\pi) - det(\varphi h|_\pi)\} + (\mu+n-2)trace(h^T)\\ &+ \frac{1}{4}\{\|(\varphi h)\|^2 - \|h^T\|^2 - (trace(\varphi h)^T)^2 + (trace(h^T))^2\} - (n-1)\lambda + trace(\alpha|_\pi)\\ &+ \sum_{r=n+1}^{2m+1}[(\sigma_{11}^r + \sigma_{22}^r)\sum_{3\leqslant i\leqslant n}\sigma_{ii}^r + \sum_{3\leqslant i<j\leqslant n}\sigma_{ii}^r\sigma_{jj}^r].\end{aligned} \tag{13}$$

Let us consider the following problem:

$$\max\{f_r(\sigma_{11}^r, \ldots, \sigma_{nn}^r) = (\sigma_{11}^r + \sigma_{22}^r)\sum_{3\leqslant i\leqslant n}\sigma_{ii}^r + \sum_{3\leqslant i<j\leqslant n}\sigma_{ii}^r\sigma_{jj}^r \,|\, \sigma_{11}^r + \ldots + \sigma_{nn}^r = k^r\}$$

where k^r is a real constant. From Lemma 3.1, we know

$$f_r \leqslant \frac{n-2}{2(n-1)}(k^r)^2, \tag{14}$$

with the equality holding if and only if

$$\sigma_{11}^r + \sigma_{22}^r = \sigma_{ii}^r = \frac{k^r}{n-1}, \; i = 3, \ldots, n. \tag{15}$$

From (13) and (14), we have

$$\begin{aligned}\tau - K(\pi) \leq& \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{8}n(n-3)(c+3) + (n-1)\kappa + \frac{3(c-1)}{8}[\|P\|^2 - 2\beta(\pi)]\\ &+ \frac{1}{4}(c+3-4\kappa)\gamma(\pi) - (\mu-1)\theta(\pi) - \frac{1}{2}[2trace(h|_\pi) + det(h|_\pi) - det((\varphi h)|_\pi)]\\ &+ (\mu+n-2)trace(h^T) + \frac{1}{4}[\|(\varphi h)^T\|^2 - \|h^T\|^2 - (trace(\varphi h)^T)^2 + (trace(h^T))^2]\\ &- (n-1)\lambda + trace(\alpha|_\pi).\end{aligned}$$

If the equality in (9) holds, then the inequalities given by (13) and (14) become equalities. In this case we have

$$\sum_{3\leqslant i\leqslant n}(\sigma_{1i}^r)^2 = 0\,, \quad \sum_{2\leqslant i<j\leqslant n}(\sigma_{ij}^r)^2 = 0\,, \; \forall r.$$

$$\sigma_{11}^r + \sigma_{22}^r = \sigma_{ii}^r\,, \; 3 \leqslant i \leqslant n\,, \; \forall r.$$

Since U is a tangent vector field to M, we know $\sigma' = \sigma$. By choosing a suitable orthonormal basis, the shape operators take the desired forms.

The converse is easy to show. □

For a Sasakian space form $\hat{M}(c)$, we have $\kappa = 1$ and $h = 0$. So using Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let M be an n-dimensional $(n \geq 3)$ submanifold in a Sasakian space form $\hat{M}(c)$ endowed with a semi-symmetric metric connection such that $\xi \in TM$. Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have*

$$\begin{aligned}\tau - K(\pi) \leqslant & \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{8}n(n-3)(c+3) + (n-1) + \frac{3(c-1)}{8}[\|P\|^2 - 2\beta(\pi)] \\ & + \frac{c-1}{4}\gamma(\pi) - (n-1)\lambda + trace(\alpha|_\pi).\end{aligned} \tag{16}$$

If U is a tangent vector field to M, then the equality in (16) holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of T_pM and orthonormal basis $\{e_{n+1}, \dots, e_{2m+1}\}$ of $T_p^\perp M$ such that

$$\pi = Span\{e_1, e_2\}$$

and the forms of shape operators $A_r \equiv A_{e_r} (r = n+1, \dots, 2m+1)$ become

$$A_{n+1} = \begin{pmatrix} \sigma_{11}^{n+1} & 0 & 0 \\ 0 & \sigma_{22}^{n+1} & 0 \\ 0 & 0 & (\sigma_{11}^r + \sigma_{22}^r)I_{n-2} \end{pmatrix},$$

$$A_r = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 \\ \sigma_{12}^r & -\sigma_{11}^r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \quad r = n+2, \cdots, 2m+1.$$

Since in case of non-Sasakian (κ, μ)-contact space form, we have $\kappa < 1$, thus $c = -2\kappa - 1$ and $\mu = \kappa + 1$. Putting these values in (9), we can have a direct corollary to Theorem 3.2.

Corollary 3.4. *Let Let M be an n-dimensional $(n \geq 3)$ submanifold in a non-Sasakian (κ, μ)-contact space form $\hat{M}(c)$ with a semi-symmetric metric connection such that $\xi \in TM$. Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have*

$$\begin{aligned}\tau - K(\pi) \leqslant & \frac{n^2(n-2)}{2(n-1)}\|H\|^2 - \frac{1}{4}n(n-3)(\kappa-1) + (n-1)\kappa - \frac{3}{4}(\kappa+1)\|P\|^2 \\ & + \frac{1}{2}[3(\kappa+1)\beta(\pi) - (3\kappa-1)\gamma(\pi) - 2\kappa\theta(\pi)] \\ & - \frac{1}{2}[2trace(h|_\pi) + det(h|_\pi) - det((\varphi h)|_\pi)] + (\kappa+n-1)trace(h^T) \\ & + \frac{1}{4}[\|(\varphi h)^T\|^2 - \|h^T\|^2 - (trace(\varphi h)^T)^2 + (trace(h^T))^2] - (n-1)\lambda + trace(\alpha|_\pi).\end{aligned} \tag{17}$$

If U is a tangent vector field to M, then the equality in (17) holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of T_pM and orthonormal basis $\{e_{n+1}, \dots, e_{2m+1}\}$ of $T_p^\perp M$ such that

$$\pi = Span\{e_1, e_2\}$$

and the forms of shape operators $A_r \equiv A_{e_r} (r = n+1, \dots, 2m+1)$ become

$$A_{n+1} = \begin{pmatrix} \sigma_{11}^{n+1} & 0 & 0 \\ 0 & \sigma_{22}^{n+1} & 0 \\ 0 & 0 & (\sigma_{11}^r + \sigma_{22}^r)I_{n-2} \end{pmatrix},$$

$$A_r = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 \\ \sigma_{12}^r & -\sigma_{11}^r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \quad r = n+2, \cdots, 2m+1.$$

4 Ricci and k-Ricci curvature

In this section, we establish inequalities between Ricci curvature and the squared mean curvature for submanifolds in a (κ, μ)-contact space form with a semi-symmetric metric connection. These inequalities are called Chen-Ricci inequalities [2].

First we give a lemma as follows.

Lemma 4.1 ([20]). *Let $f(x_1, x_2, \ldots, x_n)$ be a function in R^n defined by*

$$f(x_1, x_2, \ldots, x_n) = x_1 \sum_{i=2}^{n} x_i.$$

If $x_1 + x_2 + \ldots + x_n = 2\varepsilon$, then we have

$$f(x_1, x_2, \ldots, x_n) \leqslant \varepsilon^2,$$

with the equality holding if and only if $x_1 = x_2 + \ldots + x_n = \varepsilon$.

Theorem 4.2. *Let M be an n-dimensional $(n \geqslant 2)$ submanifold of a $(2m+1)$-dimensional (κ, μ)-contact space form $\hat{M}(c)$ endowed with a semi-symmetric metric connection such that $\xi \in TM$. Then for each point $p \in M$,*
(1) For each unit vector X in T_pM, we have

$$\begin{aligned} Ric(X) \leqslant & \frac{n^2}{4}\|H\|^2 + \frac{(n-1)(c+3)}{4} + \frac{3(c-1)}{4}\|PX\|^2 - \frac{c+3-4\kappa}{4}[1+(n-2)\eta(X)^2] \\ & + \frac{1}{2}[\|(\varphi hX)^T\|^2 - \|(hX)^T\|^2 - g(\varphi hX, X)trace((\varphi h)^T) + g(hX, X)trace(h^T)] \\ & + (\mu+n-3)g(hX, X) + [1+(\mu-1)\eta(X)^2]trace(h^T) - (n-2)\alpha(X, X) - \lambda. \end{aligned} \tag{18}$$

(2) If $H(p) = 0$, a unit tangent vector $X \in T_pM$ satisfies the equality case of (18) *if and only if $X \in N(p) = \{X \in T_pM | \sigma(X, Y) = 0,\ \forall Y \in T_pM\}$.*
(3) The equality of (18) *holds identically for all unit tangent vectors if and only if either*
(i) $n \neq 2$, $\sigma_{ij}^r = 0$, $i, j = 1, 2, \ldots, n$; $r = n+1, \ldots, 2m+1$;
or
(ii) $n = 2$, $\sigma_{11}^r = \sigma_{22}^r$, $\sigma_{12}^r = 0$, $r = 3, \ldots, 2m+1$.

Proof. (1) Let $X \in T_pM$ be an unit vector. We choose an orthonormal basis $e_1, \ldots, e_n, e_{n+1} \ldots, e_{2m+1}$ such that $e_1, \ldots, e_n$ are tangential to M at p with $e_1 = X$.

Using (10), we have

$$\begin{aligned} Ric(X) = & \frac{(n-1)(c+3)}{4} - \frac{c+3-4\kappa}{4}[1+(n-2)\eta(X)^2] + \frac{3(c-1)}{4}\|PX\|^2 \\ & + \frac{1}{2}[\|(\varphi hX)^T\|^2 - \|(hX)^T\|^2 + g(hX, X)trace(h^T) - g(\varphi hX, X)trace((\varphi h)^T)] \\ & + (\mu+n-3)g(X, hX) + (1-\eta(X)^2 + \mu\eta(X)^2)trace(h^T) \\ & - (n-2)\alpha(X, X) - \lambda + \sum_{r=n+1}^{2m+1}\sum_{i=2}^{n}[\sigma_{11}^r\sigma_{ii}^r - (\sigma_{1i}^r)^2] \\ \leqslant & \frac{(n-1)(c+3)}{4} + \frac{3(c-1)}{4}\|PX\|^2 - \frac{c+3-4\kappa}{4}[1+(n-2)\eta(X)^2] \\ & + \frac{1}{2}[\|(\varphi hX)^T\|^2 - \|(hX)^T\|^2 - g(\varphi hX, X)trace(\varphi h)^T + g(hX, X)trace(h^T)] \\ & + (\mu+n-3)g(hX, X) + [1-\eta(X)^2 + \mu\eta(X)^2]trace(h^T) \\ & - (n-2)\alpha(X, X) - \lambda + \sum_{r=n+1}^{2m+1}\sum_{i=2}^{n}\sigma_{11}^r\sigma_{ii}^r. \end{aligned} \tag{19}$$

Let us consider the function $f_r : R^n \to R$, defined by

$$f_r(\sigma^r_{11}, \sigma^r_{22}, \ldots, \sigma^r_{nn}) = \sum_{i=2}^{n} \sigma^r_{11}\sigma^r_{ii}.$$

We consider the problem

$$\max\{f_r | \sigma^r_{11} + \ldots + \sigma^r_{nn} = k^r\},$$

where k^r is a real constant. From Lemma 4.1, we have

$$f_r \leqslant \frac{(k^r)^2}{4}, \tag{20}$$

with equality holding if and only if

$$\sigma^r_{11} = \sum_{i=2}^{n} \sigma^r_{ii} = \frac{k^r}{2}. \tag{21}$$

From (19) and (20) we get

$$\begin{aligned} Ric(X) \leqslant & \frac{n^2}{4}\|H\|^2 + \frac{(n-1)(c+3)}{4} + \frac{3(c-1)}{4}\|PX\|^2 - \frac{c+3-4\kappa}{4}[1+(n-2)\eta(X)^2] \\ & + \frac{1}{2}[\|(\varphi hX)^T\|^2 - \|(hX)^T\|^2 - g(\varphi hX, X)trace(\varphi h)^T + g(hX, X)trace(h^T)] \\ & + (\mu + n - 3)g(hX, X) + [1 + (\mu - 1)\eta(X)^2]trace(h^T) - (n-2)\alpha(X, X) - \lambda. \end{aligned}$$

(2) For a unit vector $X \in T_pM$, if the equality case of (18) holds, from (19), (20) and (21) we have

$$\sigma^r_{1i} = 0\ ,\ i \neq 1\ ,\ \forall r.$$

$$\sigma^r_{11} + \sigma^r_{22} + \ldots + \sigma^r_{nn} = 2\sigma^r_{11}\ ,\ \forall r.$$

Since $H(p) = 0$, we know

$$\sigma^r_{11} = 0\ ,\ \forall r.$$

So we get

$$\sigma^r_{1j} = 0\ ,\ \forall r.$$

i.e. $X \in N(p)$

The converse is trivial.

(3) For all unit vectors $X \in T_pM$, the equality case of (18) holds if and only if

$$2\sigma^r_{ii} = \sigma^r_{11} + \ldots + \sigma^r_{nn},\ i = 1, \ldots, n;\ r = n+1, \ldots, 2m+1.$$

$$\sigma^r_{ij} = 0,\ i \neq j,\ r = n+1, \ldots, 2m+1.$$

Thus we have two cases, namely either $n \neq 2$ or $n = 2$.

In the first case we have

$$\sigma^r_{ij} = 0,\ i.j = 1, \ldots, n;\ r = n+1, \ldots, 2m+1.$$

In the second case we have

$$\sigma^r_{11} = \sigma^r_{22}\ ,\ \sigma^r_{12} = 0\ ,\ r = 3, \ldots, 2m+1.$$

The converse part is straightforward. □

Corollary 4.3. *Let M be an n-dimensional $(n \geqslant 2)$ submanifold in a Sasakian space form $\hat{M}(c)$ endowed with a semi-symmetric metric connection such that $\xi \in TM$. Then for each point $p \in M$ and for each unit vector $X \in T_pM$, we have*

$$\begin{aligned} Ric(X) \leqslant & \frac{n^2}{4}\|H\|^2 + \frac{(n-1)(c+3)}{4} + \frac{3(c-1)}{4}\|PX\|^2 - \frac{c-1}{4}[1+(n-2)\eta(X)^2] \\ & - (n-2)\alpha(X, X) - \lambda. \end{aligned}$$

Corollary 4.4. *Let M be an n-dimensional $(n \geqslant 2)$ submanifold in a non-Sasakian space form $\hat{M}(c)$ endowed with a semi-symmetric metric connection such that $\xi \in TM$. Then for each point $p \in M$ and for each unit vector $X \in T_pM$, we have*

$$\begin{aligned} Ric(X) \leqslant & \frac{n^2}{4}\|H\|^2 + \frac{(n-1)(-\kappa+1)}{2} - \frac{3(\kappa+1)}{2}\|PX\|^2 - \frac{1-3\kappa}{2}[1+(n-2)\eta(X)^2] \\ & + \frac{1}{2}[\|(\varphi hX)^T\|^2 - \|(hX)^T\|^2 - g(\varphi hX, X) trace(\varphi h)^T + g(hX, X) trace(h^T)] \\ & + (\kappa+n-2)g(hX, X) + [1+\kappa\eta(X)^2] trace(h^T) - (n-2)\alpha(X, X) - \lambda. \end{aligned}$$

Theorem 4.5. *Let M be an n-dimensional $(n \geqslant 3)$ submanifold in a $(2m+1)$-dimensional (κ, μ)-contact space form $\hat{M}(c)$ endowed with a semi-symmetric connection such that $\xi \in TM$. Then we have*

$$\begin{aligned} n(n-1)\|H\|^2 \geqslant & n(n-1)\Theta_k(p) - \frac{1}{4}\{n(n-1)(c+3) + 3(c-1)\|P\|^2 - 2(n-1)(c+3-4\kappa)\} \\ & - \frac{1}{2}\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - (trace(\varphi(h)^T))^2 + (trace(h^T))^2\} \\ & - 2[\mu + (n-2)] trace(h^T) + 2(n-1)\lambda. \end{aligned}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM. We denote by $L_{i1,\ldots,ik}$ the k-plane section spanned by $e_{i1}, \ldots, e_{ik}$. From (4) and (5), it follows that

$$\tau(L_{i1}, \ldots, e_{ik}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} Ric_{L_{i_1,\ldots,i_k}}(e_i) \tag{22}$$

and

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \tau(L_{i_1,\ldots,i_k}). \tag{23}$$

Combining (6), (22) and (23), we obtain

$$\tau(p) \geqslant \frac{n(n-1)}{2}\Theta_k(p). \tag{24}$$

We choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM such that e_{n+1} is in the direction of the mean curvature vector $H(p)$ and $\{e_1, \ldots, e_n\}$ diagonalizes the shape operator A_{n+1}. Then the shape operators take the following forms:

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ . & . & \ldots & 0 \\ . & . & \ldots & 0 \\ . & . & \ldots & 0 \\ 0 & 0 & \ldots & a_n \end{pmatrix}, \tag{25}$$

$$traceA_r = 0\,,\ r = n+2, \ldots, 2m+1. \tag{26}$$

From (10), we have

$$\begin{aligned} 2\tau = & \frac{1}{4}\{n(n-1)(c+3) + 3(c-1)\|P\|^2 - 2(n-1)(c+3-4\kappa)\} \\ & + \frac{1}{2}\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - (trace(\varphi h)^T)^2 + (trace(h^T))^2\} \\ & + 2[\mu + (n-2)] trace(h^T) - 2(n-1)\lambda + n^2\|H\|^2 - \|\sigma\|^2. \end{aligned} \tag{27}$$

Using (25) and (27), we obtain

$$\begin{aligned} n^2\|H\|^2 = & 2\tau + \sum_{i=1}^{n} a_i{}^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2 - \frac{1}{4}\{n(n-1)(c+3) + 3(c-1)\|P\|^2 \\ & - 2(n-1)(c+3-4\kappa)\} - \frac{1}{2}\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - (trace(\varphi h)^T)^2 + (trace(h^T))^2\} \\ & - 2[\mu + (n-2)] trace(h^T) + 2(n-1)\lambda. \end{aligned} \tag{28}$$

On the other hand from (25) and (26), we have

$$(n\|H\|)^2 = (\sum a_i)^2 \leqslant n\sum_{i=1}^{n} {a_i}^2. \tag{29}$$

From (28) and (29), it follows that

$$\begin{aligned} n(n-1)\|H\|^2 \geq\ & 2\tau - \frac{1}{4}\{n(n-1)(c+3) + 3(c-1)\|P\|^2 - 2(n-1)(c+3-4\kappa)\} \\ & - \frac{1}{2}\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - (trace(\varphi h)^T)^2 + (trace(h^T))^2\} \\ & - 2[\mu + (n-2)]trace(h^T) + 2(n-1)\lambda + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^r)^2 \\ \geqslant\ & 2\tau - \frac{1}{4}\{n(n-1)(c+3) + 3(c-1)\|P\|^2 - 2(n-1)(c+3-4\kappa)\} \\ & - \frac{1}{2}\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - (trace(\varphi h)^T)^2 + (trace(h^T))^2\} \\ & - 2[\mu + (n-2)]trace(h^T) + 2(n-1)\lambda. \end{aligned}$$

Using (24), we obtain

$$\begin{aligned} n(n-1)\|H\|^2 \geqslant\ & n(n-1)\Theta_k(p) - \frac{1}{4}\{n(n-1)(c+3) + 3(c-1)\|P\|^2 - 2(n-1)(c+3-4\kappa)\} \\ & - \frac{1}{2}\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - (trace(\varphi h)^T)^2 + (trace(h^T))^2\} \\ & - 2[\mu + (n-2)]trace(h^T) + 2(n-1)\lambda. \end{aligned}$$

□

Corollary 4.6. *Let M be an n-dimensional $(n \geqslant 3)$ submanifold in a Sasakian space form $\hat{M}(c)$ endowed with a semi-symmetric connection such that $\xi \in TM$. Then for each point $p \in M$ and for each unit vector $X \in T_pM$, we have*

$$n(n-1)\|H\|^2 \geqslant n(n-1)\Theta_k(p) - \frac{1}{4}\{n(n-1)(c+3) + 3(c-1)\|P\|^2 - 2(n-1)(c-1)\} + 2(n-1)\lambda.$$

Corollary 4.7. *Let M be an n-dimensional $(n \geqslant 3)$ submanifold in a non-Sasakian space form $\hat{M}(c)$ endowed with a semi-symmetric connection such that $\xi \in TM$. Then for each point $p \in M$ and for each unit vector $X \in T_pM$, we have*

$$\begin{aligned} n(n-1)\|H\|^2 \geqslant\ & n(n-1)\Theta_k(p) - \frac{1}{4}\{2n(n-1)(-\kappa+1) - 6(\kappa+1)\|P\|^2 + 4(n-1)(3\kappa-1)\} \\ & - \frac{1}{2}\{\|(\varphi h)^T\|^2 - \|h^T\|^2 - (trace(\varphi h)^T)^2 + (trace(h^T))^2\} \\ & - 2(\kappa + n - 1)trace(h^T) + 2(n-1)\lambda. \end{aligned}$$

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