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Restricted triangulation on circulant graphs

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Abstract: The restricted triangulation existence problem on a given graph decides whether there exists a triangulation on the graph's vertex set that is restricted with respect to its edge set. Let $G = C(n, S)$ be a circulant graph on n vertices with jump value set S. We consider the restricted triangulation existence problem for G. We determine necessary and sufficient conditions on S for which G admitting a restricted triangulation. We characterize a set of jump values $S(n)$ that has the smallest cardinality with $C(n, S(n))$ admits a restricted triangulation. We present the measure of non-triangulability of $K_n - G$ for a given G.

Keywords: Triangulation, Circulant Graph

MSC: 32C25

1 Introduction

A *graph* is an ordered pair $G = (V, E)$, where V is a set of vertices, and E is a set of edges. An *order* of G is the number of its vertices while a *size* of G is the number of its edges. A graph is called *geometric*, if its edges are straight-line segments.

A *triangulation* T_n of a finite set of points V in the plane is a maximally connected, straight-line planar graph with vertex set V. Each bounded face is a triangle, and the triangulation includes the boundary of the convex hull.

Let $n \geq 3$. A *circulant graph* $G = C(n, S)$ is a graph on the vertex set $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$ such that each vertex v_i is adjacent to vertices $v_{i\pm a}$ where $i = 0, 1, \dots, n-1$ and the subscript index $i \pm a$ is reduced modulo n for all $a \in S$. That is, $v_i v_j$ is an edge of $C(n, S)$ if and only if $|j - i| \in S$ or $n - |j - i| \in S$. A set $S \subseteq \{1, 2, \dots, \lfloor n/2 \rfloor\}$ is called a set of *jump* values of $G = C(n, S)$. When discussing circulant graphs, we will often assume that the vertices are the corners of a regular n-gon, labeled in clockwise order and the edges are straight line segments. Hence, circulant graphs can be considered as geometric graphs.

Circulant graphs include the family of cycles $C(n, \{1\})$ and the family of complete graphs $K_n = C(n, \{1, 2, \dots, \lfloor n/2 \rfloor\})$. Clearly, when $G_1 = C(n, S_1)$ and $G_2 = C(n, S_2)$ are two circulant graphs such that $|S_1| < |S_2|$, then the size of G_1 is smaller than the size of G_2 (i.e., $|E(G_1)| < |E(G_2)|$). Hence, for simplicity we shall say that $G_1 = C(n, S_1)$ is a *smaller size circulant graph* than $G_2 = C(n, S_2)$ when $|S_1| < |S_2|$.

Let E be some set of edges spanned by V. We say that a triangulation T_n of V is *restricted* with respect to E if $E(T_n) \subseteq E$. *The restricted triangulation existence problem*, on a given graph $G(V, E)$, is to decide whether there exists a triangulation of V that is restricted with respect to E. This problem was proven to be NP-complete

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(see [1, 2]). In section 2, we solve this problem for a certain geometric graph – a circulant graph. We give a characterization of a circulant graph G to admit a restricted triangulation.

Another related problem is beginning with a problem presented by Micha Perles on DIMACS Workshop on Geometric Graph Theory in 2002, which asks to determine the largest possible number $h(n)$ such that every geometric graph on n vertices with at least $\binom{n}{2} - h(n)$ edges has a non-crossing Hamiltonian path which has been studied by Černý et al. [3]. The authors in [4] motivated by Perles problem proved that if F_n is a subgraph of a convex complete graph K_n, where F_n contains no boundary edges of K_n and $|E(F_n)| \leq n-3$, then $K_n - F_n$ admits a restricted triangulation.

In section 3, we characterize a circulant graph G as a subgraph of a convex complete graph K_n and $|E(G)| \leq L(n)$ such that $K_n - G$ allows a restricted triangulation.

We obtain a set of jump values $S(n) = \{a_1, a_2, \dots, a_\ell\}$ that has the smallest cardinality $|S(n)|$ such that $C(n, S(n))$ admits a restricted triangulation. That is, $C(n, S(n))$ is the smallest size circulant graph admitting a restricted triangulation.

Let v_iv_j be an edge in a convex graph G, the distance between v_i and v_j in G is the length of a shortest (v_i, v_j)-path in G. A *span* of v_iv_j is defined as the distance between its two end points v_i and v_j. In other words, a natural number $d = \min\{|j-i|, n-|j-i|\}$ such that v_iv_j can be written as v_iv_{i+d} or v_iv_{i-d}, is a span of v_iv_j. Hence, we can see that $H = C(n, \{d\})$ is a circulant graph in which $v_iv_j \in E(H)$ and each edge in H has a span d. Let $E(T_n)$ be a set of edges of T_n and $D(T_n)$ be a set of spans of all edges in $E(T_n)$.

1.1 Importance of the triangulations and circulant graphs

Circulant graphs are an important class of interconnection networks in parallel and distributed computing. It can be used in the design of local area networks (see [5, 6]). On the other hand, computing a triangulation on a given graph has several important applications in different areas such as nondense matrix computations [7], database management [8] and artificial intelligence [9]. Moreover, triangulations are used in many areas of engineering and scientific applications such as finite element methods, approximation theory, numerical computation, computer-aided geometric design, computational geometry, etc. (see [10, 11]).

2 Restricted triangulation on G

In this section, we characterize a set of jump values S^* to a circulant graph $C(n, S^*)$ admitting a restricted triangulation (Theorem 2.8). Also, we define the 'smallest' cardinality set of jump values $S(n)$ such that $C(n, S(n))$ still admits a restricted triangulation. We give a characterization of a circulant graph $C(n, S)$ that can be redrawn to admit a restricted triangulation (Corollary 2.13). These results are then applied to determine the convex skewness of the circulant graphs G in section (2.2).

Lemma 2.1. *If T_n is a restricted triangulation of a circulant graph $G = C(n, S)$, then S contains $D(T_n)$.*

Proof. Suppose T_n is a restricted triangulation of $G = C(n, S)$. Let $d \in D(T_n)$. By definition of span of edges, there is an edge v_iv_j in T_n such that $v_iv_j = v_iv_{i\pm d}$ where $d = \min\{|j-i|, n-|j-i|\}$. Since T_n is a restricted triangulation of G, then $E(T_n) \subset E(G)$. Hence, $v_iv_{i\pm d} \in E(G)$. Thus, $d \in S$ (by definition of S) □

The following proposition proves that any circulant graph $C(n, S)$ does not admit a restricted triangulation unless $1, 2 \in S$.

Proposition 2.2. *Let $n \geq 4$ be a natural number. Suppose the circulant graph $C(n, S)$ admits a restricted triangulation. Then $1, 2 \in S$.*

Proof. Suppose that $C(n, S)$ is a circulant graph. Let T_n be a restricted triangulation of $C(n, S)$.

It is well known that any triangulation on a set of point in the plane includes the boundary of the convex hull. For our case, $v_iv_{i+1} \in E(T_n)$ for each $i = 0, 1, \dots, n-1$ which yields $1 \in D(T_n)$ and then $1 \in S$ by Lemma 2.1.

Every maximal outer planar graph (as a special case, triangulation on a convex polygone) has at least two vertices of degree 2 (see [12]). Hence, if v_{i+1} is a vertex of degree 2 in T, then the diagonal edge v_iv_{i+2} must be in $E(T_n)$ because v_{i+1} is incident just on two edges of T_n which are the boundary edges v_iv_{i+1} and $v_{i+1}v_{i+2}$. Thus, $2 \in D(T_n)$ since 2 is the distance between v_i and v_{i+1}. Then $2 \in S$ by Lemma 2.1. □

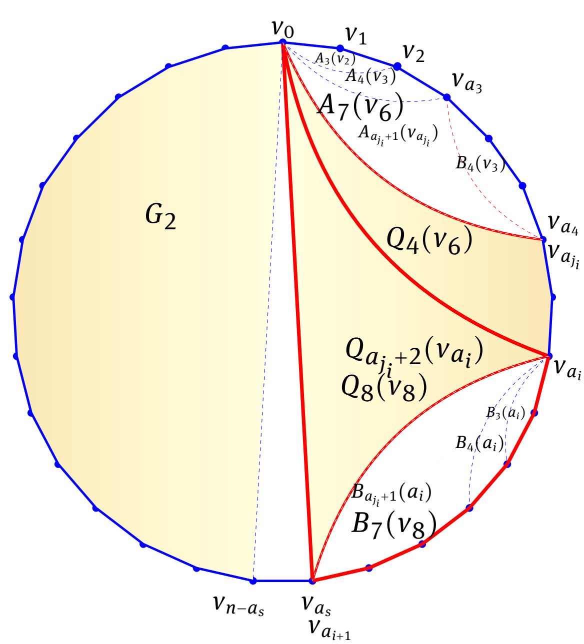
It is not difficult to verify that the converse of Proposition 2.2 is not true. For instance, $G_1 = C(12, S_1)$ where $S_1 = \{1, 2, 3\}$ does not admit any restricted triangulation, while each of $G_2 = C(12, S_2)$ and $G_3 = C(12, S_3)$ admits a restricted triangulation with $S_2 = \{1, 2, 4\}$ and $S_3 = \{1, 2, 3, 5\}$. Note that $|S_2| = |S_1|$ while $S_3 = S_1 \cup \{5\}$. Hence, the question arises: what conditions on S do guarantee that $C(n, S)$ admits a restricted triangulation and what is the *smallest* cardinality $|S|$ of the set of jump values S for which $C(n, S)$ admits a restricted triangulation?

The following proposition proves that the circulant graph $C(n, S_n)$ admits a restricted triangulation where S_n is a set of ascending values $\{a_1, a_2, \dots, a_s\}$ in which $a_1 = 1$, $a_2 = 2$, $a_s = \lfloor \frac{n}{2} \rfloor$ and $a_{i+1} - a_i \in S_n$ for each $i \in \{1, 2, \dots, s-1\}$.

Proposition 2.3. *Let $n \geq 4$ be a natural number. Suppose S_n is a set of ascending values $\{1, 2, a_3, \dots, a_s\}$ in which $a_s = \lfloor \frac{n}{2} \rfloor$ and S_n has the property $a_{i+1} - a_i \in S_n$ for each $i \in \{1, 2, \dots, s-1\}$. Then the circulant graph $C(n, S_n)$ admits a restricted triangulation.*

Proof. Suppose that $C(n, S_n)$ is a circulant graph. Since $a_s \in S_n$ is a jump value, then $v_0v_{a_s}$ and $v_{n-a_s}v_0$ are two edges in $C(n, S_n)$. Thus, we can define G_1 and G_2 to be two subgraphs of $C(n, S_n)$, induced by the vertices $v_0, v_1, v_2, \dots, v_{a_s}$ and $v_{n-a_s}, v_{n-a_s+1}, \dots, v_{n-1}, v_0$, respectively. See Figure 1.

Fig. 1. $Q_{a_{j_i}+2}(a_i)$ on G_1



We shall construct the triangulation T_{G_1} of G_1, and then G_2 can be triangulated in a similar way.

$(*)$ It is important to mention that in the following argument the property $a_{i+1} - a_i \in S_n$, for each two consecutive values a_i and a_{i+1} in S_n, and $1, 2 \in S_n$, are basic tools to construct T_{G_1}.

Let a_i, a_{i+1} be two consecutive vertices in S_n and $a_{i+1} - a_i = a_{j_i} \in S_n$ for some $a_{j_i} \in \{1, 2, a_3, \ldots, a_i\}$ (since S_n is a set of ascending values). Let $v_0 v_{a_i} v_{a_i+1} \ldots v_{a_{i+1}} v_0$ be a convex $(a_{j_i} + 2)$-gon in G_1 denoted by $Q_{a_{j_i}+2}(v_{a_i})$. Clearly $G_1 = \cup_{i=1}^{s-1} Q_{a_{j_i}+2}(v_{a_i})$. First, we prove $Q_{a_{j_i}+2}(v_{a_i})$ can be triangulated by edges of $C(n, S_n)$ for each $i = 1, \ldots, s-1$.

Note that $a_1 = 1$, $a_2 = 2$ and $a_3 \in \{3, 4\}$ (otherwise, if $a_3 = 5$ then $a_3 - a_2 = 5 - 2 = 3 \notin S_n$, a contradiction). It is easy to see that $Q_3(v_1)(= v_0 v_1 v_2 v_0)$ is triangulated by $v_0 v_1 v_2 v_0$ and when $a_3 = 3$ we have $Q_3(v_2)(= v_0 v_2 v_3 v_0)$ is triangulated by $v_0 v_2 v_3 v_0$, and also when $a_3 = 4$ we have that $Q_4(v_2)(= v_0 v_2 v_3 v_4 v_0)$ is triangulated by the boundary edges $v_0 v_2 v_3 v_4 v_0$ together with the diagonal edge $v_2 v_4$.

Now we obtain a triangulation for $Q_{a_{j_i}+2}(v_{a_i})$, $i = 3, \ldots, s-1$.

Let $A_{a_i+1}(v_{a_i}) = v_0 v_1 \ldots v_{a_i} v_0$, be a convex $(a_i + 1)$-gon in G_1. Clearly, $A_3(v_2)(= v_0 v_1 v_2 v_0)$ is a triangle that is trivially triangulated by $T_1 = v_0 v_1 v_2 v_0$. If $i = 3$, then either $a_3 = 3$ and $A_4(v_3)(= v_0 v_1 v_2 v_3 v_0)$ is a quadrilateral that is triangulated by $T_2 = v_0 v_2 \cup v_0 v_1 v_2 v_3 v_0$, or $a_3 = 4$ and $A_5(v_4)(= v_0 v_1 v_2 v_3 v_4 v_0)$ is a pentagon triangulated by $T_2 = v_0 v_2 v_4 \cup v_0 v_1 v_2 v_3 v_4 v_0$.

Let $B_{a_{j_i}+1}(v_{a_i}) = Q_{a_{j_i}+2}(v_{a_i}) - v_0$. Then $B_{a_{j_i}+1}(v_{a_i}) = v_{a_i} v_{a_i+1} \ldots v_{a_i+a_{j_i}} v_{a_i}$ (since $a_{i+1} - a_i = a_{j_i}$) which is equivalent to the convex polygon $v_0 v_1 \ldots v_{a_{j_i}} v_0$ (by considering $a_i = 0$). Thus, $B_{a_{j_i}+1}(v_{a_i})$ is equivalent $A_{a_{j_i}+1}(v_{a_{j_i}})$.

If $i = 3$, we have $a_4 - a_3 = a_{j_3} \in S_n$ for some $a_{j_3} \in \{1, 2, a_3\}$. Then, $B_{a_{j_3}+1}(a_3)(= v_{a_3} v_{a_3+1} \ldots v_{a_4} v_{a_3} = v_{a_3} v_{a_3+1} \ldots v_{a_3+a_{j_3}} v_{a_3})$ is equivalent to $A_{a_{j_3}+1}(a_{j_3})(= v_0 v_1 \ldots v_{a_{j_3}} v_0)$ which is triangulated by $T \in \{T_1, T_2\}$. Then, $B_{a_{j_3}+1}(a_3)$ can be triangulated by T' equivalent to T. Hence, $T_3 = T' \cup v_{a_3} v_0 v_{a_4}$ is a triangulation of $Q_{a_{j_3}+2}(v_{a_3})$.

Recursively, if we have $i = s-1$ then $a_s - a_{s-1} = a_{j_{s-1}} \in S_n$ for some $a_{j_{s-1}} \in \{1, 2, a_3, \ldots, a_{s-1}\}$. Then, $B_{a_{j_{s-1}}+1}(v_{a_{s-1}})$ is equivalent to $A_{a_{j_{s-1}}+1}(a_{j_{s-1}})$ which is triangulated by $T \in \{T_1, T_2, T_3, \ldots, T_{s-2}\}$ (depending on the value of $a_{j_{s-1}} \in \{1, 2, a_3, \ldots, a_{s-1}\}$). Then, $B_{a_{j_{s-1}}+1}(v_{a_{s-1}})$ can be triangulated by T' equivalent to T. Hence, $T_{s-1} = T' \cup v_{a_{s-1}} v_0 v_{a_s}$ is a triangulation of $Q_{a_{j_{s-1}}+2}(v_{a_{s-1}})$.

Since, $G_1 = \cup_{i=1}^{s-1} Q_{a_{j_i}+2}(v_{a_i})$, then let $T_{G_1} = \cup_{i=1}^{s-1} T_i$ be the triangulation of G_1. In a similar way we obtain T_{G_2}. Thus, $T_n = T_{G_1} \cup T_{G_2} \cup \{v_{a_s} v_{n-a_s}\}$, where $a_s \neq n - a_s$; otherwise $\{v_{a_s} v_{n-a_s}\} = \emptyset$ is a restricted triangulation of G. □

Corollary 2.4. *Let $n \geq 4$ be a natural number. Suppose $G = C(n, S)$ is a circulant graph. Then G admits a restricted triangulation if one of the following conditions hold.*
(1) $S = \{1, 2, a_3, a_4, \ldots, \lfloor \frac{n}{2} \rfloor\}$ where a_i's are consecutive odd values.
(2) $S = \{1, 2, a_3, a_4, \ldots, \lfloor \frac{n}{2} \rfloor\}$ where a_i's are consecutive even values.

Proof. In both cases $a_1 = 1$, $a_2 = 2$ and $\lfloor \frac{n}{2} \rfloor$ are in S. Further, for each a_{i+1}, $a_i \in S_n$, $a_{i+1} - a_i \in \{1, 2\} \subseteq S_n$. Then the circulant graph $C(n, S)$ admits a restricted triangulation by Proposition 2.3. □

Now, we shall define a smallest size circulant graph that admits a restricted triangulation. Before proceeding, we present some ingredients that will be used to prove the main results in this section.

- When n is even number, then n can be written as $n = t.2^{r'}$ for some positive natural number r' and some positive odd natural number t.
- Let S_α be a set of jump values obtaining by next algorithm where $\alpha = \lfloor \frac{t}{3} \rfloor$ and $\beta = \lceil \frac{t}{3} \rceil$ with $t = \begin{cases} n, & n \text{ is odd}; \\ t, & n = t.2^{r'}. \end{cases}$
- Let $S_1 = \{1, 2, 4, 8, \ldots, c\}$ with $c = 2^r$ where $r = \begin{cases} 0, & n \text{ is odd}; \\ r' - 1, & t = 1; \\ r', & t \geq 3. \end{cases}$
- If we have a set $B = \{b_1, b_2, \ldots, b_k\}$ of positive integers, then define $a.B$ to be $\{a.b_1, a.b_2, \ldots, a.b_k\}$ where $a \geq 1$ is an integer number.

Algorithm (A)

1- If $\alpha = 0$, then let $S_\alpha = \{\beta\}$ and Stop. If $\alpha = 1$, then let $S_\alpha = \{1, \beta\}$ and Stop. If $2 \le \alpha \le 4$, then let $S_\alpha = \{1, 2, \alpha, \beta\}$ and Stop. Otherwise, let $L = 0$ be the number of starting level. Let $S(L) = \{1, 2, \beta\}$, $a_0 = \alpha$, and let $L = L + 1$.

2- If a_0 is odd and divisible by 3, then let $a_{1,L} = \frac{a_0}{3}$ and $a_{2,L} = 2.a_{1,L}$. Otherwise, let $a_{1,L} = \lfloor \frac{a_0}{2} \rfloor$ and $a_{2,L} = \lceil \frac{a_0}{2} \rceil$.

3- Let $S(L) = \{a_{1,L}, a_{2,L}, a_0\}$. Let $S_\alpha = \bigcup_{l=0}^{L} S(l)$.

4- If $3 \in \{a_{1,L}, a_{2,L}\}$ and each of α and $\lfloor \frac{\alpha}{2} \rfloor$ is odd and not divisible by 3, then do the following:

(i)- Let $L = 0$ be the number of starting level. Let $S(L) = \{1, 2, 3, \alpha-3, \alpha, \beta\}$, $a_0 = \alpha-3$, and let $L = L+1$.

(ii)- If a_0 is odd and not divisible by 3, then let $a_{1,L} = a_{2,L} = a_0 - 3$. If a_0 is odd and divisible by 3, then let $a_{1,L} = \frac{a_0}{3}$ and $a_{2,L} = 2.a_{1,L}$. Otherwise, $a_{1,L} = a_{2,L} = \frac{a_0}{2}$.

(iii)- Let $S(L) = \{a_{1,L}, a_{2,L}, a_0\}$. Let $S_\alpha = \bigcup_{l=0}^{L} S(l)$.

(iv)- If $a_{1,L} \le 3$, then arrange S_α to be a set of ascending values and stop. Otherwise, let $a_0 = a_{1,L}$ and $L = L + 1$ and then repeat Step (ii).

5- If $a_{1,L} \le 4$, then arrange S_α to be a set of ascending values and stop. Otherwise, let $a_0 = a_{1,L}$ and $L = L+1$ and then repeat Step (2).

Note that, S_α that obtained by Algorithm (A) is a set of ascending values $1, 2, a_{i-1}, a_i, a_{i+1}, \ldots, \alpha, \beta$.

Lemma 2.5. *Suppose $S_\alpha = \{a_1, a_2, \ldots, \alpha, \beta\}$ is a set obtained by Algorithm (A). Then $1, 2 \in S_\alpha$ and $a_{i+1} - a_i \in S_\alpha$ for each a_{i+1}, a_i belong to S_α.*

Proof. By Algorithm (A) step (1), we get $1, 2 \in S_\alpha$. Suppose that, a_i, a_{i+1} are two consecutive values belonging to S_α.

If $a_{i+1} = a_0$ is an odd and divisible by 3 number, then by Algorithm (A) Step (2), $a_i = a_{2,L} = \frac{2.a_{i+1}}{3}$. Thus, $a_{i+1} - a_i = a_{i+1} - \frac{2.a_{i+1}}{3} = \frac{a_{i+1}}{3} = a_{i-1} \in S_\alpha$ (where $a_{i-1} = a_{1,L} = \frac{a_{i+1}}{3} \in S_\alpha$).

If $3 \in S_\alpha$ and $a_{i+1} = a_0$ is an odd and not divisible by 3 number, then by Algorithm (A) Step (4), $a_i = a_{i+1} - 3$. Thus, $a_{i+1} - a_i = 3 \in S_\alpha$.

Otherwise, by Algorithm (A) Step (2), $a_i = \lceil \frac{a_{i+1}}{2} \rceil$ (where $\lceil \frac{a_{i+1}}{2} \rceil = a_{2,L} \in S_\alpha$). Hence, $a_{i+1} - a_i = a_{i+1} - \lceil \frac{a_{i+1}}{2} \rceil = \lfloor \frac{a_{i+1}}{2} \rfloor = a_{i-1} \in S_\alpha$ (where $a_{i-1} = \lfloor \frac{a_{i+1}}{2} \rfloor = a_{1,L} \in S_\alpha$). In case when, $\lceil \frac{a_{i+1}}{2} \rceil = \lfloor \frac{a_{i+1}}{2} \rfloor = \frac{a_{i+1}}{2}$, then we have $a_i = \frac{a_{i+1}}{2}$. Thus, $a_{i+1} - a_i = a_{i+1} - \frac{a_{i+1}}{2} = \frac{a_{i+1}}{2} = a_i \in S_\alpha$. □

Theorem 2.6. *Let $S(n) = S_1 \cup c.S_\alpha$, where $n \ge 4$. Then $C(n, S(n))$ admits a restricted triangulation.*

Proof. Suppose that $C(n, S(n))$ is a circulant graph and let α, β, c and t is defined as above. We shall construct a restricted triangulation T_n of $C(n, S(n))$.

When n is even and $t = 1$, $S(n) = S_1$ (In this case, $\beta = 1$ and then $S_\alpha = \{1\}$ and then $c.S_\alpha = \{c\}$ and $c \in S_1$). Then let $T_n = \bigcup_{i=0}^{r} \{v_{j2^i} v_{(j+1)2^i}, j = 0, 1, \ldots, \frac{n}{2^i} - 1\}$. The circulant graph when $n = 8$ is depicted in Figure 2.

When n is even and $t \ge 3$, $S(n) = S_1 \cup c.S_\alpha$. In this case, and also when n is odd (which means $t = n$), we have t-gon which is induced by the vertices $v_0, v_c, v_{2.c}, \ldots, v_{(t-2).c}, v_{(t-1).c}$ (the shaded part in Figure 3).

We shall triangulate this t-gon by T' which is obtained by one of the following three cases depending on S_α. According to S_1, let $T = \bigcup_{i=0}^{r} \{v_{j2^i} v_{(j+1)2^i}, j = 0, 1, \ldots, \frac{n}{2^i} - 1\}$ which triangulates unshaded part, between n-gon and t-gon, in Figure 3. Then, let $T_n = T \cup T'$ or $T_n = T'$ be a triangulation to $C(n, S(n))$ when n is even with $t \ge 3$ or when n is odd, respectively.

Case (1) When $\alpha = \frac{t-2}{3}$, then $t = 2\beta + \alpha$ (since $\beta = \alpha + 1$). Let $\triangle = v_0 v_\alpha v_{\alpha+\beta} v_0$ be a triangle. Then there are three $(\alpha + 1)$-gons G_1, G_2 and G_3 induced by the vertices $v_0, v_c, v_{2.c}, \ldots, v_{\alpha.c}$; $v_{(\alpha).c}, v_{(\alpha+1).c}, \ldots, v_{2\alpha.c}$ and $v_{(\alpha+\beta).c}, v_{(2\alpha+2).c}, \ldots, v_{(3\alpha+1).c}$, respectively. See Figure 4.

Case (2) When $\alpha = \frac{t-1}{3}$, then $t = 2\alpha + \beta$ (since $\beta = \alpha + 1$). Let $\triangle = v_0 v_\alpha v_{2\alpha} v_0$ be a triangle. Then there are three $(\alpha + 1)$-gons G_1, G_2 and G_3, induced by the vertices $v_0, v_c, v_{2.c}, \ldots, v_{\alpha.c}$; $v_{(\alpha).c}, v_{(\alpha+1).c}, \ldots, v_{2\alpha.c}$ and $v_{(2\alpha).c}, v_{(2\alpha+1).c}, \ldots, v_{(3\alpha).c}$, respectively. See Figure 5.

Fig. 2. $n = 8 = 2^3$, $t = 1$, $r = r' - 1 = 2$, $T_8 = \{v_j v_{(j+1)},\ j = 0, \dots, 7\} \cup \{v_{2j} v_{2(j+1)},\ j = 0, \dots, 3\} \cup \{v_{4j} v_{4(j+1)},\ j = 0, 1\}$.

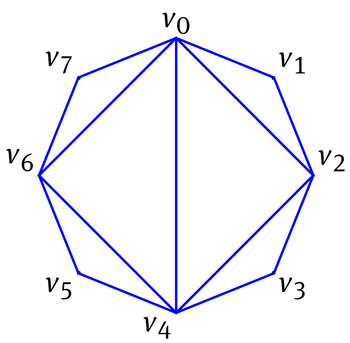


Fig. 3. $n = 56 = 7.2^3$, $t = 7$, $r = r' = 3$, $T_{56} = \{v_j v_{(j+1)},\ j = 0, \dots, 55\} \cup \{v_{2j} v_{2(j+1)},\ j = 0, \dots, 27\} \cup \{v_{4j} v_{4(j+1)},\ j = 0, \dots, 13\} \cup \{v_{8j} v_{8(j+1)},\ j = 0, \dots, 6\}$.

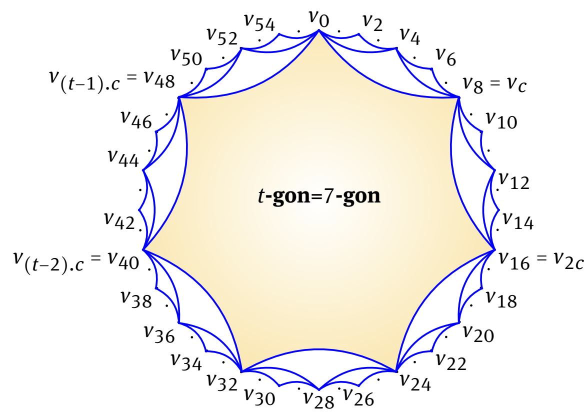


Fig. 4. $n = 164$, $t = 41$, $r = 2$, $\alpha = 13$, and $\beta = 14$.

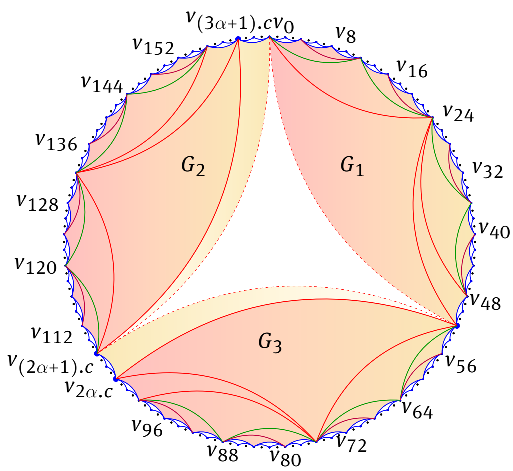
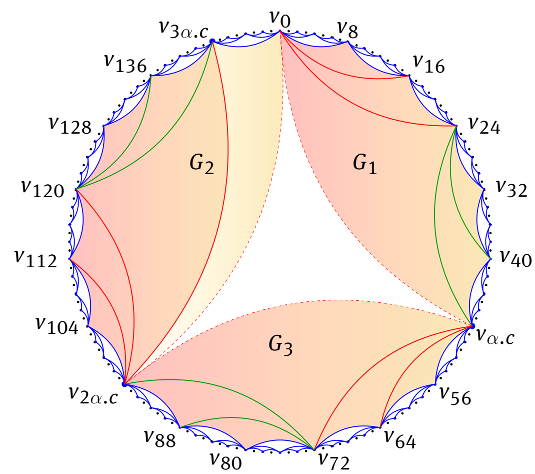


Fig. 5. $n = 152$, $t = 19$, $r = 3$, $\alpha = 6$, and $\beta = 7$.



Case (3) When $\alpha = \frac{t}{3}$, then $t = 3\alpha$. There are three $(\alpha + 1)$-gons G_1, G_2 and G_3 induced by the vertices $v_0, v_c, v_{2.c}, \dots, v_{\alpha.c}$; $v_{(\alpha).c}, v_{(\alpha+1).c}, \dots, v_{2\alpha.c}$ and $v_{2\alpha.c}, v_{(2\alpha+1).c}, \dots, v_{(3\alpha-1).c}, v_0$, respectively. See Figure 6.

In each case, G_1, G_2 and G_3 are of the same order. Suppose that R is an r-gon in G_i, for some $i \in \{1, 2, 3\}$ such that $V(R) = \{v_h, v_{h+a_i}, v_{h+a_i+1}, \dots, v_{h+a_{i+1}}\}$ for some $v_h \in V(G_i)$ and two consecutive jump values a_i and a_{i+1} in S_α.

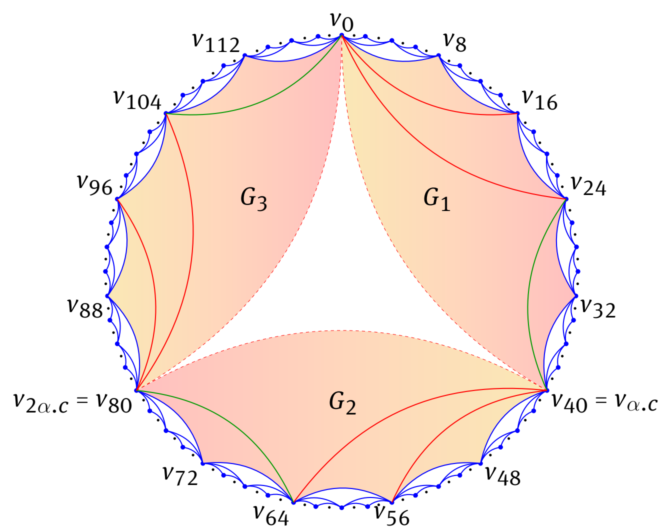
Without loss of generality assume that R is a subgraph of G_1 and $h = 0$.

By Lemma 2.5, we have $1, 2 \in S_\alpha$ and $a_{i+1} - a_i \in S_\alpha$ for each a_{i+1}, a_i belonging to S_α. Hence, we can use the argument $(*)$ of Proposition 2.3 to show that R can be triangulated by T_r. Moreover, similar as in the proof of Proposition 2.3 we can consider G_1 as a finite union of such polygons. Then we can assume that T_{G_1} is a finite union of the restricted triangulation of these polygons.

Obtain T_{G_2} (on G_2) and T_{G_3} (on G_3) by "rotating" the edges of T_{G_1}, see Figures 4, 5 and 6. Let $T' = T_{G_1} \cup T_{G_2} \cup T_{G_3} \cup \triangle$ for case (1) and case (2), and let $T' = T_{G_1} \cup T_{G_2} \cup T_{G_3}$ for case (3).

This completes the proof. □

Fig. 6. $n = 120$, $t = 15$, $r = 3$, and $\alpha = 5$.



Definition 2.7. *Let $n \geq 4$ and t be defined as above, and let a_k be a natural number such that $\lceil \frac{t}{3} \rceil \cdot c \leq a_k \leq \lfloor \frac{n}{2} \rfloor$. Then define S^* to be a set of ascending natural numbers such that,*
(1) $S^* = \{1, 2, a_3, a_4, \dots, a_k\}$,
(2) there is $\{x, z\} \subset S^*$ such that $n = x + y + z$ where $y \in \{0, a_i\}$ for some $i \in \{1, \dots, k\}$,
(3) $a_{i+1} - a_i \in S^*$ for each $i = 1, \dots, k - 1$.

It is clear that, $S^* = S_n$ when $a_k = \lfloor \frac{n}{2} \rfloor$. Our main result, Theorem 2.8, proves the sufficient and necessary condition for a circulant graph $C(n, S)$ to admit a restricted triangulation.

Theorem 2.8. *Suppose $G = C(n, S)$ is a circulant graph. G admits a restricted triangulation if and only if S contains S^*.*

Proof. Let G have a restricted triangulation T_n. We shall prove that S contains S^*. By Lemma 2.1, it is enough to prove $S^* = D(T_n)$.

Arrange the spans of $D(T_n)$ to be ascending values.

Since, the circulant graph $C(n, D(T_n))$ admits T_n, then by Proposition 2.2, we have $\{1, 2\} \subseteq D(T_n)$.

Let $|D(T_n)| = s$, then d_s is the maximum span in $D(T_n)$.

($\star$) Let t be defined as before. Suppose that $d_s < \lceil \frac{t}{3} \rceil \cdot c$. Without loss of generality assume that, $c = 1$ and $d_s = \lceil \frac{t}{3} \rceil - 1$. Thus, $d_s = \frac{t-b}{3} - 1$, $b \in \{0, 1, 2\}$ and then $3d_s + b + 3 = t$. Let R be a convex r-gon induced by the vertices $v_{d_s}, v_{2d_s}, v_{3d_s}, v_{3d_s+1}, \dots, v_{3d_s+b+1}, v_{3d_s+b+2}, v_t$ (recall that, $v_t = v_{3d_s+b+3}$). Suppose that $T_r = T_n(R)$ is a subgraph of T_n that triangulates R and let **e** be an edge in T_r such that the span of **e** is the maximum with respect to $D(T_r)$. Then:

– either $\mathbf{e} = v_{2d_s} v_t$ which yields that its span $t - 2d_s \in D(T_r) \subset D(T_n)$; but $t - 2d_s > d_s + 1$ (by above assumption, $t - 2d_s = d_s + b + 3 > d_s + 1$), which is a contradiction with maximality of $d_s \in D(T_n)$;
– or, $\mathbf{e} = v_{d_s} v_{3d_s}$ which yields that $3d_s - d_s = 2d_s \in D(T_r) \subset D(T_n)$; but $2d_s > d_s$, which also contradicts the maximality of $d_s \in D(T_n)$.

Hence, R is not triangulated by T_n which is a contradiction with $C(n, S)$ admitting a restricted triangulation T_n. Thus, $d_s \geq \lceil \frac{t}{3} \rceil \cdot c$.

Now, in order to check property (ii) of S^* we have to consider two cases:

Case (1) If $d_s = \frac{n}{2}$, then $n = 2.d_s$. Hence let $x = z = d_s$ and $y = 0$.

Case (2) If $\lceil \frac{t}{3} \rceil \cdot c \leq d_s \leq \frac{n-1}{2}$, then T_n contains a triangle $\triangle = v_h v_{h+d_s} v_{h+d_s+d_i} v_h$ for some $d_i \in D(T_n)$ and $v_h \in V(G)$. The span of the edge $v_{h+d_s+d_i} v_h \in E(T_n)$ is $n-(d_s+d_i) \in D(T_n)$ (by definition of the span of edge). Hence let $x = d_s$, $y = d_i$ and $z = n-(d_s+d_i)$, then $n = x+y+z$.

To check the property (iii) of S^*, assume that d_i and d_{i+1}, $i \in \{1, 2, \dots, s-1\}$ are two consecutive spans in $D(T_n)$.

If $d_s = \frac{n}{2}$, then let T_1 and T_2 be two subgraphs of T_n induced by the vertices $v_0, v_1, v_2, \dots, v_x$; $v_x, v_{x+1}, \dots, v_{x+y-1}, v_0$ respectively ($x = \frac{n}{2}$). Let R be a polygon induced by the vertices $v_j, v_{j+d_i}, \dots, v_{j+d_{i+1}}$ where $v_j \in V(T_i)$ and $v_j v_{j+d_i}$ and $v_j v_{j+d_{i+1}}$ are two diagonals in T_i for some $i \in \{1, 2\}$.

If $\lceil \frac{t}{3} \rceil \cdot c \leq d_s \leq \frac{n-1}{2}$, then let T_1, T_2 and T_3 be three subgraphs of T_n induced by the vertices $v_0, v_1, v_2, \dots, v_x$, $v_x, v_{x+1}, \dots, v_{x+y}$, and $v_{x+y}, v_{x+y+1}, \dots, v_{x+y+z-1}, v_0$ respectively. Let R be a polygon induced by the vertices $v_j, v_{j+d_i}, \dots, v_{j+d_{i+1}}$ where $v_j \in V(T_i)$ and $v_j v_{j+d_i}$ and $v_j v_{j+d_{i+1}}$ are two diagonals in T_i for some $i \in \{1, 2, 3\}$.

Assume without loss of generality that $v_j v_{j+d_i}$ and $v_j v_{j+d_{i+1}}$ are two diagonals in T_1. Then, $j < j+d_i < j+d_{i+1}$ (by definitions of T_1 and R). It is clear that, $v_j v_{j+d_i} \dots v_{j+d_{i+1}} v_j \subset T_1$. Let $T_r = T_1(R)$ be a subgraph of T_1 that triangulates R with the boundary edges $v_j v_{j+d_i} \dots v_{j+d_{i+1}} v_j$ of R.

Suppose that $v_{j+d_i} v_{j+d_{i+1}} \notin E(T_r)$. Then there is $d_i < d < d_{i+1}$ such that $v_j v_{j+d} \in E(T_r)$. This implies that, $d \in D(T_r) \subset D(T_n)$, a contradiction with d_i and d_{i+1} are two consecutive spans in $D(T_n)$.

Hence, $v_{j+d_i} v_{j+d_{i+1}} \in E(T_r) \subset E(T_n)$. Then $d_{i+1} - d_i \in D(T_n)$.

Thus, $D(T_n) = S^*$. This completes the proof of the necessity.

To show the sufficiency, suppose that $S^* \subseteq S$. Then, $C(n, S^*)$ is a subgraph of $C(n, S)$.

Let $\triangle = v_0 v_x v_{x+y} v_{x+y+z}$ be a triangle (since, $x+y+z = n$). Clearly, $E(\triangle) \subset E(G)$. Then there are three polygons G_1, G_2 and G_3, induced by the vertices $v_0, v_1, v_2, \dots, v_x$; $v_x, v_{x+1}, \dots, v_{x+y}$ and $v_{x+y}, v_{x+y+1}, \dots, v_{x+y+z}$ respectively, and $G_2 = \varnothing$ where $y \in \{0, 1\}$.

Suppose that R is an r-gon in G_i, for some $i \in \{1, 2, 3\}$ such that $V(R) = \{v_h, v_{h+a_i}, v_{h+a_i+1}, \dots, v_{h+a_{i+1}}\}$ for some $v_h \in V(G_i)$ and two consecutive jump values a_i and a_{i+1} in S^*.

Without loss of generality assume that R is a subgraph of G_1 and $h = 0$.

Since $S^* = \{1, 2, a_3, a_4, \dots, a_k\}$ and satisfies that $a_{i+1} - a_i \in S^*$ for each $i = 1, \dots, k-1$, then we can use argument $(*)$ of Proposition 2.3 to show that R can be triangulated by T_r. Consider G_1 as a finite union of such polygons. Then we can assume that T_{G_1} is a finite union of the restricted triangulation of those polygons.

Obtain T_{G_2} (on G_2) and T_{G_3} (on G_3) in a similar way. Then $T_{G_1} \cup T_{G_2} \cup T_{G_3}$ is a triangulation of $C(n, S^*)$. Since $C(n, S^*)$ is a spanning subgraph of G, then $T_{G_1} \cup T_{G_2} \cup T_{G_3}$ is a triangulation of G.

This completes the proof. □

Corollary 2.9. *$S(n)$ is S^*.*

Proof. By definition of $S(n)$, either $S(n) = S_1$ (when n is even and $t = 1$) or $S(n) = S_\alpha$ (when n is odd) or else $S(n) = S_1 \cup c.S_\alpha$. By definition of S_1, we have that S_1 is a set of ascending values and contains $1, 2$; also S_α is a set of ascending values by Algorithm (A) and contains $1, 2$ by Lemma 2.5. Thus $S(n)$ is a set of ascending values containing $1, 2$.

Now, let ℓ denote the cardinality of $S(n)$.

When $S(n) = S_1$, $a_\ell = 2^r = \frac{n}{2}$. Let $x = z = a_\ell$ and $y = 0$ then we have $n = x+y+z$.

When $S(n) = S_\alpha$ or $S(n) = S_1 \cup c.S_\alpha$, we have by definition of β, $a_\ell = \lceil \frac{t}{3} \rceil \cdot c$. To get $n = x+y+z$ we have three cases. When $\alpha = \frac{t-2}{3}$, $\beta = \alpha + 1$ and $t = 3.\alpha + 2$. Let $x = c.\alpha$, $y = z = c.\beta$ (since, $n = c.t$). When $\alpha = \frac{t-1}{3}$, $\beta = \alpha + 1$ and $t = 3.\alpha + 1$. Let $x = y = c.\alpha$, $z = c.\beta$. When $\alpha = \frac{t}{3}$, let $x = y = z = c.\alpha$ (since $\beta = \alpha$).

Let a_i and a_{i+1} be any two consecutive values in $S(n)$. If $a_i, a_{i+1} \in S_1$, then $a_i = 2^i$ and $a_{i+1} = 2^{i+1}$ and then $a_{i+1} - a_i = 2^{i+1} - 2^i = 2^i \in S_1$. If $a_i = 2^r$, then a_i is the last value in S_1 and the first in $c.S_\alpha$ (recall that, $c = 2^r$) and if $a_i, a_{i+1} \in S_\alpha$ then by Lemma 2.2, $a_{i+1} - a_i \in S_\alpha$. Thus, $a_{i+1} - a_i \in S(n)$ for each $a_{i+1}, a_i \in S(n)$.

This completes the proof. □

Corollary 2.10. *Let $n \geq 4$ be a natural number. Suppose $G = C(n, S)$ is a circulant graph. Then G admits a restricted triangulation if one of the following conditions hold.*

(1) When n is odd, S contains $\{2\}$ *and all odd values* $a_i \le a_s$ *where* $\lceil \frac{n}{3} \rceil \le a_s \le \lfloor \frac{n}{2} \rfloor$.
(2) When n is even, S contains $\{1\}$ *and all even values* $a_i \le a_s$ *where* $\lceil \frac{t}{3} \rceil \cdot c \le a_s \le \lfloor \frac{n}{2} \rfloor$.

Proof. In both cases $a_1 = 1$, $a_2 = 2$ belong to a set of ascending values S, and $\lceil \frac{t}{3} \rceil \cdot c \le a_s \le \lfloor \frac{n}{2} \rfloor$ (when n is an odd natural number, $\lceil \frac{t}{3} \rceil \cdot c = \lceil \frac{n}{3} \rceil$). Further, for each $a_{i+1},\ a_i \in S$, $a_{i+1} - a_i \in \{1, 2\} \subset S$.

To show the property (2) of S^* we have to consider two cases.

Case (1) When n is an odd natural number. Let $n \ge 7$ (When $n = 5$, then $S = \{1, 2\}$ and clearly $C(5, S)$ admits a restricted triangulation $T_5 = \{v_0v_1v_2v_3v_4v_0\} \cup \{v_0v_2, v_0v_3\}$). If $\lceil \frac{n}{3} \rceil$ is an odd number, then $\lceil \frac{n}{3} \rceil = a_i$ for some $i \in \{3, \dots, s\}$. If $\lceil \frac{n}{3} \rceil$ is an even number, then $\lceil \frac{n}{3} \rceil + 1 = a_i$ for some $i \in \{4, \dots, s\}$.

Whether $\lceil \frac{n}{3} \rceil$ is odd or even, we have $n - 2a_i$ is odd and $1 \le n - 2a_i \le a_i$. Then $n - 2a_i = a_j$ for some $j \in \{1, \dots, i\}$. Let $x = y = a_i$, $z = a_j$. Thus, $n = x + y + z$.

Case (2) When n is an even natural number, we have $r \ge 1$ and then $c(\ge 2)$ is even. Hence, $\lceil \frac{t}{3} \rceil \cdot c$ is even and then $\lceil \frac{t}{3} \rceil \cdot c = a_i$ for some $i \in \{2, \dots, s\}$. Now, we have $n - 2a_i$ is even and either $n - 2a_i = 0$ or $2 \le n - 2a_i \le a_i$. If $n - 2a_i = 0$, let $x = z = a_i$, $y = 0$. If $2 \le n - 2a_i \le a_i$, then $n - 2a_i = a_j$ for some $j \in \{2, \dots, i\}$ and let $x = z = a_i$, $y = a_j$. Thus, $n = x + y + z$.

By Theorem 2.8, the circulant graph $C(n, S)$ admitting a restricted triangulation. □

Proposition 2.11 ([13]). *Suppose* $G = C(n, \{a_1, a_2, \dots, a_k\})$ *and* $H = C(n, \{b_1, b_2, \dots, b_k\})$ *with* $\{a_1, a_2, \dots, a_k\} = q.\{b_1, b_2, \dots, b_k\}$, *where the multiplication is reduced modulo n and* $gcd(q, n) = 1$. *Then G is isomorphic to H.*

Example 2.12. *Let* $n = 9$ *and* $S(n) = \{1, 2, 3\}$.
When $S_1 = \{2, 3, 4\}$, *then we have* $S_1 = \{2, 6, 4\} = 2.\{1, 3, 2\} = 2.S(n)$.
When $S_2 = \{1, 3, 4\}$, *then we have* $S_2 = \{8, 12, 4\} = 4.\{2, 3, 1\} = 4.S(n)$.

The next corollary considers circulant graph $G = C(n, S)$ admits a restricted triangulation when there is an integer $q \ge 1$ with $gcd(q, n) = 1$ such that $S^* \subseteq q \cdot S$.

Corollary 2.13. *let* $G = C(n, S)$ *be a circulant graph. Suppose there is an integer* $q \ge 1$ *with* $gcd(q, n) = 1$ *such that* $S^* \subseteq q.S$ *or* $q.S^* \subseteq S$ *where the multiplication is reduced modulo n. Then G has a configuration that admits a restricted triangulation.*

Proof. Suppose that, $q \ge 1$ is an integer such that $gcd(q, n) = 1$ and $S^* \subseteq q.S$ or $q.S^* \subseteq S$. Then, there is a set $S' \subset S$ such that $S^* = q.S'$ or $q.S^* = S'$. Thus, by Proposition 2.11, the subgraph $H = C(n, S')$ of $C(n, S)$ is isomorphic to $C(n, S^*)$. By Theorem 2.8, $H = C(n, S')$ admits a restricted triangulation. Thus, G has a configuration that admits a restricted triangulation. This completes the proof. □

2.1 An application

The skewness of a graph G, denoted $sk(G)$, is the minimum number of edges to be deleted from G such that the resulting graph is planar. The convex skewness of a convex graph G, denoted $sk_c(G)$ is the minimum number of edges to be removed from G so that the resulting graph is a convex plane graph (see [14]).

Proposition 2.14. *Let* $G = C(n, S)$ *be a circulant graph and let* $q \ge 1$ *be an integer such that* $gcd(q, n) = 1$. *Then* $sk_c(G) = E(G) - (2n - 3)$ *if* $S^* \subseteq S$, *or* $qS^* \subseteq S$, *or* $S^* \subseteq qS$.

Proof. If $S^* \subseteq S$, then $G = C(n, S)$ admits a restricted triangulation, by Theorem 2.8. If $qS^* \subseteq S$ or $S^* \subseteq qS$, then $G = C(n, S)$ admits a restricted triangulation, by Corollary 2.13.

It is known that, any triangulation T of a convex n-gon has $2n - 3$ edges ($n - 3$ of them are non-boundary edges). If any new straight line segment is added to the triangulation, it will intersect with some non-boundary edge of T. Hence, we have $sk_c(G) = E(G) - (2n - 3)$. □

3 $K_n - G$ admits a restricted triangulation

In this section we turn to another question: which circulant graph G on n vertices does satisfy that $K_n - G$ admits a restricted triangulation and what is the *largest size* of G such that $K_n - G$ still admits a restricted triangulation? We answered the first question by Corollary 3.1 and Corollary 3.3. We show that $C(n, S(n))$ is a smallest size circulant graph that admits a restricted triangulation, in order to answer the second question by Theorem 3.5. In what follows, let $\mathcal{N} = \{1, 2, \dots, \lfloor n/2 \rfloor\}$.

Corollary 3.1. *Let $G = C(n, S)$ be a circulant graph. Then $K_n - G$ admits a restricted triangulation if one of the following conditions hold.*
(1) If there is S^* such that $S \cap S^* = \emptyset$.
(2) S contains all odd i of $\mathcal{N}$ except $\{1, \lfloor n/2 \rfloor\}$.
(3) S contains all even i of $\mathcal{N}$ except $\{2, \lfloor n/2 \rfloor\}$.

Proof. First, it is well known that $K_n - C(n, S) = C(n, \mathcal{N} - S)$.

Suppose that $S \cap S^* = \emptyset$. Then, $\mathcal{N} - S$ contains S^* which yields that the circulant graph $C(n, \mathcal{N} - S)$ admits the restricted triangulation by Theorem 2.8, and this shows the property (1).

In (2), $\mathcal{N} - S$ contains all even values of $\mathcal{N}$ together with $\{1, \lfloor n/2 \rfloor\}$. In (3), $\mathcal{N} - S$ contains all odd values of $\mathcal{N}$ together with $\{2, \lfloor n/2 \rfloor\}$. By Corollary 2.4, $C(n, \mathcal{N} - S)$ admits the restricted triangulation for both cases.

This completes the proof. □

Definition 3.2 ([14]). *Let K_n be a convex complete graph with n vertices. F is said to be potentially triangulable in K_n if there exists a configuration of F in K_n such that $K_n - F$ admits a triangulation.*

Corollary 3.3. *Let $G = C(n, S)$ be a circulant graph. Suppose $S^* \subseteq q.(\mathcal{N} - S)$ for some an integer $q \geq 1$ with $gcd(q, n) = 1$. Then G is potentially triangulable in K_n.*

Proof. Suppose that, $q \geq 1$ is an integer such that $gcd(q, n) = 1$ and $S^* \subseteq q.(\mathcal{N} - S)$ for some S^*. Then, there is a set $S' \subset \mathcal{N} - S$ such that $S^* = q.S'$. Then by Proposition 2.11, the subgraph $C(n, S')$ of $C(n, \mathcal{N} - S)$ is isomorphic to $C(n, S^*)$. By Theorem 2.8, $C(n, S')$ has a configuration that admits a restricted triangulation. Thus, $C(n, \mathcal{N} - S)(= K_n - G)$ admits a restricted triangulation. This completes the proof. □

To answer the second part of the question, we shall determine the largest size $L(n)$ of G for which $K_n - G$ admits a restricted triangulation. Before proceeding, let first $|E(C(n, S(n)))| = \mathcal{E}_\ell$ where $|S(n)| = \ell$. Then we deduce that,

$$\mathcal{E}_\ell = \begin{cases} n\ell - \frac{n}{2}, & t = 1; \\ n\ell, & \text{otherwise.} \end{cases}$$

The next result shows that whenever $C(n, S)$ admits a restricted triangulation then $|S| \geq |S(n)|$. That is, $C(n, S)$, is not a smaller size than $C(n, S(n))$.

Theorem 3.4. *$C(n, S(n))$ is the smallest size circulant graph admitting a restricted triangulation if $n \geq 4$.*

Proof. Recall that, $S(n) = S_1 \cup c \cdot S_\alpha$. By Theorem 2.6, $C(n, S(n))$ admits a restricted triangulation. Hence, we just show that $S(n)$ is the smallest cardinality set for which the conclusion remains true.

Assume that $C(n, S)$ is a circulant graph that admits a restricted triangulation where $S = \{b_1, b_2, b_3, \dots, b_s\}$ is a set of ascending jump values to $C(n, S)$.

By Theorem 2.8, S contains S^*. Thus, $1, 2 \in S$ and $\lceil \frac{t}{3} \rceil \cdot c \leq b_s \leq \lfloor \frac{n}{2} \rfloor$ and for any $i \in \{1, \dots, s-1\}$, $b_{i+1} - b_i = b_j \in S$ where $j \in \{1, \dots, i\}$.

In case when b_{i+1} is even we have, according to S_α (by Algorithm (A) step (2) where $b_{i+1} = a_0$ is even) and S_1 (by definition of S_1), that the difference between the two consecutive values b_i and b_{i+1} is always b_i.

If $j \in \{1, \dots, i-1\}$, then the number of values in $S(n)$ is less than the number of values in S. If $b_{i+1} - b_i = b_i \in S$, then the number of values in $S(n)$ is equal to the number of values in S. Thus, $|S| \geq |S(n)|$.

In case when b_{i+1} is odd, then $b_j \neq b_i$ and there are three cases which are depending only on S_α.

Case (1): $b_j \in \{1, 2\}$.

In this case, the difference between the two consecutive values b_i and b_{i+1} is at most 2. According to Algorithm (A), the difference between the two consecutive values is at most 2 only when $b_i = \lfloor \frac{b_{i+2}}{2} \rfloor$ and $b_{i+1} = \lceil \frac{b_{i+2}}{2} \rceil$ or when $b_i = 3$ and $b_i = 5$. Then the number of values in $S(n)$ is equal to or less than the number of values in S. Whereas Step (2) and Step (4) of Algorithm (A) give the difference between any two consecutive values being at least 3, which implies $|S| \geq |S(n)|$.

For instance, take $b_{i+1} = 13$ then $b_i \in \{11, 12\}$. Take $S \in \{S_1, S_2, S_3, S_4, S_5, S_6\}$ where $S_1 = \{1, 2, 3, 5, 10, 12, 13\}$, $S_2 = \{1, 2, 3, 6, 9, 12, 13\}$, $S_3 = \{1, 2, 3, 5, 6, 11, 13\}$, $S_4 = \{1, 2, 3, 6, 9, 11, 13\}$, $S_5 = \{1, 2, 3, 5, 10, 11, 13\}$, or $S_6 = \{1, 2, 3, 4, 7, 9, 11, 13\}$. Then the cardinality of S is 7. By Algorithm (A) step (2):

- either, $b_i = 7$ and $b_{i-1} = 6$ and then $S_\alpha = \{1, 2, 3, 6, 7, 13\}$. Thus, the cardinality of S_α is 6;
- or, $b_{i+2} = 25$ and then $b_{i+1} = 13$ and $b_i = 12$. Then, $S_\alpha = \{1, 2, 3, 6, 12, 13, 25\}$. In this case, Take $S \in \{S_1, S_2\}$ where $S_1 = \{1, 2, 3, 5, 10, 12, 13, 25\}$, and $S_2 = \{1, 2, 3, 6, 9, 12, 13, 25\}$. Note that S is of cardinality 8 while S_α is of cardinality 7.

Case (2): $j \in \{3, \ldots, i-2\}$.

In S_α, this case is satisfied in step (4) of Algorithm (A) (when $b_i = a_0$ is odd and not divisible by 3). But in this case, Algorithm (A) states b_i to be $b_{i+1} - 3$ which is always even. Then by step (2) when $b_i(= a_0)$ is even we have that $b_{i-1} = \frac{b_i}{2}$. That is, the difference between the two consecutive values b_i and b_{i-1} is always b_{i-1} (since $b_i = 2b_{i-1}$). While, in S, we have either $b_i - b_{i-1} = b_{i-1}$ and then $|S| = |S(n)|$ or $b_i - b_{i-1} = b_k \neq b_{i-1}$ and then $|S| > |S(n)|$. Thus, $|S| \geq |S(n)|$.

For instance, take $b_{i+1} = 23$ and $S = \{1, 2, 3, 6, 8, 9, 17, 23\}$. By Algorithm (A) we have $S_\alpha = \{1, 2, 3, 5, 10, 20, 23\}$. Note that S is of cardinality 8 while S_α is of cardinality 7.

Case (3): $j = i - 1$.

If b_{i+1} is not divisible by 3, then by Algorithm (A) step (2) we have $b_j = b_{i-1} = \lfloor \frac{b_{i+1}}{2} \rfloor$ and $b_i = \lceil \frac{b_{i+1}}{2} \rceil$ (where $b_{i-1} = b_{1,L}$ and $b_i = b_{2,L}$), and then the difference between the two consecutive values b_i and b_{i-1} is 1. While in S, we have either $b_i - b_{i-1} = 1$ and then $|S| = |S(n)|$ or $b_i - b_{i-1} = b_k \neq 1$ and then $|S| > |S(n)|$.

If b_{i+1} is divisible by 3, then in S_α by Algorithm (A) step (2) (where $b_{i+1} = a_0$ is odd and divisible by 3). Then $b_{i-1} = \frac{b_{i+1}}{3}$ and $b_i = 2.b_{i-1}$ (where $b_{i-1} = a_{1,L}$ and $b_i = a_{2,L}$). That is, the difference between the two consecutive values b_i and b_{i-1} is always b_{i-1} itself (since $b_i = 2.b_{i-1}$). While in S, either $b_i - b_{i-1} = b_{i-1}$ and then $|S| = |S(n)|$ or $b_i - b_{i-1} = b_k$ for some $k \in \{1, 2, \ldots i-2\}$ and then $|S| > |S(n)|$.

This completes the proof. □

Let $L(n) = \binom{n}{2} - \mathcal{E}_\ell$ (where $\binom{n}{2}$ is the size of K_n and $\mathcal{E}_\ell$ is the size of $C(n, S(n))$). Then, we conclude that

$$L(n) = \begin{cases} \frac{n(n-2\ell)}{2}, & t = 1; \\ \frac{n(n-2\ell-1)}{2}, & \text{otherwise.} \end{cases}$$

By Theorem 3.4, we have that $\mathcal{E}_\ell$ is the smallest number of edges of a circulant graph that admits a restricted triangulation. The following result is to measure the non-triangulability of $K_n - G$.

Theorem 3.5. *Let $G = C(n, S)$ be a circulant graph. Then $K_n - G$ admits no restricted triangulation if $|E(G)| > L(n)$.*

Proof. Let $|E(G)| > L(n)$. Then $|E(K_n - G)| = \binom{n}{2} - |E(G)| < \binom{n}{2} - L(n) = \mathcal{E}_\ell$. Then $|\mathcal{N} - S| < \mathcal{E}_\ell$. By Theorem 3.4, $C(n, \mathcal{N} - S)(= K_n - G)$ admits no restricted triangulation. □

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