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Research Article

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Involutory biquandles and singular knots and links

Abstract: We define a new algebraic structure for singular knots and links. It extends the notion of a bikei (or involutory biquandle) from regular knots and links to singular knots and links. We call this structure a singbikei. This structure results from the generalized Reidemeister moves representing singular isotopy. We give several examples on singbikei and we use singbikei to distinguish several singular knots and links.

1 Introduction

Singular knots and links are viewed diagrammatically as knots and links with some of the crossings being 4-valent rigid vertices. The theory of Vassiliev Invariants of knots and links shed the light on studying singular knots and links as a larger space that involves usual knots and links as a subspace. See [1 - 3]. Since then, many knot and link invariants have been generalized to singular knots and links. For example see [4 - 8].

Kei and quandles are algebraic structures, which were constructed to describe knots and links via generators and relations resulting from the arcs and the crossings of a knot or link diagram, and respecting the invariance of these diagrams under the Reidemeister moves. See [9 - 17].

Kei and quandles for singular knots were constructed in [18, 19], respectively. This paper introduces a new algebraic structure that generalizes involutory biquandles to singular knots. We call this structure singbikei. We give a plethora of non-trivial natural algebraic objects in examples that satisfy the axioms of this new algebraic structure. As a byproduct of this structure, we show how to apply the theory for distinguishing singular knots by giving several examples.

This paper is organized as follow. In Section 2 we give the basic concepts and terminology for kei and bikei. We also define singular knots and links and their isotopy invariance. In Section 3 we introduce the structure of a bikei for singular knots and links and we give several examples on this new structure with some related results. In Section 4 we give some examples of singular knots and links and distinguish them using several singular bikei colorings.

2 Basic concepts and terminology

Most of the basic concepts and terminology in this research can be found in [8].

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We begin this section with the definition of a kei, and after the definition we will see how the axioms of a kei result from the three Reidemeister moves. The coloring of a regular crossing is drawn as in the following figure.

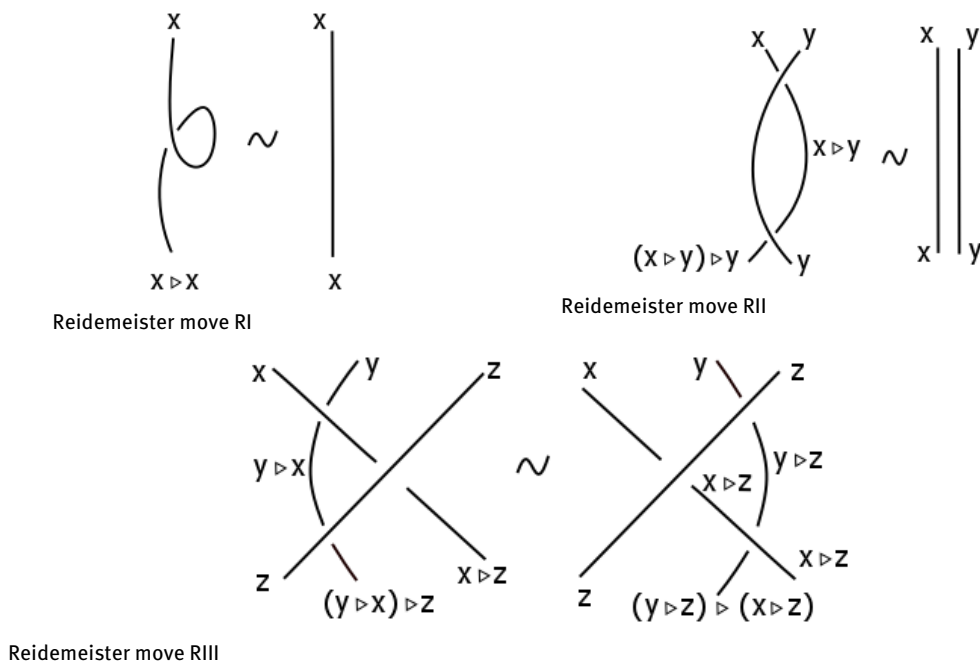**Fig. 1.** The coloring of a regular crossing with one operation
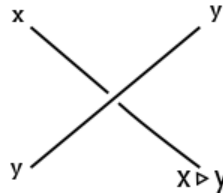
Definition 2.1. *A kei (involutory quandle) is a set X with a binary operation $\triangleright: X \times X \to X$ satifying the following three axioms:*

(i) $x \triangleright x = x$, for all $x \in X$.

(ii) $(x \triangleright y) \triangleright y = x$, for all $x, y \in X$.

(iii) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$, for all $x, y, z \in X$.
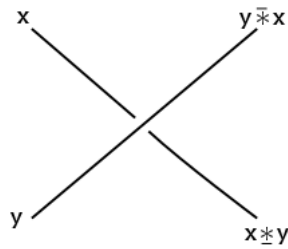
The three axioms of a kei result from the invariance of the three Reidemeister moves as in the following figures.

Fig. 2. The coloring of Reidemeister moves

Reidemeister move RI

Reidemeister move RII

Reidemeister move RIII

Now, instead of one operation at a crossing, two operations are defined, and instead of coloring the arcs from the top, the arcs are colored from left to right, so we get:

Fig. 3. The coloring of a regular crossing with two operations

Next we give the definition of a bikei. After this we will see how the axioms of a bikei result from the three Reidemeister moves.

Definition 2.2. *A bikei (involutory biquandle) is a set X with two binary operations $\overline{\ast}, \underline{\ast} : X \times X \to X$ such that for all x, y, z ∈ X, we have*

$$x \underline{\ast} x = x \overline{\ast} x \qquad (i)$$

$$x \underline{\ast} (y \overline{\ast} x) = x \underline{\ast} y \qquad (ii.i)$$
$$x \overline{\ast} (y \underline{\ast} x) = x \overline{\ast} y \qquad (ii.ii)$$
$$(x \underline{\ast} y) \underline{\ast} y = x \qquad (ii.iii)$$
$$(x \overline{\ast} y) \overline{\ast} y = x \qquad (ii.iv)$$

$$(x \underline{\ast} y) \underline{\ast} (z \underline{\ast} y) = (x \underline{\ast} z) \underline{\ast} (y \overline{\ast} z) \qquad (iii.i)$$
$$(z \underline{\ast} y) \overline{\ast} (x \underline{\ast} y) = (z \overline{\ast} x) \underline{\ast} (y \overline{\ast} x) \qquad (iii.ii)$$
$$(y \overline{\ast} x) \overline{\ast} (z \overline{\ast} x) = (y \overline{\ast} z) \overline{\ast} (x \underline{\ast} z) \qquad (iii.iii)$$

The axioms of a bikei result from the invariance of the three Reidemeister moves as in figures 4, 6 and 7.
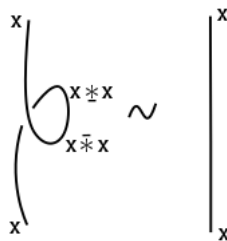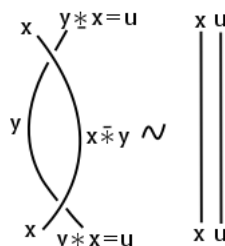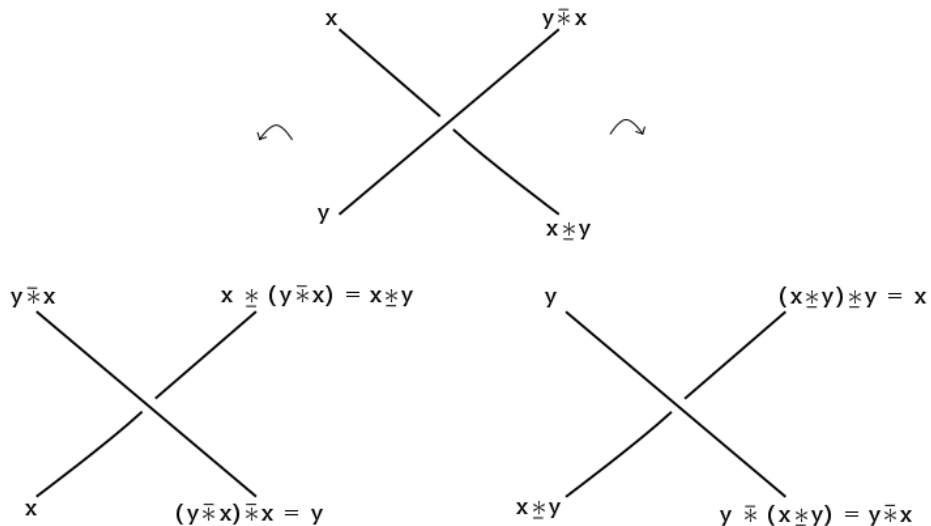
Fig. 4. Reidemeister move RI for bikei

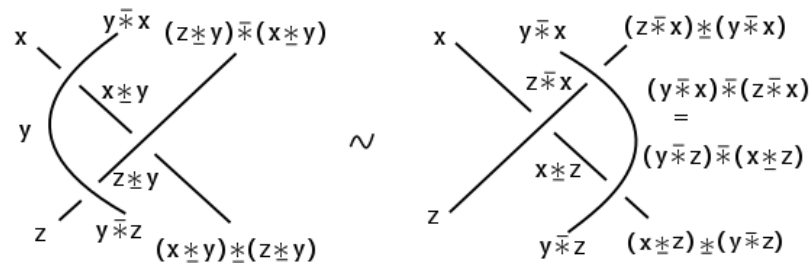**Fig. 5.** Reidemeister move RII for bikei

The Reidemeister move RII means that the over crossing and under crossing operations do not depend on which way the crossing is rotated.

Fig. 6. Rotated coloring crossing

The Reidemeister move RIII gives us what is called the exchange laws:

Fig. 7. Reidemeister move RIII for bikei

Let X and Y be bikei with operations $\underline{*}_X$, $\overline{*}_X$ and $\underline{*}_Y$, $\overline{*}_Y$ respectively. Then a map $f : X \longrightarrow Y$ is a *bikei homomorphism* if for all $x, x' \in X$ we have

$$f(x\underline{*}_X x') = f(x)\underline{*}_Y f(x') \text{ and } f(x\overline{*}_X x') = f(x)\overline{*}_Y f(x').$$

A *bikei isomorphism* is a bijective bikei homomorphism, and *two bikei are isomorphic* if there is a bikei isomorphism between them.

Remark 2.3. *Every Kei is a bikei with the operations $x\underline{*}y = x \triangleright y$ and $x\overline{*}y = x$.*

Typical examples of bikei include the following :
- A non-empty set X with operation $x\overline{*}y = x\underline{*}y = \sigma(x)$, where σ is any involution from X to X, is a bikei. It is called *a constant action bikei*.
- Let $\Lambda = Z[t, s]/(t^2 - 1, s^2 - 1, (s-t)(1-s))$, then any Λ - module X with $x\underline{*}y = tx + (s-t)y$, and $x\overline{*}y = sx$, is called *an Alexander bikei*.

—   A group $X = G$ with $x\underline{*}y = yx^{-1}y$ and $x\overline{*}y = x$ is a bikei. It is called *the core bikei of the group G.*

A *singular link* in S^3 is the image of a smooth immersion of n circles in S^3 that has finitely many double points, called *singular points.*

     Two singular knots K_1 and K_2 are *isotopy equivalent* if we can get one of them from the other by a finite sequence of the generalized Reidemeister moves RI, RII, RIII, RIVa, RIVb and RV in the following figure.

Fig. 8. Generalized Reidemeister moves RI, RII, RIII, RIVa, RIVb and RV
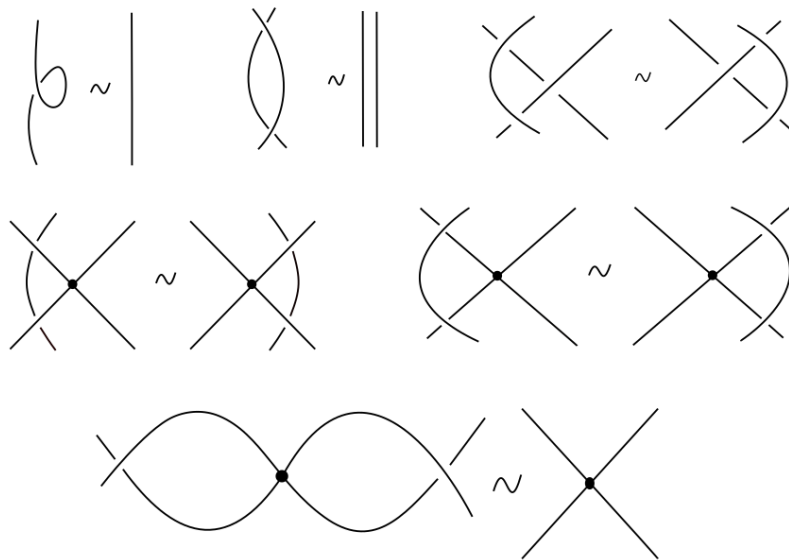
3 Construction of singbikei

We will define the notion of a singbikei and give some examples and use them to construct an invariant of singular knots and links. The invariant is the set of colorings of a given singular knot or link by a singbikei. We draw the colorings of the regular and singular crossings as in the following figure.

The colorings of a regular crossing   The colorings of a singular crossing
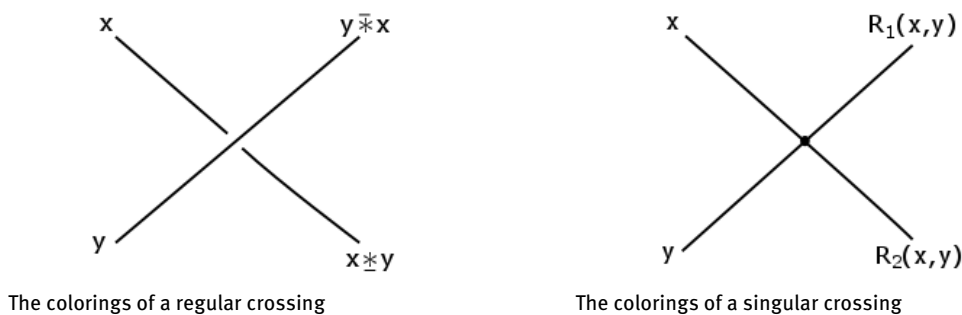
Since our singular crossings are unoriented, we need the operations to be symmetric in the sense that if we rotate the crossing in the right diagram of the above figure by 90, 180 or 270 degrees, the operations should stay the same in order for colorings to be well-defined. Therefore we get the following three axioms:

$$x = R_1(y, R_2(x, y)) = R_2(R_2(x, y), R_1(x, y)) \tag{1}$$

$$y = R_2(R_1(x, y), x) = R_1(R_2(x, y), R_1(x, y)) \tag{2}$$

$$R(x, y) = (R_2(y, R_2(x, y)), R_1(R_1(x, y), x)) \tag{3}$$

We have 5 generalized Reidemeister moves; I, II, III are for regular crossings; IV, V are for singular crossings. Next we show how the generalized Reidemeister moves induce relations considering the colorings of the singular crossings.
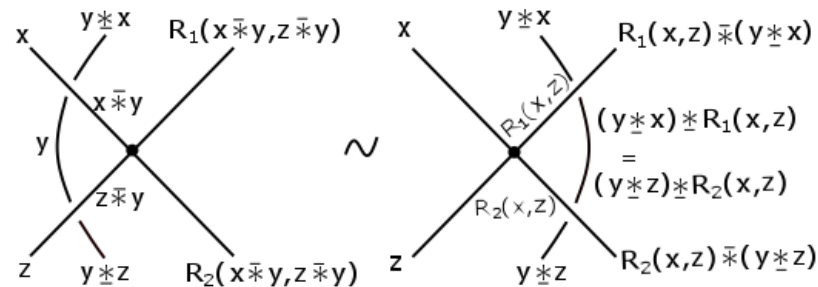
Fig. 9. Reidemeister move RIVa

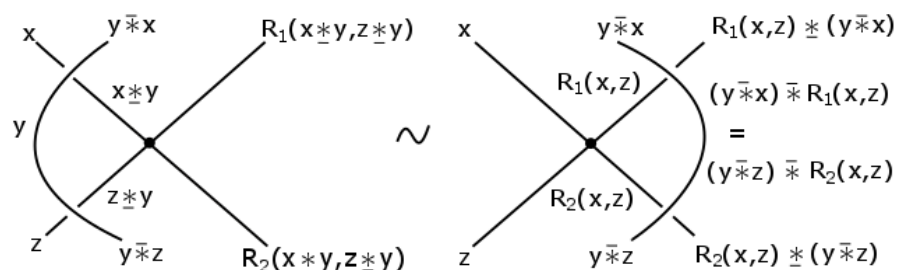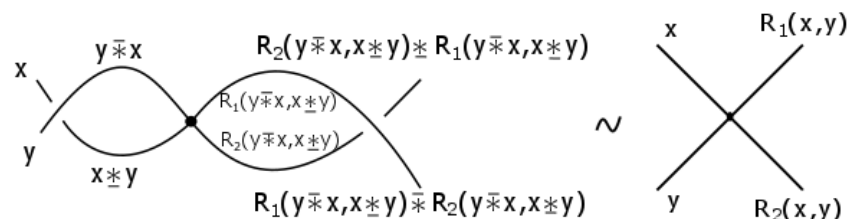**Fig. 10.** Reidemeister move RIVb

Fig. 11. Reidemeister move RV

The following definition is coming from the generalized Reidemeister moves and the axioms are justified from the figures [9-11].

Definition 3.1. *Let $(X, \bar{\ast}, \underline{\ast})$ be a bikei. Let R_1 and R_2 be two maps from $X \times X$ to X. Then $(X, \bar{\ast}, \underline{\ast}, R_1, R_2)$ is called a singbikei if, in addition to the three axioms 3.1, 3.2 and 3.3, the following axioms are satisfied*

$$R_1(x \bar{\ast} y, z \bar{\ast} y) = R_1(x, z) \bar{\ast} (y \underline{\ast} x) \tag{4}$$

$$R_2(x \bar{\ast} y, z \bar{\ast} y) = R_2(x, z) \bar{\ast} (y \underline{\ast} z) \tag{5}$$

$$(y \underline{*} x) \underline{*} R_1(x, z) = (y \underline{*} z) \underline{*} R_2(x, z) \tag{6}$$

$$R_1(x \underline{*} y, z \underline{*} y) = R_1(x, z) \underline{*} (y \overline{*} x) \tag{7}$$

$$R_2(x \underline{*} y, z \underline{*} y) = R_2(x, z) \underline{*} (y \overline{*} z) \tag{8}$$

$$(y \overline{*} x) \overline{*} R_1(x, z) = (y \overline{*} z) \overline{*} R_2(x, z) \tag{9}$$

$$R_1(x, y) = R_2(y \overline{*} x, x \underline{*} y) \underline{*} R_1(y \overline{*} x, x \underline{*} y) \tag{10}$$

$$R_2(x, y) = R_1(y \overline{*} x, x \underline{*} y) \overline{*} R_2(y \overline{*} x, x \underline{*} y) \tag{11}$$

The following straightforward lemma makes the set of colorings of a singular knot or link by a singbikei an invariant of singular knots and links.

Lemma 3.2. *The set of colorings of a singular knot by a singbikei does not change by the Reidemeister moves RI, RII, RIII, RIVa, RIVb and RV.*

Example 3.3. *It is known that every kei is a bikei with the operations $x \underline{*} y = x \triangleright y$ and $x \overline{*} y = x$. Then X is a bikei with these operations so $(X, \underline{*}, \overline{*}, R_1, R_2)$ is a singbikei if $R_1, R_2 : X \times X \longrightarrow X$ satisfy the following equations:*

$$x = R_1(y, R_2(x, y)) = R_2(R_2(x, y), R_1(x, y))$$

$$y = R_2(R_1(x, y), x) = R_1(R_2(x, y), R_1(x, y))$$

$$R(x, y) = (R_2(y, R_2(x, y)), R_1(R_1(x, y), x))$$

$$(y \triangleright x) \triangleright R_1(x, z) = (y \triangleright z) \triangleright R_2(x, z)$$

$$R_1(x \triangleright y, z \triangleright y) = R_1(x, z) \triangleright y$$

$$R_2(x \triangleright y, z \triangleright y) = R_2(x, z) \triangleright y$$

$$R_1(x, y) = R_2(y, x \triangleright y) \triangleright R_1(y, x \triangleright y)$$

$$R_2(x, y) = R_1(y, x \triangleright y)$$

Example 3.4. *Let X be a set and $\sigma : X \longrightarrow X$ be any involution on X, (i.e any map such that $\sigma^2 = Id_X$) with $x \overline{*} y = x \underline{*} y = \sigma(x)$.*
Let $R_1, R_2 : X \times X \longrightarrow X$ be two maps, then $(X, \overline{*}, \underline{*}, R_1, R_2)$ is a singbikei if R_1 and R_2 satisfy the following equations:

$$x = R_1(y, R_2(x, y)) = R_2(R_2(x, y), R_1(x, y))$$

$$y = R_2(R_1(x, y), x) = R_1(R_2(x, y), R_1(x, y))$$

$$R(x, y) = (R_2(y, R_2(x, y)), R_1(R_1(x, y), x))$$

$$R_1(\sigma(x), \sigma(z)) = \sigma(R_1(x, z))$$

$$R_2(\sigma(x), \sigma(z)) = \sigma(R_2(x, z))$$

$$R_1(x, y) = \sigma(R_2(\sigma(y), \sigma(x))$$

$$R_2(x, y) = \sigma(R_1(\sigma(y), \sigma(x))$$

Proposition 3.5. *Let $X = Z_n$ and $\sigma : Z_n \longrightarrow Z_n$ be given by one of the following rules*

$$\sigma(x) = (n - 1)x + d, \text{ where d is arbitrary in } Z_n$$

or

$$\sigma(x) = x$$

or

$$\sigma(x) = x + \frac{n}{2}, \text{ where n is even}$$

then σ is an involution.

Proof. 1. If σ is given by $\sigma(x) = (n-1)x + d$, then

$$
\begin{aligned}
\sigma(\sigma(x)) &= \sigma((n-1)x + d) \\
&= (n-1)[(n-1)x + d] + d \\
&= (n-1)^2 x + (n-1)d + d \\
&= (n-1)^2 x + nd \\
&= x
\end{aligned}
$$

2. If σ is given by $\sigma(x) = x$, then

$$
\begin{aligned}
\sigma(\sigma(x)) &= \sigma(x) \\
&= x
\end{aligned}
$$

3. If σ is given by $\sigma(x) = x + \dfrac{n}{2}$, where n is even, then

$$
\begin{aligned}
\sigma(\sigma(x)) &= \sigma\left(x + \frac{n}{2}\right) \\
&= \left[x + \frac{n}{2}\right] + \frac{n}{2} \\
&= x + n \\
&= x
\end{aligned}
$$

This completes the proof. $\qquad\qquad\square$

Sometimes these are the only linear (i.e functions of the form $f(x) = ax + b$) involutions σ on Z_n and sometimes Z_n has other linear involutions. For example if $X = Z_8$, in addition to the previous solutions,

$$
\begin{aligned}
\sigma(x) &= 5x + 4 \\
\sigma(x) &= 3x + 2 \\
\sigma(x) &= 3x + 6
\end{aligned}
$$

are also linear involutions in Z_8.

Lemma 3.6. *If n is prime, then the only linear formulas for an involution σ on Z_n are:*

$$
\sigma(x) = (n-1)x + d,\ d \in Z_n \text{ or } \sigma(x) = x.
$$

Proof. The general linear formula of σ is $\sigma(x) = cx + d$, where $c, d \in Z_n$.
Since σ is an involution, we have

$$
\begin{aligned}
\sigma(\sigma(x)) &= c[cx + d] + d \\
&= c^2 x + cd + d \\
&= c^2 x + (c+1)d \\
&= x
\end{aligned}
$$

Since $c^2 = 1$ in Z_n and n is prime, we have

$$
\begin{aligned}
c^2 &\equiv 1 \quad (mod\ n) \\
(c^2 - 1) &\equiv 0 \quad (mod\ n) \\
(c-1)(c+1) &\equiv 0 \quad (mod\ n)
\end{aligned}
$$

$$n/(c-1)(c+1) \Longrightarrow n/(c-1) \text{ or } n/(c+1)$$
$$c - 1 \equiv 0 \ (mod \ n) \text{ or } c + 1 \equiv 0 \ (mod \ n)$$
$$c \equiv 1 \ (mod \ n) \text{ or } c \equiv -1 \ (mod \ n)$$
$$c \equiv 1 \ (mod \ n) \text{ or } c \equiv n - 1 \ (mod \ n)$$
$$c = 1 \text{ or } c = n - 1 \text{ in } Z_n$$

So, we have two cases:

If $c = n - 1$, then

$$\sigma(\sigma(x)) = (n-1)^2 x + nd$$
$$= n^2 x - 2nx + x$$
$$= x.$$

Therefore $\sigma(x) = (n-1)x + d$, where $d \in Z_n$.

If $c = 1$, then

$$\sigma(\sigma(x)) = x + 2d$$
$$= x, \text{ when } 2d = 0 \text{ we get } d = 0 \text{ or } 2d = n.$$

– If $d = 0$, then $\sigma(x) = x$.
– If $d = \dfrac{n}{2}$, then n is even and prime. Therefore $n = 2$ so $d = 1$, then $\sigma(x) = x + 1$, which is included in the case when $c = n - 1$.

So σ must be $\sigma(x) = (n-1)x + d, d \in Z_n$ or $\sigma(x) = x$.   □

Theorem 3.7. *Let $X = Z_n$ and $\sigma : Z_n \longrightarrow Z_n$ be an involution and $x \overline{\ast} y = x \underline{\ast} y = \sigma(x)$. Then $R_1, R_2 : Z_n \times Z_n \longrightarrow Z_n$ given below make $(Z_n, \overline{\ast}, \underline{\ast}, R_1, R_2)$ a singbikei.*
(1) *If $\sigma(x) = (n-1)x + d, d \in Z_n$ then*
   $R_1(x, y) = (n-1)y + c$ *when $(n-2)d = (n-2)c$ and*
   $R_2(x, y) = (n-1)x + c.$
(2) *If $\sigma(x) = x$, then*
   $R_1(x, y) = (n-1)y + c$ *and*
   $R_2(x, y) = (n-1)x + c.$
(3) *If $\sigma(x) = x + \dfrac{n}{2}$, where n is even, then*
   $R_1(x, y) = (n-1)y + c$ *when $(n-1)\dfrac{n}{2} = \dfrac{n}{2}$ and*
   $R_2(x, y) = (n-1)x + c.$

Proof. We show that R_1 and R_2 satisfy all the equations in Example 3.4,

1. If $\sigma(x) = (n-1)x + d$, where $d \in Z_n$, then

$$R_1(y, R_2(x, y)) = (n-1)^2 x + (n-1)c + c$$
$$= n^2 x - 2nx + x + nc$$
$$= x$$

$$R_2(R_2(x, y), R_1(x, y)) = (n-1)^2 x + (n-1)c + c$$
$$= n^2 x - 2nx + x + nc$$
$$= x$$

$$R_2(R_1(x, y), x) = (n-1)^2 y + (n-1)c + c$$

$$= n^2y - 2ny + y + nc$$
$$= y$$

$$R_1(R_2(x, y), R_1(x, y)) = (n-1)^2y + (n-1)c + c$$
$$= n^2y - 2ny + y + nc$$
$$= y$$

$$(R_2(y, R_2(x, y)), R_1(R_1(x, y), x)) = ((n-1)y + c, (n-1)x + c)$$
$$= R(x, y)$$

$$R_1(\sigma(x), \sigma(z)) = R_1((n-1)x + d, (n-1)z + d)$$
$$= (n-1)[(n-1)z + d] + c$$
$$= (n-1)^2z + (n-1)d + c$$
$$= (n-1)^2z + (n-2)d + d + c$$
$$= (n-1)^2z + (n-2)c + d + c$$
$$= (n-1)^2z + (n-1)c + d$$
$$= (n-1)[(n-1)z + c] + d$$
$$= (n-1)R_1(x, z) + d$$
$$= \sigma(R_1(x, z))$$

$$R_2(\sigma(x), \sigma(z)) = R_2((n-1)x + d, (n-1)z + d)$$
$$= (n-1)[(n-1)x + d] + c$$
$$= (n-1)^2x + (n-1)d + c$$
$$= (n-1)^2x + (n-2)d + d + c$$
$$= (n-1)^2x + (n-2)c + d + c$$
$$= (n-1)^2x + (n-1)c + d$$
$$= (n-1)[(n-1)x + c] + d$$
$$= (n-1)R_2(x, z) + d$$
$$= \sigma(R_2(x, z))$$

Let $I = \sigma(R_2(\sigma(y), \sigma(x)))$.

$$I = \sigma(R_2((n-1)y + d, (n-1)x + d))$$
$$= (n-1)[(n-1)[(n-1)y + d] + c] + d$$
$$= (n-1)[(n-1)^2y + (n-1)d + c] + d$$
$$= (n-1)^3y + (n-1)^2d + (n-1)c + d$$
$$= -y + 2d + (n-1)c$$
$$= (n-1)y + 2d + c + (n-2)c$$
$$= (n-1)y + 2d + c + (n-2)d$$
$$= (n-1)y + c$$
$$= R_1(x, y)$$

Let $H = \sigma(R_1(\sigma(y), \sigma(x)))$.

$$H = \sigma(R_1((n-1)y + d, (n-1)x + d))$$

$$= (n-1)[(n-1)[(n-1)x+d]+c]+d$$
$$= (n-1)[(n-1)^2 x+(n-1)d+c]+d$$
$$= (n-1)^3 x+(n-1)^2 d+(n-1)c+d$$
$$= -x+2d+(n-1)c$$
$$= (n-1)x+2d+c+(n-2)c$$
$$= (n-1)x+2d+c+(n-2)d$$
$$= (n-1)x+c$$
$$= R_2(x,y)$$

2. If $\sigma(x) = x$, then

$$R_1(y, R_2(x,y)) = R_1(y, (n-1)x+c)$$
$$= (n-1)[(n-1)x+c]+c$$
$$= (n-1)^2 x+(n-1)c+c$$
$$= n^2 x-2nx+x+nc$$
$$= x$$

$$R_2(R_2(x,y), R_1(x,y)) = R_2((n-1)x+c, (n-1)y+c)$$
$$= (n-1)[(n-1)x+c]+c$$
$$= (n-1)^2 x+(n-1)c+c$$
$$= n^2 x-2nx+x+nc$$
$$= x$$

$$R_2(R_1(x,y), x) = R_2((n-1)y+c, x)$$
$$= (n-1)[(n-1)y+c]+c$$
$$= (n-1)^2 y+(n-1)c+c$$
$$= n^2 y-2ny+y+nc$$
$$= y$$

$$R_1(R_2(x,y), R_1(x,y)) = R_1((n-1)x+c, (n-1)y+c)$$
$$= (n-1)[(n-1)y+c]+c$$
$$= (n-1)^2 y+(n-1)c+c$$
$$= n^2 y-2ny+y+nc$$
$$= y$$

$$(R_2(y, R_2(x,y)), R_1(R_1(x,y), x)) = (R_2(y, (n-1)x+c), R_1((n-1)y+c, x))$$
$$= ((n-1)y+c, (n-1)x+c)$$
$$= R(x,y)$$

$$R_1(\sigma(x), \sigma(z)) = R_1(x,z)$$
$$= \sigma(R_1(x,z))$$

$$R_2(\sigma(x), \sigma(z)) = R_2(x,z)$$

$$= \sigma(R_2(x, z))$$

$$\sigma(R_2(\sigma(y), \sigma(x))) = \sigma(R_2(y, x))$$
$$= R_2(y, x)$$
$$= (n - 1)y + c$$
$$= R_1(x, y)$$

$$\sigma(R_1(\sigma(y), \sigma(x))) = \sigma(R_1(y, x))$$
$$= R_1(y, x)$$
$$= (n - 1)x + c$$
$$= R_2(x, y)$$

3. If $\sigma(x) = x + \dfrac{n}{2}$, where n is even, then

$$R_1(y, R_2(x, y)) = R_1(y, (n - 1)x + c)$$
$$= (n - 1)[(n - 1)x + c] + c$$
$$= (n - 1)^2 x + (n - 1)c + c$$
$$= n^2 x - 2nx + x + nc$$
$$= x$$

$$R_2(R_2(x, y), R_1(x, y)) = R_2((n - 1)x + c, (n - 1)y + c)$$
$$= (n - 1)[(n - 1)x + c] + c$$
$$= (n - 1)^2 x + (n - 1)c + c$$
$$= n^2 x - 2nx + x + nc$$
$$= x$$

$$R_2(R_1(x, y), x) = R_2((n - 1)y + c, x)$$
$$= (n - 1)[(n - 1)y + c] + c$$
$$= (n - 1)^2 y + (n - 1)c + c$$
$$= n^2 y - 2ny + y + nc$$
$$= y$$

$$R_1(R_2(x, y), R_1(x, y)) = R_1((n - 1)x + c, (n - 1)y + c)$$
$$= (n - 1)[(n - 1)y + c] + c$$
$$= (n - 1)^2 y + (n - 1)c + c$$
$$= n^2 y - 2ny + y + nc$$
$$= y$$

$$(R_2(y, R_2(x, y)), R_1(R_1(x, y), x)) = (R_2(y, (n - 1)x + c), R_1((n - 1)y + c, x))$$
$$= ((n - 1)y + c, (n - 1)x + c)$$
$$= R(x, y)$$

$$R_1(\sigma(x), \sigma(z)) = R_1(x + \frac{n}{2}, z + \frac{n}{2})$$

$$= (n-1)\left[z + \frac{n}{2}\right] + c$$

$$= (n-1)z + (n-1)\frac{n}{2} + c$$

$$= R_1(x,z) + \frac{n}{2}$$

$$= \sigma(R_1(x,z))$$

$$R_2(\sigma(x), \sigma(z)) = R_2\left(x + \frac{n}{2}, z + \frac{n}{2}\right)$$

$$= (n-1)\left[x + \frac{n}{2}\right] + c$$

$$= (n-1)x + (n-1)\frac{n}{2} + c$$

$$= R_2(x,z) + \frac{n}{2}$$

$$= \sigma(R_2(x,z))$$

$$\sigma(R_2(\sigma(y), \sigma(x))) = \sigma\left(R_2\left(y + \frac{n}{2}, x + \frac{n}{2}\right)\right)$$

$$= (n-1)\left[y + \frac{n}{2}\right] + c + \frac{n}{2}$$

$$= (n-1)y + (n-1)\frac{n}{2} + c + \frac{n}{2}$$

$$= (n-1)y + c$$

$$= R_1(x,y)$$

$$\sigma(R_1(\sigma(y), \sigma(x))) = \sigma\left(R_1\left(y + \frac{n}{2}, x + \frac{n}{2}\right)\right)$$

$$= (n-1)\left[x + \frac{n}{2}\right] + c + \frac{n}{2}$$

$$= (n-1)x + (n-1)\frac{n}{2} + c + \frac{n}{2}$$

$$= (n-1)x + c$$

$$= R_2(x,y)$$

So $(Z_n, \overline{*}, \underline{*}, R_1, R_2)$ is a singbikei.   □

Example 3.8. *As examples on the last theorem, we give*
  1. *let $X = Z_{10}$, with*

$$\sigma(x) = 9x + 3$$
$$R_1(x,y) = 9y + 8$$
$$R_2(x,y) = 9x + 8$$

  then $(Z_{10}, \overline{*}, \underline{*}, R_1, R_2)$ is a singbikei.
  2. *let $X = Z_8$, with*

$$\sigma(x) = x + 4$$
$$R_1(x,y) = 7y + 5$$
$$R_2(x,y) = 7x + 5$$

  then $(Z_8, \overline{*}, \underline{*}, R_1, R_2)$ is a singbikei.

Example 3.9. *Let $\Lambda = Z[t,s]/(t^2-1, s^2-1, (s-t)(1-s))$ be the quotient of the ring of two-variable polynomials with integer coefficients such that $s^2 = t^2 = 1$ by the ideal generated by (s-t)(1-s). Let $R_1, R_2 : X \times X \longrightarrow X$ be two maps. Then any Λ - module X with $x \underline{*} y = tx + (s-t)y$, and $x \overline{*} y = sx$, is a singbikei if R_1 and R_2 satisfy the following equations:*

$$x = R_1(y, R_2(x,y)) = R_2(R_2(x,y), R_1(x,y))$$
$$y = R_2(R_1(x,y), x) = R_1(R_2(x,y), R_1(x,y))$$
$$R(x,y) = (R_2(y, R_2(x,y)), R_1(R_1(x,y), x))$$
$$R_1(sx, sz) = sR_1(x,z)$$
$$R_2(sx, sz) = sR_2(x,z)$$
$$t(ty + (s-t)x) + (s-t)R_1(x,z) = t(ty + (s-t)z) + (s-t)R_2(x,z)$$
$$R_1(tx + (s-t)y, tz + (s-t)y) = tR_1(x,z) + (s-t)(sy)$$
$$R_2(tx + (s-t)y, tz + (s-t)y) = tR_2(x,z) + (s-t)(sy)$$
$$R_1(x,y) = tR_2(sy, tx+(s-t)y) + (s-t)R_1(sy, tx + (s-t)y)$$
$$R_2(x,y) = sR_1(sy, tx + (s-t)y)$$

We call such a singbikei an Alexander singbikeis. For example,

1. Z_{10} with $x \underline{*} y = 9x + 2y$ and $x \overline{*} y = x$, $R_1(x,y) = 8x + 3y$ and $R_2(x,y) = 7x + 4y$ satisfy all the equations in Example 3.9. So $(Z_{10}, \overline{*}, \underline{*}, R_1, R_2)$ is an Alexander singbikei.
2. Z_{13} with $x \underline{*} y = 12x + 2y$ and $x \overline{*} y = x$, $R_1(x,y) = 9x + 5y$ and $R_2(x,y) = 8x + 6y$ satisfy all the equations in Example 3.9. So $(Z_{13}, \overline{*}, \underline{*}, R_1, R_2)$ is an Alexander singbikei.

Example 3.10. *Let X = G be a group with $x \underline{*} y = yx^{-1}y$ and $x \overline{*} y = x$.*

Let $R_1, R_2 : X \times X \longrightarrow X$ be two maps, then $(X, \overline{*}, \underline{*}, R_1, R_2)$ is a singbikei if R_1 and R_2 satisfy the following equations:

$$x = R_1(y, R_2(x,y)) = R_2(R_2(x,y), R_1(x,y))$$
$$y = R_2(R_1(x,y), x) = R_1(R_2(x,y), R_1(x,y))$$
$$R(x,y) = (R_2(y, R_2(x,y)), R_1(R_1(x,y), x))$$
$$R_1(x,z)(x^{-1}yx^{-1})R_1(x,z) = R_2(x,z)(z^{-1}yz^{-1})R_2(x,z)$$
$$R_1(yx^{-1}y, yz^{-1}y) = yR_1^{-1}(x,z)y$$
$$R_2(yx^{-1}y, yz^{-1}y) = yR_2^{-1}(x,z)y$$
$$R_1(x,y) = R_1(y, yx^{-1}y)R_2^{-1}(y, yx^{-1}y)R_1(y, yx^{-1}y)$$
$$R_2(x,y) = R_1(y, yx^{-1}y)$$

Proposition 3.11. *The following maps*

$$R_1(x,y) = (xy^{-1})^n x$$
$$R_2(x,y) = (xy^{-1})^{n-1}x, \quad n \in N$$

with the condition

$$(yx^{-1})^{n^2+1} = 1$$

are solutions for the system in Example 3.10.

Proof. We show that R_1 and R_2 satisfy the equations in Example 3.10.

$$
\begin{aligned}
R_1(y, R_2(x, y)) &= R_1(y, (xy^{-1})^{n-1}x) \\
&= [yx^{-1}(yx^{-1})^{n-1}]^n y \\
&= (yx^{-1})^{n^2} y \\
&= xy^{-1}y \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
R_2(R_2(x, y), R_1(x, y)) &= R_2((xy^{-1})^{n-1}x, (xy^{-1})^n x) \\
&= [(xy^{-1})^{n-1}xx^{-1}(yx^{-1})^n]^{n-1}(xy^{-1})^{n-1}x \\
&= [yx^{-1}]^{n-1}(xy^{-1})^{n-1}x \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
R_2(R_1(x, y), x) &= R_2((xy^{-1})^n x, x) \\
&= [(xy^{-1})^n xx^{-1}]^{n-1}(xy^{-1})^n x \\
&= [xy^{-1}]^{n^2-n}(xy^{-1})^n x \\
&= [xy^{-1}]^{n^2} x \\
&= yx^{-1}x \\
&= y
\end{aligned}
$$

$$
\begin{aligned}
R_1(R_2(x, y), R_1(x, y)) &= R_1((xy^{-1})^{n-1}x, (xy^{-1})^n x) \\
&= [(xy^{-1})^{n-1}xx^{-1}(yx^{-1})^n]^n(xy^{-1})^{n-1}x \\
&= [yx^{-1}]^n(xy^{-1})^{n-1}x \\
&= yx^{-1}x \\
&= y
\end{aligned}
$$

$$
\begin{aligned}
(R_2(y, R_2(x, y)), R_1(R_1(x, y), x)) &= (R_2(y, (xy^{-1})^{n-1}x), R_1((xy^{-1})^n x, x)) \\
&= ([yx^{-1}(yx^{-1})^{n-1}]^{n-1}y, [(xy^{-1})^n xx^{-1}]^n(xy^{-1})^n x) \\
&= ([(yx^{-1})^n]^{n-1}y, [(xy^{-1})^n]^n(xy^{-1})^n x) \\
&= ((yx^{-1})^{n^2-n}y, (xy^{-1})^{n^2}(xy^{-1})^n x) \\
&= (xy^{-1}(xy^{-1})^n y, (xy^{-1})^{n^2+1}(xy^{-1})^{n-1}x) \\
&= ((xy^{-1})^{n+1}y, (xy^{-1})^{n-1}x) \\
&= ((xy^{-1})^n xy^{-1}y, (xy^{-1})^{n-1}x) \\
&= ((xy^{-1})^n x, (xy^{-1})^{n-1}x) \\
&= R(x, y)
\end{aligned}
$$

$$
\begin{aligned}
R_1(x, z)(x^{-1}yx^{-1})R_1(x, z) &= R_2(x, z)(z^{-1}yz^{-1})R_2(x, z) \\
(xz^{-1})^n x[x^{-1}yx^{-1}](xz^{-1})^n x &= (xz^{-1})^{n-1}x[z^{-1}yz^{-1}](xz^{-1})^{n-1}x \\
(xz^{-1})^n yx^{-1}(xz^{-1})^n x &= (xz^{-1})^n yz^{-1}(xz^{-1})^{n-1}x \\
&= (xz^{-1})^n yx^{-1}xz^{-1}(xz^{-1})^{n-1}x
\end{aligned}
$$

$$= (xz^{-1})^n yx^{-1}(xz^{-1})^n x$$

$$R_1(yx^{-1}y, yz^{-1}y) = yR_1^{-1}(x, z)y$$
$$(yx^{-1}yy^{-1}zy^{-1})^n yx^{-1}y = y[(xz^{-1})^n x]^{-1}y$$
$$y(x^{-1}z)^n x^{-1}y = y(xz^{-1})^n x^{-1}y$$

$$R_2(yx^{-1}y, yz^{-1}y) = yR_2^{-1}(x, z)y$$
$$(yx^{-1}yy^{-1}zy^{-1})^{n-1} yx^{-1}y = y[(xz^{-1})^{n-1}x]^{-1}y$$
$$yx^{-1}(zx^{-1})^{n-1}y = yx^{-1}(zx^{-1})^{n-1}y$$

$$R_1(x, y) = R_1(y, yx^{-1}y)R_2^{-1}(y, yx^{-1}y)R_1(y, yx^{-1}y)$$
$$(xy^{-1})^n x = (yy^{-1}xy^{-1})^n y[(yy^{-1}xy^{-1})^{n-1}y]^{-1}(yy^{-1}xy^{-1})^n y$$
$$= (xy^{-1})^n yy^{-1}(yx^{-1})^{n-1}(xy^{-1})^n y$$
$$= (xy^{-1})^n xy^{-1}y$$
$$= (xy^{-1})^n x$$

$$R_2(x, y) = R_1(y, yx^{-1}y)$$
$$(xy^{-1})^{n-1} x = (yy^{-1}xy^{-1})^n y$$
$$= (xy^{-1})^n y$$
$$= (xy^{-1})^{n-1}xy^{-1}y$$
$$= (xy^{-1})^{n-1}x$$

This completes the proof.   □

Remark 3.12. *From the proposition above, note that:*
(a) *If $n = 1$, then $R_1(x, y) = xy^{-1}x$ and $R_2(x, y) = x$ with the condition $(yx^{-1})^2 = 1$ is a solution for Example 3.10.*
(b) *If $n = 2$, then $R_1(x, y) = xy^{-1}xy^{-1}x$ and $R_2(x, y) = xy^{-1}x$ with the condition $(yx^{-1})^5 = 1$ is a solution for Example 3.10.*
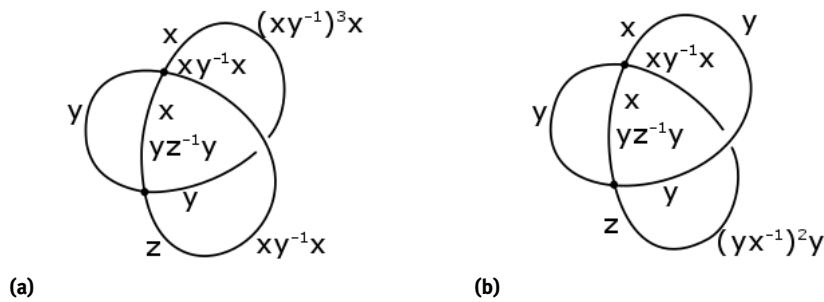
4 Applications

In this section we use singbikei and coloring invariants to distinguish singular knots and links.

Example 4.1. *Consider the two singular knots in the graph below.*
(a) Let $X = G$ be a group generated by $(yx^{-1})^2 = 1$ with $x \underline{*} y = yx^{-1}y$ and $x \overline{*} y = x$, $R_1(x, y) = xy^{-1}x$ and $R_2(x, y) = x$.

Fig. 12. The singular knots in Example 4.1 (a).

(a)   (b)

In Figure 12 (a), the relations at the crossings give

$$x = (xy^{-1})^3 x$$
$$x = yz^{-1}y$$
$$z = xy^{-1}x$$

Thus the set of colorings is $\{(x, y, z) \in G \times G \times G : x = y = z\}$.

In Figure 12 (b), the relations at the crossings give

$$x = y$$
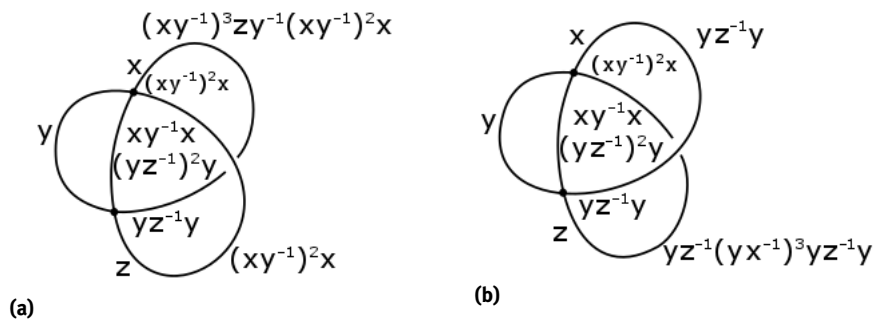$$x = yz^{-1}y$$
$$z = (yx^{-1})^2 y$$

Thus the set of colorings is $\{(x, y, z) \in G \times G \times G : x = y = z\}$.

The solution set is the same for both of the sets of colorings above. Therefore, this coloring invariant fails to distinguish these two singular knots.

(b) Let $X = G$ be a group generated by $(yx^{-1})^5 = 1$ with $x \underline{*} y = yx^{-1}y$ and $x \overline{*} y = x$, $R_1(x, y) = (xy^{-1})^2 x$ and $R_2(x, y) = xy^{-1}x$.

Fig. 13. The singular knots in Example 4.1 (b).

(a)   (b)

In Figure 13 (a), the relations at the crossings give

$$x = (xy^{-1})^3 zy^{-1}(xy^{-1})^2 x$$
$$(xy^{-1})^2 = (yz^{-1})^2$$
$$z = (xy^{-1})^2 x$$

Thus the set of colorings is $\{(x, y, z) \in G \times G \times G : 1 = (xy^{-1})^3, (xy^{-1})^2 = (yz^{-1})^2\}$.
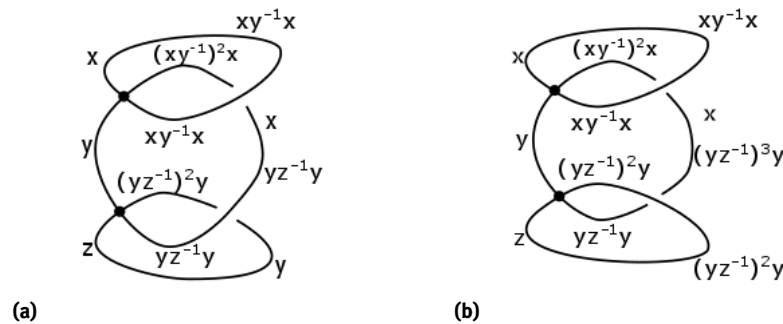
In Figure 13 (b), the relations at the crossings give

$$x = yz^{-1}y$$
$$(xy^{-1})^2 = (yz^{-1})^2$$
$$z = yz^{-1}(yx^{-1})^3 yz^{-1}y.$$

Thus the set of colorings is $\{(x, y, z) \in G \times G \times G : xy^{-1} = yz^{-1}\}$.

One can always choose a group G such that these two coloring sets are distinct.

Example 4.2. *Consider the two singular links in the graph below, let $X = G$ be a group generated by $(yx^{-1})^5 = 1$ with $x \underline{*} y = yx^{-1}y$ and $x \overline{*} y = x$, $R_1(x, y) = (xy^{-1})^2 x$ and $R_2(x, y) = xy^{-1}x$.*

Fig. 14. The singular links in Example 4.2.

(a)   (b)

In Figure 14 (a), the relations at the crossings give

$$x = xy^{-1}x$$
$$x = yz^{-1}y$$
$$z = y$$

Thus the set of colorings is $\{(x, y, z) \in G \times G \times G : x = y = z\}$.

In Figure 14 (b), the relations at the crossings give

$$x = xy^{-1}x$$
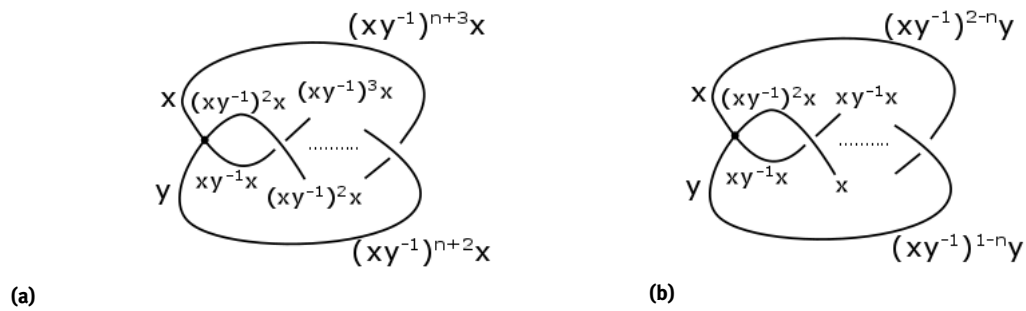$$x = (yz^{-1})^3 y$$
$$z = (yz^{-1})^2 y$$

Thus the set of colorings is $\{(x, y, z) \in G \times G \times G : x = y, 1 = (yz^{-1})^3\}$.

One can always choose a group G such that these two coloring sets are distinct.

Example 4.3. *Consider the two singular links in the graph below. Each of them has one singular crossing followed by $(n + 1)$ regular crossings, let $X = G$ be a group generated by $(yx^{-1})^5 = 1$ with $x \underline{*} y = yx^{-1}y$ and $x \overline{*} y = x$, $R_1(x, y) = (xy^{-1})^2 x$ and $R_2(x, y) = xy^{-1}x$.*

Fig. 15. The singular links in Example 4.3.

    (a)   **(b)**

In Figure 15 (a), the relations at the crossings give

$$x = (xy^{-1})^{n+3}x$$
$$y = (xy^{-1})^{n+2}x$$

Thus the set of colorings is $\{(x,y) \in G \times G : 1 = (xy^{-1})^{n+3}\}$.
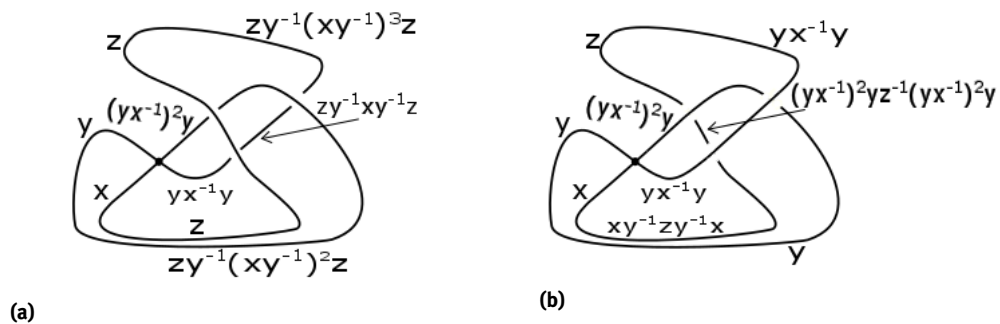
   In Figure 15 (b), the relations at the crossings give

$$x = (xy^{-1})^{2-n}y$$
$$y = (xy^{-1})^{1-n}y$$

Thus the set of colorings is $\{(x,y) \in G \times G : 1 = (xy^{-1})^{1-n}\}$.

   One can always choose a group G such that these two coloring sets are distinct.

Example 4.4. *Consider the two singular knots in the graph below, let $X = G$ be a group generated by $(yx^{-1})^5 = 1$ with $x \underline{*} y = yx^{-1}y$ and $x \overline{*} y = x$, $R_1(x, y) = (xy^{-1})^2 x$ and $R_2(x, y) = xy^{-1}x$.*

Fig. 16. The singular knots in Example 4.4.

    (a)   **(b)**

In Figure 16 (a), the relations at the crossings give

$$x = z$$
$$y = zy^{-1}(xy^{-1})^2 z$$
$$z = zy^{-1}(xy^{-1})^3 z$$

Thus the set of colorings is $\{(x,y,z) \in G \times G \times G : x = z, 1 = (xy^{-1})^4\}$.

In Figure 16 (b), the relations at the crossings give
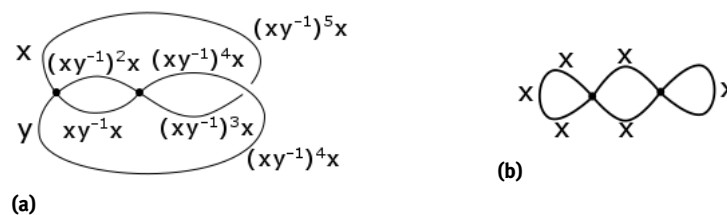
$$x = xy^{-1}zy^{-1}x$$
$$y = y$$
$$z = yx^{-1}y$$

Thus the set of colorings is $\{(x, y, z) \in G \times G \times G : z = yx^{-1}y\}$.

Therefore, this coloring invariant distinguishes these two singular knots.

Example 4.5. *Consider the two singular knots in the graph below, let $X = G$ be a group generated by $(yx^{-1})^5 = 1$ with $x \underline{*} y = yx^{-1}y$ and $x \overline{*} y = x$, $R_1(x, y) = (xy^{-1})^2 x$ and $R_2(x, y) = xy^{-1}x$.*

Fig. 17. The singular knots in Example 4.5.

(a)

(b)

In Figure 17 (a), the relations at the crossings give

$$x = (xy^{-1})^5 x$$
$$y = (xy^{-1})^4 x$$

Thus the set of colorings is $G \times G$.

In Figure 17 (b), the set of colorings is G. Therefore, this coloring invariant distinguishes these two singular knots.

References

[1]   Birman JS., New points of view in knot theory, *Bulletin of the American Mathematical Society*, 1993, 28, 253-87.

[2]   Birman JS., Lin XS., Knot polynomials and Vassiliev's invariants, *Inventiones mathematicae*, 1993, 111, 225-70.

[3]   Vassiliev VA., Cohomology of knot spaces, *Theory of singularities and its applications*, 1990, 1, 23-69.

[4]   Bataineh K., Elhamdadi M., Hajij M., The colored Jones polynomial of singular knots, *New York J. Math.*, 2016, 22, 1439-56.

[5]   Fiedler T., The Jones and Alexander polynomials for singular links, *Journal of Knot Theory and Its Ramifications*, 2010, 19, 859-66.

[6]   Kauffman LH., Invariants of graphs in three-space, *Transactions of the American Mathematical Society*, 1989, 311, 697-710.

[7]   Juyumaya J., Lambropoulou S., An invariant for singular knots, *Journal of Knot Theory and Its Ramifications*, 2009, 18, 825-40.

[8]   Kauffman LH., Vogel P., Link polynomials and a graphical calculus, *Journal of Knot Theory and Its Ramifications*, 1992, 1, 59-104.

[9]   Carter JS., Silver DS., Williams SG., Elhamdadi M., Saito M., Virtual knot invariants from group biquandles and their cocycles, *Journal of Knot Theory and Its Ramifications*, 2009, 18, 957-72.

[10]   Clark WE., Elhamdadi M., Saito M., Yeatman T., Quandle colorings of knots and applications, *Journal of knot theory and its ramifications*, 2014, 23, 1450035.

[11]   Elhamdadi M., Nelson S., Quandles, *American Mathematical Soc.*, 2015, 47.

[12]   Henrich A., Nelson S., Semiquandles and flat virtual knots, *Pacific journal of mathematics*, 2010, 248, 155-70.

[13]   Chien J., Nelson S., Virtual Links with Finite Medial Bikei, arXiv preprint arXiv:1704.00812, 2017.

[14]   Nelson S., Rivera P., Bikei invariants and gauss diagrams for virtual knotted surfaces, *Journal of Knot Theory and Its Ramifications*, 2016, 25, 1640008.

[15]   Joyce D., A classifying invariant of knots, the knot quandle, *Journal of Pure and Applied Algebra*, 1982, 23, 37-65.

[16]   Przytycki J., 3-coloring and other elementary invariants of knots, *Banach Center Publications*, 1998, 42, 275-95.

[17]   Wada M., Group invariants of links, *Topology*, 1992, 31, 399-406.

[18]   Churchill IR., Elhamdadi M., Hajij M., Nelson S., Singular knots and involutive quandles, *Journal of Knot Theory and Its Ramifications*, 2017, 26, 1750099.

[19]   Bataineh K., Elhamdadi M., Hajij M., Youmans W., Generating sets of Reidemeister moves of oriented singular links and quandles, arXiv preprint arXiv:1702.01150, 2017.