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A Geršgorin-type eigenvalue localization set with n parameters for stochastic matrices

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Abstract: A set in the complex plane which involves n parameters in $[0, 1]$ is given to localize all eigenvalues different from 1 for stochastic matrices. As an application of this set, an upper bound for the moduli of the subdominant eigenvalues of a stochastic matrix is obtained. Lastly, we fix n parameters in $[0, 1]$ to give a new set including all eigenvalues different from 1, which is tighter than those provided by Shen et al. (Linear Algebra Appl. 447 (2014) 74-87) and Li et al. (Linear and Multilinear Algebra 63(11) (2015) 2159-2170) for estimating the moduli of subdominant eigenvalues.

Keywords: Stochastic Matrix, Geršgorin set, Subdominant eigenvalue

MSC: 65F15, 15A18, 15A51

1 Introduction

Stochastic matrices and eigenvalue localization of stochastic matrices play key roles in many application fields, such as Computer Aided Geometric Design [1], Birth-Death Processes [2–5], and Markov chains [6]. An entrywise nonnegative matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called row stochastic (or simply stochastic) if all its row sums are 1,that is,

$$\sum_{j=1}^{n} a_{ij} = 1, \ for\ each\ i \in N = \{1, 2, \ldots, n\}.$$

Let us denote the ith deleted column sum of the moduli of off-diagonal entries of A by

$$C_i(A) = \sum_{j\neq i} |a_{ji}|.$$

Obviously, 1 is an eigenvalue of a stochastic matrix with a corresponding eigenvector $e = [1, 1, \ldots, 1]^T$. From the Perron-Frobenius Theorem [7], for any eigenvalue λ of A, that is, $\lambda \in \sigma(A)$, we have $|\lambda| \leq 1$ [8]. Here we call $|\lambda|$ a moduli of subdominant eigenvalue of a stochastic matrix A if $1 > |\lambda| > |\eta|$ for every eigenvalue η different from 1 and λ [8–10].

Since the subdominant eigenvalue of a stochastic matrix is crucial for bounding the convergence rate of stochastic processes [8, 11–14], it is interesting to give a set to localize all eigenvalues different from 1, or an upper bound for the moduli of its subdominant eigenvalue [8, 15].

One can use the well-known Geršgorin circle set [16] to localize all eigenvalues for a stochastic matrix. However, this set always includes the trival eigenvalue 1, and thus it is not always precise for capturing

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all eigenvalues different from 1 of a stochastic matrix. Therefore, several authors have tried to modify the Geršgorin circle set to localize more precisely all eigenvalues different from 1. In [8], Cvetković et al. gave the following set.

Theorem 1.1 ([8, Theorem 3.4]). *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. If $\lambda \in \sigma(A)\backslash\{1\}$, then*

$$\lambda \in \Gamma(A) = \{z \in \mathbb{C} : |z - \gamma(A)| < 1 - trace(A) + (n-1)\gamma(A)\},$$

where $\gamma(A) = \max_{i\in N}(a_{ii} - l_i(A))$, $l_i(A) = \min_{j\neq i} a_{ji}$ and $trace(A)$ is the trace of A.

However, the set provided by Theorem 1.1 is not effective in some cases, such as, for the class of stochastic matrices

$$SM_0 = \{A \in \mathbb{R}^{n\times n} : A\ is\ stochastic,\ and\ a_{ii} = l_i = 0,\ for\ each\ i \in N\},$$

for more details, see [15]. To overcome this drawback, Li and Li [15] provided another set as follows.

Theorem 1.2 ([15, Theorem 6]). *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. If $\lambda \in \sigma(A)\backslash\{1\}$, then*

$$\lambda \in \tilde{\Gamma}(A) = \{z \in \mathbb{C} : |z + \tilde{\gamma}(A)| < trace(A) + (n-1)\tilde{\gamma}(A) - 1\},$$

where $\tilde{\gamma}(A) = \max_{i\in N}(L_i(A) - a_{ii})$ and $L_i(A) = \max_{j\neq i} a_{ji}$.

Recently, by taking respectively

$$l_i(A) = \min_{j\neq i} a_{ji},\ v_i(A) = \max\left\{0, \frac{1}{2}\min_{\substack{k,m\neq i,\\ k\neq m}}\{a_{ki} + a_{mi}\}\right\} = \frac{1}{2}\min_{\substack{k,m\neq i,\\ k\neq m}}\{a_{ki} + a_{mi}\},$$

and

$$q_i(A) = \frac{1}{n-1}\sum_{j\neq i} a_{ji}$$

to modify the Geršgorin circle set, Shen et al. [12], and Li et al. [11] gave three sets to localize all eigenvalues different from 1.

Theorem 1.3 ([11, 12]). *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. If $\lambda \in \sigma(A)\backslash\{1\}$, then*

$$\lambda \in \Gamma^{stol}(A) = \bigcup_{i\in N}\left(\Gamma_i^{stol}(A) = \{z \in \mathbb{C} : |a_{ii} - z - l_i(A)| < Cl_i(A)\}\right),$$

$$\lambda \in \Gamma^{stov}(A) = \bigcup_{i\in N}\left(\Gamma_i^{stov}(A) = \{z \in \mathbb{C} : |a_{ii} - z - v_i(A)| < Cv_i(A)\}\right)$$

and

$$\lambda \in \Gamma^{stoq}(A) = \bigcup_{i\in N}\left(\Gamma_i^{stoq}(A) = \{z \in \mathbb{C} : |a_{ii} - z - q_i(A)| < Cq_i(A)\}\right),$$

where

$$Cl_i(A) = \sum_{j\neq i}|a_{ji} - l_i(A)| = \sum_{j\neq i} a_{ji} - \sum_{j\neq i} l_i(A) = C_i(A) - (n-1)l_i(A),$$

$$Cv_i(A) = \sum_{j\neq i}|a_{ji} - v_i(A)| = C_i(A) - (n-3)v_i(A) - 2l_i(A)$$

and $Cq_i(A) = \sum_{j\neq i}|a_{ji} - q_i(A)|$.

Remark here that Shen et al. [12] used these three sets to localize any real eigenvalue different from 1, which are generalized to localize all eigenvalues different from 1 by Li et al. [11].

Also in [11], Li et al. provided another two modifications of the Geršgorin circle set by taking respectively

$$L_i(A) = \max_{j\neq i} a_{ji},\ and\ V_i(A) = \frac{1}{2}\max_{\substack{k,m\neq i,\\ k\neq m}}\{a_{ki} + a_{mi}\}.$$

Theorem 1.4 ([11, Theorems 3.3 and 3.8]). *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. If $\lambda \in \sigma(A)\backslash\{1\}$, then*

$$\lambda \in \Gamma^{stoL}(A) = \bigcup_{i\in N}\left(\Gamma_i^{stoL}(A) = \{z \in \mathbb{C} : |L_i(A) - a_{ii} + z| < CL_i(A)\}\right)$$

and

$$\lambda \in \Gamma^{stoV}(A) = \bigcup_{i\in N}\left(\Gamma_i^{stoV}(A) = \{z \in \mathbb{C} : |V_i(A) - a_{ii} + z| < CV_i(A)\}\right),$$

where

$$CL_i(A) = \sum_{j\neq i}|L_i(A) - a_{ji}| = (n-1)L_i(A) - C_i(A)$$

and

$$CV_i(A) = \sum_{j\neq i}|V_i(A) - a_{ji}| = (n-3)V_i(A) + 2L_i(A) - C_i(A).$$

Note that $l_i(A)$, $v_i(A)$, $q_i(A)$, $V_i(A)$ and $L_i(A)$ are all in the interval $[\min_{j\neq i} a_{ji}, \max_{j\neq i} a_{ji}]$. So it is natural to ask whether or not there is an optimal value in $[\min_{j\neq i} a_{ji}, \max_{j\neq i} a_{ji}]$ such that the set, which is obtained by using this value to modify the Geršgorin circle set, captures all eigenvalues different from 1 of a stochastic matrix most precisely. To answer this question, we give a set in Section 2 with n parameters in $[0, 1]$ to localize all eigenvalues different from 1 for a stochastic matrix, and show that this set would reduce to $\Gamma^{stol}(A)$, $\Gamma^{stov}(A)$, $\Gamma^{stoq}(A)$, $\Gamma^{stoV}(A)$ and $\Gamma^{stoL}(A)$ by taking some fixed parameters. And we use this set in Section 3 to give an upper bound for the moduli of its subdominant eigenvalue for a stochastic matrix. In section 4, by choosing special values of these n parameters in $[0, 1]$ for the upper bound obtained in Section 3, we give a new set including all eigenvalues different from 1, which is better than $\Gamma^{stol}(A)$ and $\Gamma^{stoL}(A)$ in the sense of estimating the moduli of subdominant eigenvalues.

2 A Geršgorin-type eigenvalue localization set with *n* parameters

We first begin with an important lemma, which is used to give some modifications of the Geršgorin circle set.

Lemma 2.1 ([8, 11, 12]). *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. For any $d = [d_1, d_2, \ldots, d_n]^T \in \mathbb{R}^n$, if $\mu \in \sigma(A)\backslash\{1\}$, then $-\mu$ is an eigenvalue of the matrix*

$$B = ed^T - A.$$

Lemma 2.1 shows that once an eigenvalue localization set for $B = ed^T - A$ is given, we can get a set to localize all eigenvalues different from 1 for the stochastic matrix A [11]. Now we present the following choice of d:

$$d = L^{\alpha_i}(A), \tag{1}$$

where $L^{\alpha_i}(A) = [L_1^{\alpha_1}(A), L_2^{\alpha_2}(A), \ldots, L_n^{\alpha_n}(A)]^T$, $\alpha_i \in [0, 1]$ for $i \in N$ and

$$L_i^{\alpha_i}(A) = \alpha_i L_i(A) + (1-\alpha_i)l_i(A) = \alpha_i \max_{j\neq i, j\in N} a_{ji} + (1-\alpha_i)\min_{j\neq i, j\in N} a_{ji}, \; i \in N.$$

By Lemma 2.1 and (1), we can obtain the following set to localize all eigenvalues different from 1 of a stochastic matrix.

Theorem 2.2. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. If $\lambda \in \sigma(A)\backslash\{1\}$, then for any $\alpha_i \in [0, 1]$, $i \in N$,*

$$\lambda \in \Gamma^{stoL^{\alpha}}(A) = \bigcup_{i\in N}\Gamma_i^{stoL^{\alpha_i}}(A),$$

where

$$\Gamma_i^{stoL^{\alpha_i}}(A) = \{z \in \mathbb{C} : |\alpha_i L_i(A) + (1-\alpha_i)l_i(A) - a_{ii} + z| \leq CL_i^{\alpha_i}(A)\}$$

and

$$CL_i^{\alpha_i}(A) = \sum_{j\neq i} |L_i^{\alpha_i}(A) - a_{ji}| = \sum_{j\neq i} |\alpha_i L_i(A) + (1-\alpha_i) l_i(A) - a_{ji}|. \tag{2}$$

Proof. Let $B^{\alpha_i} = ed^T - A = [b_{ij}^{\alpha_i}]$, where $d = L^{\alpha_i}(A) = [L_1^{\alpha_1}(A), L_2^{\alpha_2}(A), \ldots, L_n^{\alpha_n}(A)]^T$. By applying the Geršgorin circle theorem to B^{α_i}, we have that for any $\hat{\lambda} \in \sigma(B^{\alpha_i})$,

$$\hat{\lambda} \in \bigcup_{i\in N} \{z \in \mathbb{C} : |b_{ii}^{\alpha_i} - z| \leq C_i(B^{\alpha_i})\}.$$

By Lemma 2.1, we have for $\lambda \in \sigma(A)\backslash\{1\}$, then $-\lambda \in \sigma(B^{\alpha_i})$, that is,

$$-\lambda \in \bigcup_{i\in N} \{z \in \mathbb{C} : |b_{ii}^{\alpha_i} - z| \leq C_i(B^{\alpha_i})\}.$$

Furthermore, note that for any $i \in N$,

$$b_{ii}^{\alpha_i} = L_i^{\alpha_i}(A) - a_{ii} = \alpha_i L_i(A) + (1-\alpha_i) l_i(A) - a_{ii}$$

and

$$C_i(B^{\alpha_i}) = \sum_{j\neq i} |L_i^{\alpha_i}(A) - a_{ji}| = CL_i^{\alpha_i}(A).$$

Hence,

$$\lambda \in \Gamma^{stoL^{\alpha}}(A) = \bigcup_{i\in N} \Gamma_i^{stoL^{\alpha_i}}(A),$$

where $\Gamma_i^{stoL^{\alpha_i}}(A) = \{z \in \mathbb{C} : |\alpha_i L_i(A) + (1-\alpha_i) l_i(A) - a_{ii} + z| \leq CL_i^{\alpha_i}(A)\}$. □

Example 2.3. *Consider the first 50 stochastic matrices generated by the MATLAB code*

$$k = 10; A = rand(k,k); A = inv(diag(sum(A'))) * A,$$

and take $\alpha_i \in [0,1]$ for $i = 1, 2, \ldots, 10$ by the MATLAB code

$$alpha = rand(1,k).$$

By drawing the sets $\Gamma^{stoL^{\alpha}}(A)$ in Theorem 2.2 and

$$\Gamma = \left(\Gamma(A) \bigcap \tilde{\Gamma}(A)\right)$$

in Theorems 1.1 and 1.2, it is not difficult to see that the number of $\Gamma^{stoL^{\alpha}}(A) \subset \Gamma$ is 46, that if $1 \notin \Gamma$, then $1 \notin \Gamma^{stoL^{\alpha}}(A)$, and that if $1 \in \Gamma$, then $\Gamma^{stoL^{\alpha}}(A)$ may not contain the trivial eigenvalue 1 (also see Table 1). So, by these examples, we conclude that the set in Theorem 2.2 captures all eigenvalues different from 1 of a stochastic matrix more precisely than the sets in Theorem 1.1 and Theorem 1.2 in some cases.

Table 1. Comparisons of $\Gamma^{stoL^{\alpha}}(A)$ and $\Gamma = \left(\Gamma(A) \cap \tilde{\Gamma}(A)\right)$

	$1 \in \Gamma^{stoL^{\alpha}}(A)$	$1 \notin \Gamma$	$\Gamma^{stoL^{\alpha}}(A) \nsubseteq \Gamma$, $\Gamma \nsubseteq \Gamma^{stoL^{\alpha}}(A)$	$\Gamma^{stoL^{\alpha}}(A) \subset \Gamma$
Number	**8**	**2**	**4**	**46**
The i-th happens	**3,4,7,14, 38,39,40,42**	**25, 31**	**3, 7, 11, 35**	**otherwise**

Remark 2.4. *(I) When $\alpha_i = 1$ for each $i \in N$, then $L_i^{\alpha_i}(A) = L_i(A)$ and $CL_i^{\alpha_i}(A) = CL_i(A)$ for any $i \in N$, which implies $\Gamma^{stoL^{\alpha}}(A)$ reduces to $\Gamma^{stoL}(A)$ in Theorem 1.4;*

(II) When $\alpha_i = \frac{V_i(A)-l_i(A)}{L_i(A)-l_i(A)} \in [0,1]$ *and* $L_i(A) > l_i(A)$ *for each* $i \in N$*, then* $L_i^{\alpha_i}(A) = V_i(A)$ *and* $CL_i^{\alpha_i}(A) = CV_i(A)$ *for any* $i \in N$*. On the other hand, if for some* $i \in N$*,* $L_i(A) = l_i(A)$*, then for any* $\alpha_i \in [0,1]$ *we also have* $L_i^{\alpha_i}(A) = V_i(A)$ *and* $CL_i^{\alpha_i}(A) = CV_i(A)$*. These imply* $\Gamma^{stoL^\alpha}(A)$ *reduces to* $\Gamma^{stoV}(A)$ *in Theorem 1.4;*

(III) When $\alpha_i = \frac{q_i(A)-l_i(A)}{L_i(A)-l_i(A)} \in [0,1]$ *and* $L_i(A) > l_i(A)$ *for each* $i \in N$*, then* $L_i^{\alpha_i}(A) = q_i(A)$ *and* $CL_i^{\alpha_i}(A) = Cq_i(A)$ *for any* $i \in N$*. On the other hand, if for some* $i \in N$*,* $L_i(A) = l_i(A)$*, then for any* $\alpha_i \in [0,1]$ *we also have* $L_i^{\alpha_i}(A) = q_i(A)$ *and* $CL_i^{\alpha_i}(A) = Cq_i(A)$*. These imply* $\Gamma^{stoL^\alpha}(A)$ *reduces to* $\Gamma^{stoq}(A)$ *in Theorem 1.3;*

(IV) When $\alpha_i = \frac{v_i(A)-l_i(A)}{L_i(A)-l_i(A)} \in [0,1]$ *and* $L_i(A) > l_i(A)$ *for each* $i \in N$*, then* $L_i^{\alpha_i}(A) = v_i(A)$ *and* $CL_i^{\alpha_i}(A) = Cv_i(A)$ *for any* $i \in N$*. On the other hand, if for some* $i \in N$*,* $L_i(A) = l_i(A)$*, then for any* $\alpha_i \in [0,1]$ *we also have* $L_i^{\alpha_i}(A) = v_i(A)$ *and* $CL_i^{\alpha_i}(A) = Cv_i(A)$*. These imply* $\Gamma^{stoL^\alpha}(A)$ *reduces to* $\Gamma^{stov}(A)$ *in Theorem 1.3;*

(V) When $\alpha_i = 0$ *for each* $i \in N$*, then* $L_i^{\alpha_i}(A) = l_i(A)$ *and* $CL_i^{\alpha_i}(A) = Cl_i(A)$ *for any* $i \in N$*, which implies* $\Gamma^{stoL^\alpha}(A)$ *reduces to* $\Gamma^{stol}(A)$ *in Theorem 1.3.*

Hence, we say that the set $\Gamma^{stoL^\alpha}(A)$ is a generalization of $\Gamma^{stol}(A)$, $\Gamma^{stov}(A)$ and $\Gamma^{stoq}(A)$ in Theorem 1.3, and $\Gamma^{stoV}(A)$ and $\Gamma^{stoL}(A)$ in Theorem 1.4. Moreover, according to $\alpha_i \in [0,1]$ in Theorem 2.2, we can get the following result easily.

Remark 2.5. *Let* $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ *be a stochastic matrix. If* $\lambda \in \sigma(A)\backslash\{1\}$*, then*

$$\lambda \in \Gamma^{[0,1]}(A) = \bigcap_{\alpha\in[0,1]} \Gamma^{stoL^\alpha}(A).$$

Furthermore, $\Gamma^{[0,1]}(A) \subseteq \left(\Gamma^{stoL}(A) \bigcap \Gamma^{stoV}(A) \bigcap \Gamma^{stoq}(A) \bigcap \Gamma^{stov}(A) \bigcap \Gamma^{stol}(A)\right)$*.*

The set $\Gamma^{[0,1]}(A)$ in Remark 2.5 is not of much practical use because it involves some parameters α_i. In fact, we can take some special α_i in practice, which is illustrated by the following example.

Example 2.6. *Consider the third stochastic matrix A in Example 2.3. By Table 1, we have that*

$$1 \in \Gamma^{stoL^\alpha}(A),\ \Gamma^{stoL^\alpha}(A) \nsubseteq \Gamma,\ \text{and}\ \Gamma \nsubseteq \Gamma^{stoL^\alpha}(A),$$

which is shown in Figure 1, where $\Gamma^{stoL^\alpha}(A)$ *is drawn slightly thicker than* Γ*. Furthermore, we take the first* 3 *vectors*

$$\alpha^{(j)} = [\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_{10}^{(j)}],\ j = 1,2,3$$

generated by the MATLAB code $alpha = rand(1,10)$*, that is,*

$$\alpha^{(1)} = [0.8147, 0.9058, 0.1270, 0.9134, 0.6324, 0.0975, 0.2785, 0.5469, 0.9575, 0.9649],$$

$$\alpha^{(2)} = [0.1576, 0.9706, 0.9572, 0.4854, 0.8003, 0.1419, 0.4218, 0.9157, 0.7922, 0.9595],$$

and

$$\alpha^{(3)} = [0.6557, 0.0357, 0.8491, 0.9340, 0.6787, 0.7577, 0.7431, 0.3922, 0.6555, 0.1712].$$

By Remark 2.5, we have that for any $\lambda \in \sigma(A)\backslash\{1\}$*,*

$$\lambda \in \left(\Gamma^{stoL^{\alpha^{(1)}}}(A) \bigcap \Gamma^{stoL^{\alpha^{(2)}}}(A) \bigcap \Gamma^{stoL^{\alpha^{(3)}}}(A)\right).$$

We draw this set in the complex plane, see Figure 2. It is easy to see

$$1 \notin \left(\Gamma^{stoL^{\alpha^{(1)}}}(A) \bigcap \Gamma^{stoL^{\alpha^{(2)}}}(A) \bigcap \Gamma^{stoL^{\alpha^{(3)}}}(A)\right)$$

and

$$\left(\Gamma^{stoL^{\alpha^{(1)}}}(A) \bigcap \Gamma^{stoL^{\alpha^{(2)}}}(A) \bigcap \Gamma^{stoL^{\alpha^{(3)}}}(A)\right) \subset \Gamma.$$

Fig. 1. $\Gamma^{stoL^{\alpha}}(A) \nsubseteq \Gamma$, *and* $\Gamma \nsubseteq \Gamma^{stoL^{\alpha}}(A)$

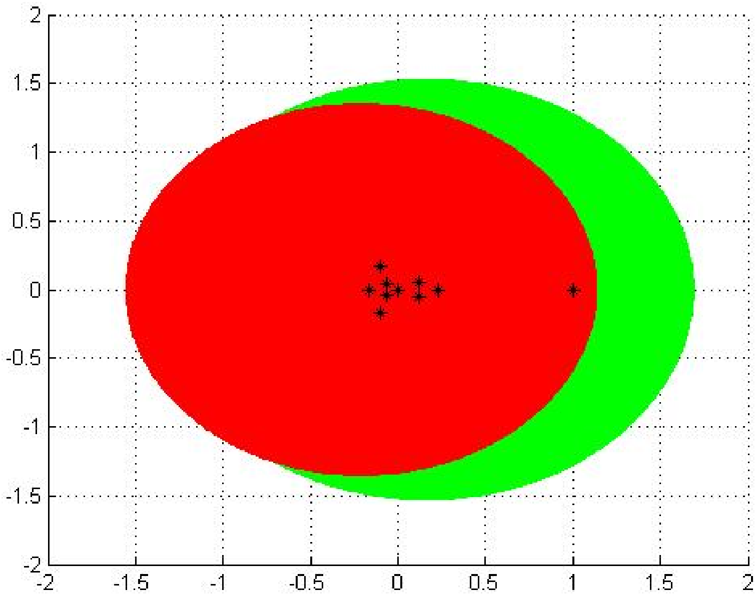
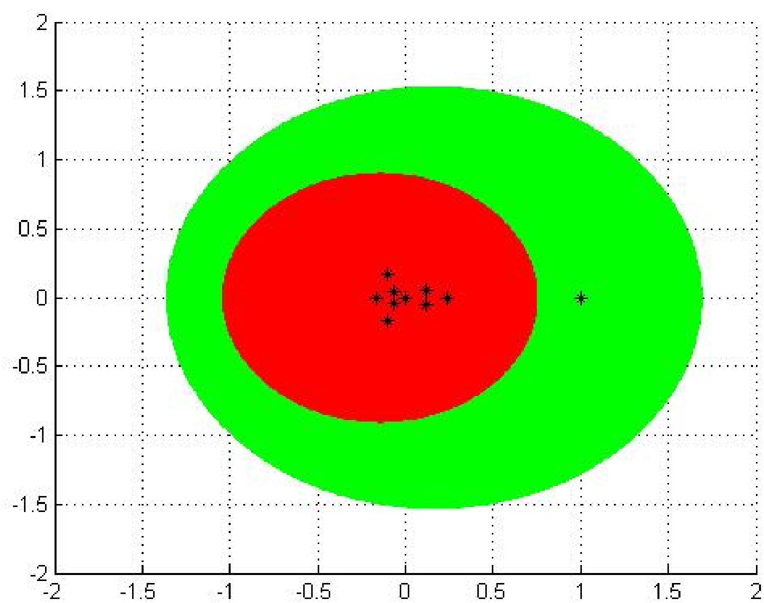


Fig. 2. $\left(\Gamma^{stoL^{\alpha^{(1)}}}(A) \bigcap \Gamma^{stoL^{\alpha^{(2)}}}(A) \bigcap \Gamma^{stoL^{\alpha^{(3)}}}(A)\right) \subset \Gamma$



This example shows that we can take some special α_i to get a set which is tighter than the sets in Theorems 1.1 and 1.2.

It is well-known that an eigenvalue inclusion set leads to a sufficient condition for nonsingular matrices, and vice versa [12, 16]. Hence, from Theorem 2.2 or Remark 2.5, we can get a nonsingular condition for stochastic matrices.

Proposition 2.7. *Let* $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ *be a stochastic matrix. If for some* $\bar{\alpha}_i \in [0, 1]$, $i \in N$,

$$|\bar{\alpha}_i L_i(A) + (1 - \bar{\alpha}_i) l_i(A) - a_{ii}| > CL_i^{\bar{\alpha}_i}(A), \; i \in N, \tag{3}$$

where $CL_i^{\bar{\alpha}_i}$ *is defined as (2), then* A *is nonsingular.*

Proof. Suppose that A is singular, that is, $0 \in \sigma(A)$. From Theorem 2.2, we have that for any $\alpha_i \in [0,1]$, $i \in N$,

$$0 \in \Gamma^{stoL^{\alpha}}(A) = \bigcup_{i\in N} \Gamma_i^{stoL^{\alpha_i}}(A).$$

In particular,

$$0 \in \Gamma^{stoL^{\bar{\alpha}}}(A) = \bigcup_{i\in N} \Gamma_i^{stoL^{\bar{\alpha}_i}}(A).$$

Hence, there is an index $i_0 \in N$ such that

$$|\bar{\alpha}_{i_0} L_{i_0}(A) + (1-\bar{\alpha}_{i_0}) l_{i_0}(A) - a_{i_0 i_0}| \le CL_{i_0}^{\bar{\alpha}_i}(A).$$

This contradicts (3). The conclusion follows. □

3 An upper bound for the moduli of subdominant eigenvalues

By using the set $\Gamma^{stoL^{\alpha}}(A)$ in Theorem 2.2, we can give a bound to estimate the moduli of subdominant eigenvalues of a stochastic matrix.

Theorem 3.1. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. If $\lambda \in \sigma(A)\backslash\{1\}$, then*

$$|\lambda| \le \rho^{[0,1]}, \tag{4}$$

where $\rho^{[0,1]} = \max_{i\in N} \min_{\alpha_i\in[0,1]} \left\{ \sum_{j=1}^{n} |\alpha_i L_i(A) + (1-\alpha_i) l_i(A) - a_{ji}| \right\}$.

Proof. Let

$$\begin{aligned} f_i(\alpha_i) &= \sum_{j=1}^{n} |\alpha_i L_i(A) + (1-\alpha_i) l_i(A) - a_{ji}| \\ &= CL_i^{\alpha_i}(A) + |\alpha_i L_i(A) + (1-\alpha_i) l_i(A) - a_{ii}|, \alpha_i \in [0,1], i \in N, \end{aligned}$$

where

$$CL_i^{\alpha_i}(A) = \sum_{j\ne i} |\alpha_i L_i(A) + (1-\alpha_i) l_i(A) - a_{ji}|.$$

Therefore, each $f_i(\alpha_i)$, $i \in N$ is a continuous function of $\alpha_i \in [0,1]$, and there are $\tilde{\alpha}_i \in [0,1]$, $i \in N$ such that

$$f_i(\tilde{\alpha}_i) = \min_{\alpha_i\in[0,1]} \left\{ CL_i^{\alpha_i}(A) + |\alpha_i L_i(A) + (1-\alpha_i) l_i(A) - a_{ii}| \right\}, i \in N. \tag{5}$$

For these $\tilde{\alpha}_i \in [0,1]$, $i \in N$, by Theorem 2.2 we have

$$\lambda \in \Gamma^{stoL^{\tilde{\alpha}}}(A) = \bigcup_{i\in N} \Gamma_i^{stoL^{\tilde{\alpha}_i}}(A).$$

Hence, there is an index $i_0 \in N$ such that

$$|\tilde{\alpha}_{i_0} L_{i_0}(A) + (1-\tilde{\alpha}_{i_0}) l_{i_0}(A) - a_{i_0 i_0} + \lambda| \le CL_{i_0}^{\tilde{\alpha}_{i_0}}(A)\},$$

which gives

$$|\lambda| \le CL_{i_0}^{\tilde{\alpha}_{i_0}}(A) + |\tilde{\alpha}_{i_0} L_{i_0}(A) + (1-\tilde{\alpha}_{i_0}) l_{i_0}(A) - a_{i_0 i_0}|. \tag{6}$$

By (5) we have

$$|\lambda| \le \min_{\alpha_{i_0}\in[0,1]} \left\{ CL_{i_0}^{\alpha_{i_0}}(A) + |\alpha_{i_0} L_{i_0}(A) + (1-\alpha_{i_0}) l_{i_0}(A) - a_{i_0 i_0}| \right\},$$

which implies

$$\begin{aligned}|\lambda| &\le \max_{i\in N}\min_{\alpha_i\in[0,1]}\left\{CL_i^{\alpha_i}(A)+|\alpha_i L_i(A)+(1-\alpha_i)l_i(A)-a_{ii}|\right\}\\ &=\max_{i\in N}\min_{\alpha_i\in[0,1]}\left\{\sum_{j=1}^{n}|\alpha_i L_i(A)+(1-\alpha_i)l_i(A)-a_{ji}|\right\}\\ &=\rho^{[0,1]}.\end{aligned}$$

The conclusion follows. □

As in the proof of Theorem 3.1, we can give another bound to estimate the moduli of subdominant eigenvalues by using the sets $\Gamma^{stol}(A)$, $\Gamma^{stov}(A)$ and $\Gamma^{stoq}(A)$in Theorem 1.3, $\Gamma^{stoL}(A)$ and $\Gamma^{stoV}(A)$ in Theorem 1.4, respectively.

Theorem 3.2. *Let $A=[a_{ij}]\in\mathbb{R}^{n\times n}$ be a stochastic matrix. If $\lambda\in\sigma(A)\backslash\{1\}$, then*

$$|\lambda|\le\min\{\rho_L,\rho_V,\rho_q,\rho_v,\rho_l\}, \tag{7}$$

where

$$\rho_L=\max_{i\in N}\{a_{ii}+nL_i(A)-C_i(A)\},$$
$$\rho_V=\max_{i\in N}\{a_{ii}+(n-2)V_i(A)+2L_i(A)-C_i(A)\},$$
$$\rho_q=\max_{i\in N}\left\{\sum_{j=1}^{n}|a_{ji}-q_i(A)|\right\},$$
$$\rho_v=\max_{i\in N}\{a_{ii}-(n-4)v_i(A)-2l_i(A)+C_i(A)\}$$

and

$$\rho_l=\max_{i\in N}\{a_{ii}-(n-2)l_i(A)+C_i(A)\}.$$

Proof. We first prove $|\lambda|\le\rho_L$. From Theorem 1.4,

$$\lambda\in\Gamma^{stoL}(A)=\bigcup_{i\in N}\Gamma_i^{stoL}(A).$$

As in the proof of Theorem 3.1, we have that there is an index $i_0\in N$ such that

$$|L_{i_0}(A)-a_{i_0i_0}+\lambda|\le CL_{i_0}(A),$$

and

$$\begin{aligned}|\lambda| &\le |a_{i_0i_0}-L_{i_0}(A)|+CL_{i_0}(A)\\ &\le a_{i_0i_0}+L_{i_0}(A)+(n-1)L_{i_0}(A)-C_{i_0}(A)\\ &= a_{i_0i_0}+nL_{i_0}(A)-C_{i_0}(A)\\ &\le \max_{i\in N}\{a_{ii}+nL_i(A)-C_i(A)\},\end{aligned}$$

i.e., $|\lambda|\le\rho_L$. Similarly, by

$$\lambda\in\Gamma^{stoV}(A),\lambda\in\Gamma^{stoq}(A),\lambda\in\Gamma^{stov}(A),\ and\ \lambda\in\Gamma^{stol}(A),$$

we can get respectively

$$|\lambda|\le\rho_V,|\lambda|\le\rho_q,|\lambda|\le\rho_v,\ and\ |\lambda|\le\rho_l.$$

The conclusion follows. □

By the choices of α_i in Remark 2.4, it is easy to get the relationships between $\rho^{[0,1]}$, $\rho_L,\rho_V,\rho_q,\rho_v$ and ρ_l as follows.

Theorem 3.3. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. Then*

$$\rho^{[0,1]} \le \min\{\rho_L, \rho_V, \rho_q, \rho_v, \rho_l\},$$

where $\rho^{[0,1]}$, ρ_L, ρ_V, ρ_q, ρ_v, and ρ_l are defined in Theorem 3.1 and Theorem 3.2, respectively.

As in the proof of Theorem 3.2, by Theorems 1.1 and 1.2 two upper bounds for the subdominant eigenvalue of a stochastic matrix are obtained easily.

Proposition 3.4. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix and $\lambda \in \sigma(A)\backslash\{1\}$ be its subdominant eigenvalue. Then*

$$|\lambda| \le 1 - trace(A) + n\gamma(A), \text{ and } |\lambda| \le trace(A) + n\tilde{\gamma}(A) - 1,$$

consequently,

$$|\lambda| \le \min\{1 - trace(A) + n\gamma(A),\ trace(A) + n\tilde{\gamma}(A) - 1\}. \tag{8}$$

For the comparison of $\rho^{[0,1]}$ and the upper bound

$$\varLambda := \min\{1 - trace(A) + n\gamma(A),\ trace(A) + n\tilde{\gamma}(A) - 1\}$$

in (8), we conclude here that by taking some special α_i and the fact that $\varLambda$ is given by Theorems 1.1 and 1.2, an upper bound can be obtained, which is better than

$$\min\{1 - trace(A) + n\gamma(A),\ trace(A) + n\tilde{\gamma}(A) - 1\}.$$

4 Special choices of α_i for the set $\varGamma^{stoL^\alpha}(A)$

In this section, we choose α_i for the set $\varGamma^{stoL^\alpha}(A)$ to give a set, which is tighter than the sets $\varGamma^{stol}(A)$ and $\varGamma^{stoL}(A)$ by determining the optimal value of α_i for estimating the moduli of subdominant eigenvalues of a stochastic matrix.

For a given stochastic matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, let

$$N^+(A) = \{i \in N : \varDelta_i(A) \ge 0\}$$

and

$$N^-(A) = \{i \in N : \varDelta_i(A) < 0\},$$

where $\varDelta_i(A) = nL_i(A) + (n-2)l_i(A) - 2C_i(A)$. Obviously, $N = N^+(A) \cup N^-(A)$.

Proposition 4.1. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. If $\lambda \in \sigma(A)\backslash\{1\}$, then*

$$|\lambda| \le \rho^{0,1}, \tag{9}$$

where

$$\rho^{0,1} = \max\left\{\max_{i\in N^+(A)}\{a_{ii} - (n-2)l_i(A) + C_i(A)\},\ \max_{i\in N^-(A)}\{a_{ii} + nL_i(A) - C_i(A)\}\right\}.$$

Proof. Note that

$$\begin{aligned} &CL_i^{\alpha_i}(A) + |\alpha_i L_i(A) + (1-\alpha_i)l_i(A) - a_{ii}| \\ &= \sum_{j\ne i}|\alpha_i L_i(A) + (1-\alpha_i)l_i(A) - (\alpha_i a_{ji} + (1-\alpha_i)a_{ji})| \\ &\quad + |\alpha_i L_i(A) + (1-\alpha_i)l_i(A) - a_{ii}| \\ &\le \alpha_i \sum_{j\ne i}|L_i(A) - a_{ji}| + (1-\alpha_i)\sum_{j\ne i}|l_i(A) - a_{ji}| \end{aligned}$$

$$
\begin{aligned}
&+\alpha_i L_i(A) + (1-\alpha_i) l_i(A) + a_{ii} \\
&= (n-1)\left(\alpha_i L_i(A) - (1-\alpha_i) l_i(A)\right) + (1-2\alpha_i) C_i(A) \\
&\quad +\alpha_i L_i(A) + (1-\alpha_i) l_i(A) + a_{ii} \\
&= a_{ii} - (n-2) l_i(A) + C_i(A) + \alpha_i \left(n L_i(A) + (n-2) l_i(A) - 2C_i(A)\right) \\
&= a_{ii} - (n-2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A).
\end{aligned}
$$

Hence, from Theorem 3.1, we have

$$
\begin{aligned}
|\lambda| &\le \max_{i\in N} \min_{\alpha_i \in [0,1]} \left\{ \sum_{j=1}^{n} |\alpha_i L_i(A) + (1-\alpha_i) l_i(A) - a_{ji}| \right\} \\
&= \max_{i\in N} \min_{\alpha_i \in [0,1]} \left\{ CL_i^{\alpha_i}(A) + |\alpha_i L_i(A) + (1-\alpha_i) l_i(A) - a_{ii}| \right\} \\
&\le \max_{i\in N} \min_{\alpha_i \in [0,1]} \left\{ a_{ii} - (n-2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A) \right\} \\
&= \max \left\{ \max_{i\in N^+(A)} \min_{\alpha_i \in [0,1]} \left\{ a_{ii} - (n-2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A) \right\}, \right. \\
&\qquad \left. \max_{i\in N^-(A)} \min_{\alpha_i \in [0,1]} \left\{ a_{ii} - (n-2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A) \right\} \right\}. \qquad (10)
\end{aligned}
$$

Furthermore, let

$$
f(\alpha) = a_{ii} - (n-2) l_i(A) + C_i(A) + \alpha \Delta_i(A), \ \alpha \in [0,1].
$$

Then when $\Delta_i(A) \ge 0$, $f(\alpha)$ reaches its minimum $a_{ii} - (n-2) l_i(A) + C_i(A)$ at $\alpha = 0$, and when $\Delta_i(A) < 0$, $f(\alpha)$ reaches its minimum

$$
a_{ii} - (n-2) l_i(A) + C_i(A) + \Delta_i(A) = a_{ii} + n L_i(A) - C_i(A)
$$

at $\alpha = 1$. Therefore, Inequality (10) is equivalent to

$$
\begin{aligned}
|\lambda| \le \max &\left\{ \max_{i\in N^+(A)} \left\{ a_{ii} - (n-2) l_i(A) + C_i(A) \right\}, \right. \\
&\left. \max_{i\in N^-(A)} \left\{ a_{ii} + n L_i(A) - C_i(A) \right\} \right\}.
\end{aligned}
$$

The conclusion follows. □

By the proof of Proposition 4.1, it is not difficult to see that the upper bound $\rho^{0,1}$ is larger than $\rho^{[0,1]}$ in Theorem 3.1, but $\rho^{0,1}$ depends only on the entries of a stochastic matrix. Moreover, $\rho^{0,1} \le \rho_L$ and $\rho^{0,1} \le \rho_l$, which are given as follows.

Proposition 4.2. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. Then*

$$
\rho^{0,1} \le \min\{\rho_l, \rho_L\},
$$

where ρ_l, ρ_L and $\rho^{0,1}$ are defined in Theorem 3.2 and Proposition 4.1, respectively.

Proof. By the proof of Proposition 4.1, we have that $\rho^{0,1}$ is equivalent to the last of Inequality (10), that is,

$$
\begin{aligned}
\rho^{0,1} = \max &\left\{ \max_{i\in N^+(A)} \min_{\alpha_i \in [0,1]} \left\{ a_{ii} - (n-2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A) \right\}, \right. \\
&\left. \max_{i\in N^-(A)} \min_{\alpha_i \in [0,1]} \left\{ a_{ii} - (n-2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A) \right\} \right\}.
\end{aligned}
$$

Also let

$$
f(\alpha) = a_{ii} - (n-2) l_i(A) + C_i(A) + \alpha \Delta_i(A), \ \alpha \in [0,1].
$$

Then when $\Delta_i(A) \geq 0$, $f(\alpha)$ is a monotonically increasing function of α, and when $\Delta_i(A) < 0$, $f(\alpha)$ is a monotonically decreasing function of α.

For the case that $L_i(A) = \max_{j\neq i, j\in N} a_{ji} > l_i(A) = \min_{j\neq i, j\in N} a_{ji}$, $i \in N$, we will prove $\rho^{0,1} < \rho_L$. Note that

$$f(0) = a_{ii} - (n-2)l_i(A) + C_i(A),\ and\ f(1) = a_{ii} + nL_i(A) - C_i(A).$$

Since $f(\alpha)$ is increasing when $\Delta_i(A) \geq 0$, we have

$$\begin{aligned}\max_{i\in N^+(A)} \{a_{ii} - (n-2)l_i(A) + C_i(A)\} &= \max_{i\in N^+(A)} \min_{\alpha_i\in[0,1]} \{a_{ii} - (n-2)l_i(A) + C_i(A) + \alpha_i\Delta_i(A)\} \\ &\leq \max_{i\in N^+(A)} \{a_{ii} - (n-2)l_i(A) + C_i(A) + \Delta_i(A)\} \\ &= \max_{i\in N^+(A)} \{a_{ii} + nL_i(A) - C_i(A)\},\end{aligned}$$

which implies

$$\rho^{0,1} \leq \max_{i\in N} \{a_{ii} + nL_i(A) - C_i(A)\} = \rho_L.$$

Then similarly as in the proof of $\rho^{0,1} \leq \rho_L$, we can obtain easily $\rho^{0,1} \leq \rho_l$.

For the case that $L_i(A) = l_i(A)$ for some $i \in N$, we have $\Delta_i(A) = 0$ and

$$\begin{aligned}a_{ii} + nL_i(A) - C_i(A) &= a_{ii} + (n-2)V_i(A) + 2L_i(A) - C_i(A) \\ &= \sum_{j=1}^{n} |a_{ji} - q_i(A)| \\ &= a_{ii} - (n-4)v_i(A) - 2l_i(A) + C_i(A) \\ &= a_{ii} - (n-2)l_i(A) + C_i(A).\end{aligned}$$

Similarly as in the case $L_i(A) > l_i(A)$, $i \in N$, we can also obtain easily

$$\rho^{0,1} \leq \rho_L\ and\ \rho^{0,1} \leq \rho_l.$$

The conclusion follows. □

By propositions 4.1 and 4.2, we know that the optimal values of α_i, $i \in N$ for the bound

$$\max_{i\in N} \min_{\alpha_i\in[0,1]} \{a_{ii} - (n-2)l_i(A) + C_i(A) + \alpha_i\Delta_i(A)\},$$

which could be obtained by using the set $\Gamma^{stoL^{\alpha}}(A)$ in Theorem 2.2, are $\alpha_i = 0$ for $i \in N^+(A)$ and $\alpha_i = 1$ for $i \in N^-(A)$ such that

$$\rho^{0,1} = \max\left\{ \max_{i\in N^+(A)} \{a_{ii} - (n-2)l_i(A) + C_i(A)\},\ \max_{i\in N^-(A)} \{a_{ii} + nL_i(A) - C_i(A)\} \right\}.$$

is less than or equal to the bounds obtained by using the sets in Theorem 1.3 and 1.4 respectively. This provides a choice of α_i, $i \in N$ for the set $\Gamma^{stoL^{\alpha}}(A)$ to localize all eigenvalues different from 1 of a stochastic matrix.

For a stochastic matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ and

$$d = L^{\alpha_i}(A) = [L_1^{\alpha_1}(A), L_2^{\alpha_2}(A), \ldots, L_n^{\alpha_n}(A)]^T$$

defined as (1), we take $\alpha_i = 0$ for $i \in N^+(A)$ and $\alpha_i = 1$ for $i \in N^-(A)$, that is,

$$L_i^{\alpha_i}(A) = \begin{cases} L_i^0(A) = l_i(A),\ i \in N^+(A), \\ L_i^1(A) = L_i(A),\ i \in N^-(A). \end{cases}$$

For this choice, the set $\Gamma^{stoL^{\alpha}}(A)$ reduces to

$$\Gamma^{stoL^{0,1}}(A) := \left(\bigcup_{i\in N^+(A)} \Gamma_i^{stoL^0}(A) \right) \bigcup \left(\bigcup_{i\in N^-(A)} \Gamma_i^{stoL^1}(A) \right),$$

where $\Gamma_i^{stoL^0}(A) = \Gamma_i^{stol}(A)$ and $\Gamma_i^{stoL^1}(A) = \Gamma_i^{stoL}(A)$. Hence, we have the following result.

Theorem 4.3. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. If $\lambda \in \sigma(A)\backslash\{1\}$, then*

$$\lambda \in \Gamma^{stoL^{0,1}}(A) = \left(\bigcup_{i\in N^+(A)} \Gamma_i^{stol}(A)\right) \bigcup \left(\bigcup_{i\in N^-(A)} \Gamma_i^{stoL}(A)\right). \tag{11}$$

Example 4.4. *Consider the stochastic matrix*

$$A = \begin{bmatrix} 0.2656 & 0.0471 & 0.1452 & 0.0758 & 0.2199 & 0.2463 \\ 0.2634 & 0.3368 & 0.0475 & 0.1143 & 0.1354 & 0.1026 \\ 0.0591 & 0.2002 & 0.1831 & 0.1916 & 0.1814 & 0.1846 \\ 0.2699 & 0.2753 & 0.1655 & 0.1941 & 0.0788 & 0.0165 \\ 0.1443 & 0.0598 & 0.1205 & 0.2582 & 0.2839 & 0.1332 \\ 0.2355 & 0.1027 & 0.1399 & 0.2358 & 0.2111 & 0.0750 \end{bmatrix}.$$

By computations, we have that $N^+(A) = \{2,4,6\}$, $N^-(A) = \{1,3,5\}$, and

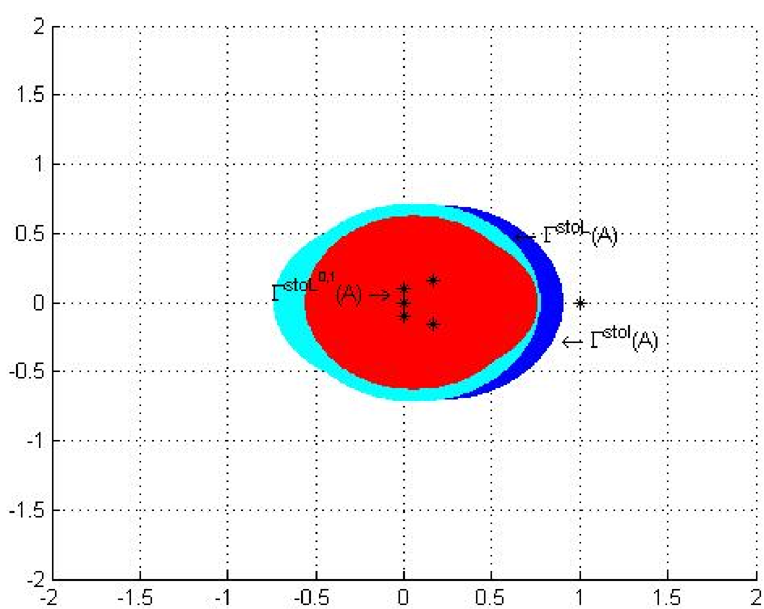
$$\Gamma^{stoL^{0,1}}(A) = \Gamma_2^{stol}(A) \bigcup \Gamma_4^{stol}(A) \bigcup \Gamma_6^{stol}(A) \bigcup \Gamma_1^{stoL}(A) \bigcup \Gamma_3^{stoL}(A) \bigcup \Gamma_5^{stoL}(A).$$

By drawing the sets $\Gamma^{stol}(A)$, $\Gamma^{stoL}(A)$ and $\Gamma^{stoL^{0,1}}(A)$ in the complex plane (see Figure 3), it is not difficult to see that for any $\lambda \in \sigma(A)\backslash\{1\}$,

$$\lambda \in \Gamma^{stoL^{0,1}}(A),$$

and that although $\Gamma^{stoL^{0,1}}(A) \nsubseteq \Gamma^{stol}(A)$ and $\Gamma^{stol}(A) \nsubseteq \Gamma^{stoL^{0,1}}(A)$, the set $\Gamma^{stoL^{0,1}}(A)$ is better than $\Gamma^{stol}(A)$ and $\Gamma^{stoL}(A)$ for estimating the moduli of subdominant eigenvalues.

Fig. 3. $\Gamma^{stol}(A)$, $\Gamma^{stoL}(A)$ and $\Gamma^{stoL^{0,1}}(A)$



5 Conclusions

In this paper, a set with n parameters in $[0, 1]$ is given to localize all eigenvalues different from 1 for a stochastic matrix A, that is,

$$\sigma(A)\backslash\{1\} \subseteq \Gamma^{stoL^{\alpha}}(A), \text{ for any } \alpha_i \in [0,1], i \in N.$$

In particular, when $\alpha_i = 0$ for each $i \in N$, $\Gamma^{stoL^{\alpha}}(A)$ reduces to the set $\Gamma^{stol}(A)$, which consists of n sets $\Gamma_i^{stol}(A)$, and when $\alpha_i = 1$ for each $i \in N$, $\Gamma^{stoL^{\alpha}}(A)$ reduces to the set $\Gamma^{stoL}(A)$, which consists of n sets $\Gamma_i^{stoL}(A)$. The sets $\Gamma^{stol}(A)$ and $\Gamma^{stoL}(A)$ are used to estimate the moduli of subdominant eigenvalues, that is, for any $\lambda \in \sigma(A)\backslash\{1\}$,

$$|\lambda| \le \rho_l = \max_{i\in N}\{a_{ii} - (n-2)l_i(A) + C_i(A)\}$$

and

$$|\lambda| \le \rho_L = \max_{i\in N}\{a_{ii} + nL_i(A) - C_i(A)\}.$$

Moreover, by taking $\alpha_i = 0$ for $i \in N^+(A)$ and $\alpha_i = 1$ for $i \in N^-(A)$, we give a set $\Gamma^{stoL^{0,1}}(A)$, which consists of $|N^+(A)|$ sets $\Gamma_i^{stol}(A)$ and $|N^-(A)|$ sets $\Gamma_i^{stoL}(A)$ where $|N^+(A)| + |N^-(A)| = n$. By using $\Gamma^{stoL^{0,1}}(A)$, we can get an upper bound for the moduli of subdominant eigenvalues which is better than ρ_l and ρ_L, i.e, for any $\lambda \in \sigma(A)\backslash\{1\}$,

$$|\lambda| \le \rho^{0,1} \le \min\{\rho_l, \rho_L\},$$

where

$$\rho^{0,1} = \max\left\{\max_{i\in N^+(A)}\{a_{ii} - (n-2)l_i(A) + C_i(A)\},\ \max_{i\in N^-(A)}\{a_{ii} + nL_i(A) - C_i(A)\}\right\}.$$

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