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On new strong versions of Browder type theorems

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Abstract: An operator T acting on a Banach space X satisfies the property (UW_{Π}) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$, where $\sigma_a(T)$ is the approximate point spectrum of T, $\sigma_{SF_+^-}(T)$ is the upper semi-Weyl spectrum of T and $\Pi(T)$ the set of all poles of T. In this paper we introduce and study two new spectral properties, namely (V_{Π}) and (V_{Π_a}), in connection with Browder type theorems introduced in [1], [2], [3] and [4]. Among other results, we have that T satisfies property (V_{Π}) if and only if T satisfies property (UW_{Π}) and $\sigma(T) = \sigma_a(T)$.

Keywords: Semi-Fredholm operator, Browder's theorem, property (W_{Π}), Property (UW_{Π}), Property (V_{Π})

MSC: 47A10, 47A11, 47A53

1 Introduction and preliminaries

Throughout this paper, $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. We refer to [5] for details about notations and terminologies. However, we give the following notations that will be useful in the sequel:

- Browder spectrum: $\sigma_b(T)$
- Weyl spectrum: $\sigma_W(T)$
- Upper semi-Browder spectrum: $\sigma_{ub}(T)$
- Upper semi-Weyl spectrum: $\sigma_{SF_+^-}(T)$
- Drazin invertible spectrum: $\sigma_D(T)$
- B-Weyl spectrum: $\sigma_{BW}(T)$
- Left Drazin invertible spectrum: $\sigma_{LD}(T)$
- Upper semi B-Weyl spectrum: $\sigma_{SBF_+^-}(T)$
- approximate point spectrum: $\sigma_a(T)$
- surjectivity spectrum: $\sigma_s(T)$

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In this paper, we introduce two new spectral properties of Browder type theorems, namely, the properties (V_{Π}) and (V_{Π_a}), respectively. In addition, we establish the precise relationships between these properties and others variants of Browder's theorem have been recently introduced in [1], [2], [3] and [4].

Denote by iso K, the set of all isolated points of $K \subseteq \mathbb{C}$. If $T \in L(X)$ define

$$\begin{aligned} E^0(T) &= \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}, \\ E^0_a(T) &= \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(\lambda I - T) < \infty\}, \\ E(T) &= \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T)\}, \\ E_a(T) &= \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(\lambda I - T)\}. \end{aligned}$$

Also, define

$$\Pi^0(T) = \sigma(T) \setminus \sigma_b(T), \quad \Pi^0_a(T) = \sigma_a(T) \setminus \sigma_{ub}(T),$$
$$\Pi(T) = \sigma(T) \setminus \sigma_D(T), \quad \Pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T).$$

Recall that $T \in L(X)$ is said to satisfy *a-Browder's theorem* (resp., *generalized a-Browder's theorem*) if $\sigma_a(T) \setminus \sigma_{SF^-_+}(T) = \Pi^0_a(T)$ (resp., $\sigma_a(T) \setminus \sigma_{SBF^-_+}(T) = \Pi_a(T)$). From [6, Theorem 2.2] (see also [7, Theorem 3.2(ii)]), a-Browder's theorem and generalized a-Browder's theorem are equivalent. It is well known that a-Browder's theorem for T implies Browder's theorem for T, i.e. $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$. Also by [6, Theorem 2.1], Browder's theorem for T is equivalent to generalized Browder's theorem for T, i.e. $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$.

Now, we describe several spectral properties introduced recently in [8], [2], [3], [9] and [10].

Definition 1.1. *An operator $T \in L(X)$ is said to have:*
(i) property (gab) *[8] if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$.*
(ii) property (az) *[9] if $\sigma(T) \setminus \sigma_{SF^-_+}(T) = \Pi^0_a(T)$.*
(iii) property (gaz) *[9] if $\sigma(T) \setminus \sigma_{SBF^-_+}(T) = \Pi_a(T)$.*
(iv) property (ah) *[10] if $\sigma(T) \setminus \sigma_{SF^-_+}(T) = \Pi^0_a(T)$.*
(v) property (gah) *[10] if $\sigma(T) \setminus \sigma_{SBF^-_+}(T) = \Pi_a(T)$.*
(vi) property (Sb) *[2] if $\sigma(T) \setminus \sigma_{SBF^-_+}(T) = \Pi^0(T)$.*
(vii) property (Sab) *[3] if $\sigma(T) \setminus \sigma_{SBF^-_+}(T) = \Pi^0_a(T)$.*

According to [9, Corollary 3.5], [10, Theorem 2.14] and [10, Corollary 2.15], we have that the properties (az), (gaz), (ah) and (gah) are equivalent. It was proved in [3, Corollary 2.9], that properties (Sb) and (Sab) are equivalent.

2 Properties (V_{Π}) and (V_{Π_a})

According to [1], $T \in L(X)$ has *property* (W_{Π}) (resp. *property* (UW_{Π_a})) if $\sigma(T) \setminus \sigma_W(T) = \Pi(T)$ (resp. $\sigma_a(T) \setminus \sigma_{SF^-_+}(T) = \Pi_a(T)$). It was shown in [1, Theorem 2.4] (resp. [1, Theorem 2.2]) that property (W_{Π}) (resp. (UW_{Π_a})) implies generalized Browder's theorem (resp. property (W_{Π})) but not conversely. Following [1], an operator $T \in L(X)$ is said to have *property* (UW_{Π}) if $\sigma_a(T) \setminus \sigma_{SF^-_+}(T) = \Pi(T)$. It was shown in [1, Theorem 3.5] that property (UW_{Π}) implies property (W_{Π}) but not conversely. According to [4], $T \in L(X)$ has *property* (Z_{Π_a}) if $\sigma(T) \setminus \sigma_W(T) = \Pi_a(T)$. In this section, we introduce and study two equivalent spectral properties that are stronger than the properties (UW_{Π_a}), (UW_{Π}) and (Z_{Π_a}).

Definition 2.1. *An operator $T \in L(X)$ is said to have* property (V_{Π}) *if $\sigma(T) \setminus \sigma_{SF^-_+}(T) = \Pi(T)$.*

Example 2.2. *1. Let $T \in L(\ell^2(\mathbb{N}))$ be defined by*

$$T(x_1, x_2, x_3, \cdots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \cdots\right).$$

Since $\sigma(T) = \sigma_{SF^-_+}(T) = \{0\}$ and $\Pi(T) = \varnothing$, then $\sigma(T) \setminus \sigma_{SF^-_+}(T) = \Pi(T)$ and hence T has property (V_{Π}).

2. Consider the Volterra operator V on the Banach space $C[0,1]$ defined by $V(f)(x) = \int_0^x f(t)dt$ for all $f \in C[0,1]$. Note that V is injective and quasinilpotent. Thus, $\sigma(V) = \{0\}$ and $\Pi(V) = \emptyset$. Since the range $R(V)$ is not closed, then $\sigma_{SF_+^-}(V) = \{0\}$. Therefore, $\sigma(V) \setminus \sigma_{SF_+^-}(V) = \Pi(V)$, that means V has property (V_Π).

According to [11], an operator $T \in L(X)$ is said to have *property* (V_E) if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T)$.

The next result gives the relationship between the properties (V_E) and (V_Π).

Theorem 2.3. *For $T \in L(X)$, the following statements are equivalent:*
(i) *T has property (V_E),*
(ii) *T has property (V_Π) and $E(T) = \Pi(T)$.*

Proof. (i)⇒(ii). Suppose that T satisfies property (V_E), i.e. $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T)$. Since by [11, Theorem 2.10], we have $E(T) = \Pi(T)$, it follows that T has property (V_Π) and $E(T) = \Pi(T)$.

(ii)⇒(i). Suppose that T satisfies property (V_Π) and $E(T) = \Pi(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T) = E(T)$ and T has property (V_E). □

Remark 2.4. *By Theorem 2.3, property (V_E) implies property (V_Π). However, the converse is not true in general. Consider the operator T in Example 2.2, then T satisfies property (V_Π) and as $E(T) = \{0\}$, it follows that T does not satisfy property (V_E).*

Theorem 2.5. *For $T \in L(X)$, the following statements are equivalent:*
(i) *T has property (V_Π),*
(ii) *T has property (UW_Π) and $\sigma(T) = \sigma_a(T)$,*
(iii) *T has property (UW_{Π_a}) and $\sigma(T) = \sigma_a(T)$.*

Proof. (i)⇒(ii). Suppose that T satisfies property (V_Π) and let $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Since $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subseteq \sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$, we have $\lambda \in \Pi(T)$ and so, $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subseteq \Pi(T)$.

To show the opposite inclusion $\Pi(T) \subseteq \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$, let $\lambda \in \Pi(T)$. Then, $\lambda \in E_a(T)$, it follows that $\lambda \in$ iso œa(T) and $\alpha(\lambda I - T) > 0$, so $\lambda \in \sigma_a(T)$. As T satisfies property (V_Π) and $\lambda \in \Pi(T)$, it follows that $\lambda I - T$ is upper semi-Weyl. Therefore, $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Thus, $\Pi(T) \subseteq \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ and T satisfies property (UW_Π). Consequently, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$ and $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$. Therefore, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ and $\sigma(T) = \sigma_a(T)$.

(ii)⇒(i). Suppose that T satisfies property (UW_Π) and $\sigma(T) = \sigma_a(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$. Thus, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$ and T satisfies property (V_Π).

(ii)⇔(iii). Obvious. □

The next example shows that, in general, property (UW_{Π_a}) does not imply property (V_Π).

Example 2.6. *Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$ and $U \in L(\ell^2(\mathbb{N}))$ be defined by*

$$U(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, \cdots).$$

Define an operator T on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus U$. Then, $\sigma(T) = \mathbf{D}(0,1)$, the closed unit disc on $\mathbb{C}$, $\sigma_a(T) = \Gamma \cup \{0\}$, where Γ denotes the unit circle of $\mathbb{C}$ and $\sigma_{SF_+^-}(T) = \Gamma$. Moreover, $\Pi_a(T) = \{0\}$ and $\Pi(T) = \emptyset$. Therefore, $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a(T)$ and $\sigma(T) \setminus \sigma_{SF_+^-}(T) \neq \Pi(T)$. Thus, T satisfies properties (UW_{Π_a}), but T does not satisfy property (V_Π).

The next example shows that, in general, property (UW_Π) does not imply property (V_Π).

Example 2.7. *Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$. Define an operator T on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = 0 \oplus R$. Then, $\sigma(T) = \mathbf{D}(0,1)$, $\sigma_a(T) = \sigma_{SF_+^-}(T) = \Gamma \cup \{0\}$ and $\Pi(T) = \emptyset$. Therefore, $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$ and $\sigma(T) \setminus \sigma_{SF_+^-}(T) \neq \Pi(T)$. Thus, T satisfies property (UW_Π), but T does not satisfy property (V_Π).*

The next result gives the relationship between the properties (V_Π) and (W_Π).

Theorem 2.8. *Let $T \in L(X)$. Then T has property (V_Π) if and only if T has property (W_Π) and $\sigma_{SF_+^-}(T) = \sigma_W(T)$.*

Proof. Sufficiency: Suppose that T satisfies property (V_Π), then by Theorem 2.5, T satisfies property (UW_Π). Property (UW_Π) implies by [1, Theorem 3.5] that T satisfies property (W_Π). Consequently, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$ and $\sigma(T) \setminus \sigma_W(T) = \Pi(T)$. Therefore, $\sigma_{SF_+^-}(T) = \sigma_W(T)$.

Necessity: Suppose that T satisfies property (W_Π) and $\sigma_{SF_+^-}(T) = \sigma_W(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_W(T) = \Pi(T)$, and so T satisfies property (V_Π). □

The next example shows that, in general, property (W_Π) does not imply property (V_Π).

Example 2.9. *Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$. Since $\sigma(R) = \sigma_W(R) = \mathbf{D}(0,1)$, $\sigma_{SF_+^-}(R) = \Gamma$ and $\Pi(R) = \varnothing$, then*

$$\sigma(R) \setminus \sigma_W(R) = \Pi(R), \qquad \sigma(R) \setminus \sigma_{SF_+^-}(R) \neq \Pi(R).$$

Hence, R satisfies property (W_Π), but R does not satisfy property (V_Π).

Theorem 2.10. *Suppose that $T \in L(X)$ has property (V_Π). Then:*

(i) *T has property (Z_{Π_a}),*
(ii) *$\Pi_a^0(T) = \Pi_a(T) = \Pi^0(T) = \Pi(T)$.*

Proof. (i) Property (V_Π) implies by Theorem 2.5 that $\sigma(T) = \sigma_a(T)$, and also implies by Theorem 2.8 that $\sigma_{SF_+^-}(T) = \sigma_W(T)$. Hence, $\sigma(T) \setminus \sigma_W(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T) = \Pi_a(T)$ and so T satisfies property (Z_{Π_a}).

(ii) Follows from (i) and [4, Lemma 2.9]. □

The next example shows that, in general, property (Z_{Π_a}) does not imply property (V_Π).

Example 2.11. *Let R be the unilateral right shift operator defined on $\ell^2(\mathbb{N})$. Since $\sigma(R) = \sigma_W(R) = \mathbf{D}(0,1)$, $\Pi(R) = \Pi_a(R) = \varnothing$ and $\sigma_{SF_+^-}(R) = \Gamma$, then R satisfies property (Z_{Π_a}), but does not satisfy property (V_Π).*

Remark 2.12. *By [4, Lemma 2.9], property (Z_{Π_a}) implies that $\Pi_a(T) = \Pi(T)$. Hence, if $T \in L(X)$ satisfies property (Z_{Π_a}), then $\sigma(T) \setminus \sigma_W(T) = \Pi_a(T) = \Pi(T)$ and follows that T satisfies property (W_Π). However, the converse is not true in general. For this, consider the operator T in Example 2.7, since $\sigma(T) = \sigma_W(T) = \mathbf{D}(0,1)$, $\Pi(T) = \varnothing$ and $\Pi_a(T) = \{0\}$, then T satisfies property (W_Π), but T does not satisfy property (Z_{Π_a}).*

Theorem 2.13. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (V_Π),*
(ii) *T has property (gah) and $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$,*
(iii) *T has property (gaz) and $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$,*
(iv) *T has property (ah) and $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$,*
(v) *T has property (az) and $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$.*

Proof. The equivalences (ii)⇔(iii), (iv)⇔(v) and (ii)⇔(iv) were shown in [10, Corllary 2.15], [10, Theorem 2.14] and [10, Theorem 2.10], respectively.

(i)⇒(ii). Assume that T satisfies property (V_Π). By Theorem 2.5, T satisfies property (UW_{Π_a}). Property (UW_{Π_a}) implies by [1, Theorem 2.6] that T satisfies generalized a-Browder's theorem and $\sigma_{SF_+^-}(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a \setminus \Pi_a^0$, but by Theorem 2.10, it follow that $\Pi_a \setminus \Pi_a^0 = \varnothing$ and hence, $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$. Consequently, $\Pi(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T)$, and hence T satisfies property (gah).

(ii)⇒(i). Suppose that T satisfies property (gah) and $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$. Then $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$, and hence T satisfies property (V_Π). □

The following example shows that, in general, property (gah) (resp. (ah)) does not imply property (V_Π).

Example 2.14. *Consider the operator $T = 0$ defined on the Hilbert space $\ell^2(\mathbb{N})$. Then, $\sigma(T) = \sigma_{SF_+^-}(T) = \{0\}$, $\sigma_{SBF_+^-}(T) = \varnothing$ and $\Pi(T) = \{0\}$. Therefore, $\sigma(T) \setminus \sigma_{SF_+^-}(T) \neq \Pi(T)$ and T does not satisfy property (V_Π). On the other hand, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$, that means T satisfies property (gah), in consequence T also satisfies property (ah).*

The next result gives the relationship between the properties (V_Π) and (Sb).

Corollary 2.15. *For $T \in L(X)$, the following statements are equivalent:*
(i) *T has property (V_Π),*
(ii) *T has property (Sab),*
(iii) *T has property (Sb).*

Proof. Follows directly from Theorem 2.13, [3, Theorem 2.4] and [3, Corollary 2.9]. □

The next result gives the relationship between the property (V_Π) and generalized Browder's theorem.

Theorem 2.16. *For $T \in L(X)$, the following statements are equivalent:*
(i) *T has property (V_Π),*
(ii) *T satisfies generalized Browder's theorem and $\sigma_{SF_+^-}(T) = \sigma_{BW}(T)$,*
(iii) *T satisfies Browder's theorem and $\sigma_{SF_+^-}(T) = \sigma_{BW}(T)$.*

Proof. (i)⇒(ii). Property (V_Π) implies by Theorem 2.8 that T satisfies property (W_Π), and property (W_Π) implies by [1, Theorem 2.4] that T satisfies generalized Browder's theorem. Consequently, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$ and $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$. Therefore, $\sigma_{SF_+^-}(T) = \sigma_{BW}(T)$.

(ii)⇒(i). Assume that T satisfies generalized Browder's theorem and $\sigma_{SF_+^-}(T) = \sigma_{BW}(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$, that means T satisfies property (V_Π).

(ii)⇔(iii). It follows from the equivalence between generalized Browder's theorem and Browder's theorem. □

Remark 2.17. *From Theorem 2.16, property (V_Π) implies generalized Browder's theorem. However, the converse is not true in general. Consider the operator R in Example 2.9, since R satisfies property (W_Π), then it also satisfies generalized Browder's theorem, but does not satisfy property (V_Π).*

Definition 2.18. *An operator $T \in L(X)$ is said to have* property (V_{Π_a}) *if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a(T)$.*

Example 2.19. *Let L be the unilateral left shift operator on $\ell^2(\mathbb{N})$. It is well known that $\sigma(L) = \sigma_{SF_+^-}(L) = \mathbf{D}(0,1)$, the closed unit disc on $\mathbb{C}$ and $\Pi_a(L) = \varnothing$. Therefore, $\sigma(L) \setminus \sigma_{SF_+^-}(L) = \Pi_a(L)$, and so L satisfies property (V_{Π_a}).*

Theorem 2.20. *Let $T \in L(X)$. Then T has property (V_{Π_a}) if and only if T has property (UW_{Π_a}) and $\sigma(T) = \sigma_a(T)$.*

Proof. Sufficiency: Assume that T satisfies property (V_{Π_a}). Then $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subseteq \sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a(T)$ and so $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subseteq \Pi_a(T)$.

To show the opposite inclusion $\Pi_a(T) \subseteq \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$, let $\lambda \in \Pi_a(T)$. Then, $\lambda \in E_a(T)$ and hence $\lambda \in \sigma_a(T)$. As T satisfies property (V_{Π_a}) and $\lambda \in \Pi_a(T)$, it follows that $\lambda I - T$ is upper semi-Weyl. Therefore, $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Thus, $\Pi_a(T) \subseteq \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ and T satisfies property (UW_{Π_a}). Consequently, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a(T)$ and $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a(T)$. Therefore, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ and $\sigma(T) = \sigma_a(T)$.

Necessity: Suppose that T satisfies property (UW_{Π_a}) and $\sigma(T) = \sigma_a(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a(T)$, in consequence T satisfies property (V_{Π_a}). □

Corollary 2.21. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property* (V_{Π_a}),
(ii) *T has property* (V_Π),
(iii) *T has property* (Sab),
(iv) *T has property* (Sb).

Proof. (i)⇒(ii). Suppose that T satisfies property (V_{Π_a}). By Theorem 2.20, $\sigma(T) = \sigma_a(T)$, it follows that $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a(T) = \Pi(T)$, hence T satisfies property (V_Π).

(ii)⇒(i). Assume that T satisfies property (V_Π). By Theorem 2.5, $\sigma(T) = \sigma_a(T)$ and so, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T) = \Pi_a(T)$. Therefore, T satisfies property (V_{Π_a}).

The rest of the proof follows from Corollary 2.15. □

The next result gives the relationship between property (V_{Π_a}) (or equivalently (V_Π)) and property (Z_{Π_a}).

Theorem 2.22. *Let $T \in L(X)$. Then T has property (V_{Π_a}) if and only if T has property (Z_{Π_a}) and $\sigma_{SF_+^-}(T) = \sigma_W(T)$.*

Proof. Sufficiency: Assume that T satisfies property (V_{Π_a}). By Corollary 2.21, property (V_{Π_a}) is equivalent to property (V_Π), and by Theorem 2.8, property (V_Π) implies that $\sigma_{SF_+^-}(T) = \sigma_W(T)$. Consequently, $\sigma(T) \setminus \sigma_W(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a(T)$. Therefore, T satisfies property (Z_{Π_a}).

Necessity: Assume that T satisfies property (Z_{Π_a}) and $\sigma_{SF_+^-}(T) = \sigma_W(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_W(T) = \Pi_a(T)$, that means T satisfies property (V_{Π_a}). □

Similar to Theorem 2.22, we have the following result.

Theorem 2.23. *Let $T \in L(X)$. Then T has property (V_{Π_a}) if and only if T has property (gab) and $\sigma_{SF_+^-}(T) = \sigma_{BW}(T)$.*

For $T \in L(X)$, define $\Pi_+^0(T) = \sigma(T) \setminus \sigma_{ub}(T)$. The following theorem describes the relationship between a-Browder's theorem and property (V_Π).

Theorem 2.24. *For $T \in L(X)$, the following statements are equivalent:*
(i) *T has property* (V_Π),
(ii) *T satisfies a-Browder's theorem and* $\Pi_+^0(T) = \Pi(T)$.
(iii) *T satisfies generalized a-Browder's theorem and* $\Pi_+^0(T) = \Pi(T)$.

Proof. (i)⇒(ii) Assume that T satisfies property (V_Π). Then T satisfies property (UW_Π) and $\Pi(T) = \Pi_a^0(T)$ by Theorems 2.5 and 2.10, respectively. Hence $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T) = \Pi_a^0(T)$. Consequently, T satisfies a-Browder's theorem and $\Pi_+^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$.

(ii)⇒(i) If T satisfies a-Browder's theorem and $\Pi_+^0(T) = \Pi(T)$, then $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_{ub}(T) = \Pi_+^0(T) = \Pi(T)$. Therefore, T satisfies property (V_Π).

(ii)⇔(iii). It follows from the equivalence between generalized a-Browder's theorem and a-Browder's theorem. □

Remark 2.25. *By Theorem 2.24, property (V_E) implies a-Browder's theorem. However, the converse is not true in general. Indeed, the operator R defined in Example 2.9, does not satisfy property (V_Π), but $\sigma_a(R) = \sigma_{SF_+^-}(R) = \Gamma$ and $\Pi_a^0(R) = \varnothing$, it follows that R satisfies a-Browder's theorem.*

Recall that an operator $T \in L(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0, the only analytic function $f : \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}_{\lambda_0}$$

is $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$ (see [12]). The operator T is said to have SVEP, if it has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, every $T \in L(X)$ has SVEP at each point of the resolvent set $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. (See [13] for more details about this concept).

Corollary 2.26. *If $T \in L(X)$ has SVEP at each $\lambda \notin \sigma_{SF_+^-}(T)$, then T has property (V_Π) if and only if $\Pi(T) = \Pi_+^0(T)$.*

Proof. By [14, Teorema 2.3], the hypothesis T has SVEP at each $\lambda \notin \sigma_{SF_+^-}(T)$ is equivalent to T satisfies a-Browder's theorem. Therefore, if $\Pi(T) = \Pi_+^0(T)$, then $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_{ub}(T) = \Pi_+^0(T) = \Pi(T)$. □

The following three tables summarizes the meaning of various theorems and properties that are related with property (V_Π).

Table 1

(W_Π) [1]	$\sigma(T) \setminus \sigma_W(T) = \Pi(T)$	$\mathcal{B}$ [15]	$\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$
(Z_{Π_a}) [4]	$\sigma(T) \setminus \sigma_W(T) = \Pi_a(T)$	(ab) [8]	$\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$
$g\mathcal{B}$ [16]	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$	(Bb) [17]	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$
(gab) [8]	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$	(Bab) [18]	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$
(gah) [10]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$	(Sb) [2]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi^0(T)$
(gaz) [9]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$	(Sab) [3]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a^0(T)$
(UW_Π) [1]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$	(b) [19]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$
(UW_{Π_a}) [1]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a(T)$	$a\mathcal{B}$ [20]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$
(gb) [19]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$	(Bgb) [17]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi^0(T)$
$ga\mathcal{B}$ [16]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$	$(SBab)$ [21]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a^0(T)$
(V_Π)	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$	(ah) [10]	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$
(V_{Π_a})	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a(T)$	(az) [9]	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$

Table 2

$\mathcal{W}$ [22]	$\sigma(T) \setminus \sigma_W(T) = E^0(T)$
(aw) [8]	$\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$
(Bw) [23]	$\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$
(Baw) [18]	$\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$
(v) [5]	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$
(z) [9]	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$
(Sw) [2]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$
(Saw) [3]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$
(w) [24]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$
$(a\mathcal{W})$ [25]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$
(Bgw) [17]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$
$(SBaw)$ [21]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$

Theorem 2.27. *Suppose that $T \in L(X)$ has property (V_Π). Then:*

(i) $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_{SF_+^-}(T) = \sigma_W(T) = \sigma_{LD}(T) = \sigma_D(T) = \sigma_{ub}(T) = \sigma_b(T)$ *and* $\sigma(T) = \sigma_a(T)$.

(ii) $\Pi^0(T) = \Pi_a^0(T) = \Pi(T) = \Pi_a(T)$, $E^0(T) = E_a^0(T)$ *y* $E(T) = E_a(T)$.

(iii) *All properties given in Table 1 are equivalent, and T satisfies each of these properties.*

(iv) *All properties given in Table 2 are equivalent.*

(v) *All properties given in Table 3 are equivalent.*

Table 3

(W_E) [26]	$\sigma(T) \setminus \sigma_W(T) = E(T)$
(Z_{E_a}) [4]	$\sigma(T) \setminus \sigma_W(T) = E_a(T)$
$g\mathcal{W}$ [16]	$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$
(gaw) [8]	$\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$
(gv) [5]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$
(gz) [9]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$
(UW_E) [1]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E(T)$
(UW_{E_a}) [26]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a(T)$
(gw) [27]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$
$ga\mathcal{W}$ [16]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$

Proof. Since property (V_Π) is equivalent to property (Sab), then (i) and (ii) follows from [3, Theorem 2.31].

(iii) By Theorem 2.5, T satisfies property (UW_Π), and the equivalence between all properties given in Table 1 follows from (i) and (ii).

(iv) and (v) Follows directly from (i) and (ii). □

In the following diagram the arrows signify implications between the Browder type theorems defined above. The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (the numbers in square brackets).

$$
\begin{tikzcd}
 & & & (W_\Pi) \arrow[r, "{[1]}"] & (g\mathcal{B}) \arrow[r, Leftrightarrow, "{[6]}"] & (\mathcal{B}) \\
 & & & (Z_{\Pi_a}) \arrow[u, "2.12"'] \arrow[r, "{[4]}"] & (gab) \arrow[u, "{[8]}"'] \arrow[r, "{[8]}"] & (ab) \arrow[u, "{[8]}"'] \\
 & (W_\Pi) & (UW_\Pi) \arrow[l, "{[1]}"'] & (V_\Pi) \arrow[l, "2.5"'] \arrow[u, "2.10"'] & (gb) \arrow[u, "{[8]}"'] & (b) \arrow[l, "{[8]}"'] \arrow[u, "{[8]}"'] \\
 (b) \arrow[d, "{[19]}"'] & (UW_{\Pi_a}) \arrow[u, "{[1]}"'] \arrow[d, Leftrightarrow, "{[1]}"] & (V_{\Pi_a}) \arrow[l, "2.19"'] \arrow[r, Leftrightarrow, "2.20"] & (Sb) \arrow[u, Leftrightarrow, "2.15"'] \arrow[d, Leftrightarrow, "{[3]}"] & (gah) \arrow[u, "{[10]}"'] \arrow[r, Leftrightarrow, "{[10]}"] \arrow[d, Leftrightarrow, "{[10]}"] & (ah) \arrow[u, "{[10]}"'] \arrow[d, Leftrightarrow, "{[10]}"] \\
 (a\mathcal{B}) \arrow[d, Leftrightarrow, "{[6]}"] & (SBab) \arrow[l, "{[21]}"'] & (Bgb) \arrow[l, "{[21]}"'] \arrow[d, "{[21,17]}"] & (Sab) \arrow[l, "{[3]}"'] \arrow[r, "{[3]}"] \arrow[d, "{[3]}"] & (gaz) \arrow[r, Leftrightarrow, "{[9]}"] & (az) \\
 (ga\mathcal{B}) \arrow[d, "{[16]}"] & (gb) \arrow[l, "{[19]}"'] & (Bb) \arrow[d, Leftrightarrow, "{[1]}"] & (Bab) \arrow[l, "{[18]}"'] \arrow[d, "{[18]}"] & (Z_{\Pi_a}) \arrow[l, "{[4]}"'] & \\
 (g\mathcal{B}) & (Bgb) \arrow[u, "{[21]}"'] & (W_\Pi) & (ab) & &
\end{tikzcd}
$$

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