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Restriction conditions on PL$(7, 2)$ codes ($3 \leq |\mathcal{G}_i| \leq 7$)

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Abstract: The Golomb-Welch conjecture states that there is no perfect r-error correcting Lee code of word length n over $\mathbb{Z}$ for $n \geq 3$ and $r \geq 2$. This problem has received great attention due to its importance in applications in several areas beyond mathematics and computer sciences. Many results on this subject have been achieved, however the conjecture is only solved for some particular values of n and r, namely: $3 \leq n \leq 5$ and $r \geq 2$; $n = 6$ and $r = 2$. Here we give an important contribution for the case $n = 7$ and $r = 2$, establishing cardinality restrictions on codeword sets.

Keywords: Perfect Lee codes, Golomb-Welch conjecture, Space tilings

MSC: 05B40, 05E99

1 Introduction

Problems involving space tilings are common in coding theory. In fact, special types of tilings can be regarded as error correcting codes which are essential on correct transmission of information over a noisy channel, see [1, 2].

In this paper we deal with tilings of $\mathbb{Z}^n$ by Lee spheres, where n is a positive integer number. The study of these tilings was introduced by Golomb and Welch, see [1, 3], where they related these tilings with error correcting codes considering the center of a Lee sphere as a codeword and the other elements of the sphere as words which are decoded by the central codeword. When a Lee sphere of radius r tiles the n-dimensional space, the set of all centers of the Lee spheres, that is, the set of all codewords, produces a perfect r-error correcting Lee code of word length n, which will be denoted by PL(n, r) code. The interest in Lee codes has been increasing due to their several applications, see, for instance, [4–7].

The question "*for what values of n and r does the n-dimensional Lee sphere of radius r tile a n-dimensional space?*" was formulated by Golomb and Welch in [1], where they proved: (i) n-dimensional Lee sphere of radius 1 tiles the n-dimensional space for any positive integer n; (ii) for each $r \geq 1$, there exists a tiling of the n-dimensional space by Lee spheres of radius r for $n = 1, 2$. In other words, there exist PL$(n, 1)$, PL$(1, r)$ and PL$(2, r)$ codes for any positive integer numbers n and r, respectively. These codes have been extensively studied by other authors, see, for instance, Stein and Szabó [8].

According to Golomb and Welch, it seems that there is no PL(n, r) code for other values of n and r, that is:

Conjecture (Golomb-Welch). *There is no PL(n, r) code for $n \geq 3$ and $r \geq 2$.*

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There exists an extensive literature on the subject, however the Golomb-Welch conjecture is still far from being solved. Actually, the conjecture is proved for $3 \leq n \leq 5$ and $r \geq 2$, see [9–11], and for $n = 6$ and $r = 2$, see [12]. The difficulty to prove the conjecture has led some authors to consider special types of $\mathrm{PL}(n, r)$ codes, such as linear and periodic ones, see [13–15]. It should be pointed out that Horak and Grosek, in [13], have proved, using a new approach, the nonexistence of **linear** $\mathrm{PL}(n, 2)$ codes for $7 \leq n \leq 12$.

As stated previously, a Lee sphere of radius 1 tiles the n-dimensional space for any positive integer n. It seems that the most difficult cases of the Golomb-Welch conjecture are those in which $r = 2$. Following an intuitive and geometric reasoning, it seems that the bigger is the radius of the Lee sphere the more difficult is to tile the space with this sphere.

Here we will give a contribution for the case $n = 7$ and $r = 2$ presenting a possible strategy to prove the non-existence of $\mathrm{PL}(7, 2)$ codes. We believe that this strategy will allow us, in the future, to get the proof of the non-existence of such codes. Our strategy does not use computational methods and is faithful to the geometric idea of the problem. By contradiction, we consider the existence of a $\mathrm{PL}(7, 2)$ code and it is assumed that $O = (0, \ldots, 0)$ is a codeword. Since O covers all words $W \in \mathbb{Z}^n$ which are distant two or less units from it, we focus our attention on the codewords which cover all words which are distant three units from O. Our idea is mostly based in cardinality restrictions on subsets of these codewords, being a natural adaptation of the one given by Horak in [12].

The next sections are organized as follows. In Section 2 some definitions, terminology and notation are given. Section 3 is devoted to the establishment of necessary conditions for the existence of $\mathrm{PL}(n, 2)$ codes for any positive integer $n \geq 7$. Necessary conditions for the existence of $\mathrm{PL}(7, 2)$ codes are given in Section 4.

2 Definitions and notation

In this section we introduce some definitions and notation. The notation follows the one used by Horak [12].

Let $(\mathcal{S}, \mu)$ be a metric space, where $\mathcal{S}$ is a nonempty set and μ a metric on $\mathcal{S}$. Any subset $\mathcal{M}$ of $\mathcal{S}$ satisfying $|\mathcal{M}| \geq 2$ is a **code**. The elements of $\mathcal{S}$ are called **words** and, in particular, the elements of a code $\mathcal{M}$ are called **codewords**.

A sphere centered at $W \in \mathcal{S}$ with radius r, denoted by $S(W, r)$, is defined as follows

$$S(W, r) = \{V \in \mathcal{S} : \mu(V, W) \leq r\}.$$

If $W \in \mathcal{M}$ and $V \in S(W, r)$, with $V \neq W$, then we say that **the codeword** W **covers the word** V.

Definition 2.1. *A code $\mathcal{M}$ is a **perfect r-error correcting code** if:*
i) $S(W, r) \cap S(V, r) = \emptyset$ *for any two distinct codewords W and V in $\mathcal{M}$;*
ii) $\bigcup_{W \in \mathcal{M}} S(W, r) = \mathcal{S}$.

In other words, $\mathcal{M}$ is a perfect r-error correcting code if the spheres of radius r centered at codewords of $\mathcal{M}$ form a partition of $\mathcal{S}$. Equivalently, $\mathcal{M}$ is a perfect r-error correcting code if the spheres of radius r centered at codewords of $\mathcal{M}$ tile $\mathcal{S}$.

When a code $\mathcal{M}$ satisfies the condition $i)$ in Definition 2.1, we say that $\mathcal{M}$ is a **r-error correcting code**.

We are interested in dealing with metric spaces $(\mathbb{Z}^n, \mu_L)$, where $\mathbb{Z}^n$ is the n-fold Cartesian product of the set of the integer numbers, with n a positive integer number, and μ_L is the **Lee metric**, that is, for any $W, V \in \mathbb{Z}^n$, with $W = (w_1, \ldots, w_n)$ and $V = (v_1, \ldots, v_n)$, the Lee distance between W and V, shortly $\mu_L(W, V)$, is given by

$$\mu_L(W, V) = \sum_{i=1}^{n} |w_i - v_i|.$$

If $\mathcal{M} \subset \mathbb{Z}^n$ is a perfect r-error correcting code of $(\mathbb{Z}^n, \mu_L)$, then $\mathcal{M}$ is called a **perfect r-error correcting Lee code of word length n over $\mathbb{Z}$**, shortly a **PL(n,r) code**.

We detach the following necessary and sufficient condition on the Lee distance between two words to avoid superposition of spheres centered at them: *Given $W, V \in \mathbb{Z}^n$, with $W \neq V$, and r a positive integer number, $S(W, r) \cap S(V, r) = \varnothing$ if and only if $\mu_L(W, V) \ge 2r + 1$.*

Having in mind the Golomb-Welch conjecture, our aim is to give a contribution for the proof of the non-existence of PL(7, 2) codes. Our strategy is based on the assumption that their existence will bring strong cardinality restrictions on the cardinality of same codeword sets that we must identify and control.

Let us assume the existence of a PL$(n, 2)$ code $\mathcal{M} \subset \mathbb{Z}^n$, $n \ge 7$, and suppose, without loss of generality, that $O \in \mathcal{M}$, with $O = (0, \dots, 0)$. Thus, all words $W \in \mathbb{Z}^n$ such that $\mu_L(W, O) \le 2$ are covered by the codeword O. Taking into account Definition 2.1, for each word $W \in \mathbb{Z}^n$ satisfying $\mu_L(W, O) = 3$ there exists a unique codeword $V \in \mathcal{M}$ such that $\mu_L(W, V) \le 2$. The conditions for the existence of PL$(n, 2)$ codes derive essentially from the analysis of the codewords which cover all words $W \in \mathbb{Z}^n$ which are distant three units from O.

Let $W \in \mathbb{Z}^n$ such that $\mu_L(W, O) = 3$. Then, $W = (w_1, \dots, w_n)$ is of one and only one of the types:

- $[\pm 3]$, if there exists $i \in \{1, \dots, n\}$ so that $|w_i| = 3$ and $w_j = 0$ for all $j \in \{1, \dots, n\} \setminus \{i\}$;
- $[\pm 2, \pm 1]$, if $|w_i| = 2$ and $|w_j| = 1$ for some $i, j \in \{1, \dots, n\}$, and $w_k = 0$ for all $k \in \{1, \dots, n\} \setminus \{i, j\}$;
- $[\pm 1^3]$, if $|w_i| = |w_j| = |w_k| = 1$ for some $i, j, k \in \{1, \dots, n\}$, and $w_l = 0$ for all $l \in \{1, \dots, n\} \setminus \{i, j, k\}$.

Let $\mathcal{T} \subset \mathcal{M}$ be the set of the codewords which cover all the words $W \in \mathbb{Z}^n$ satisfying $\mu_L(W, O) = 3$. Any codeword $V \in \mathcal{T}$ is such that $\mu_L(V, O) = 5$. In fact, since O and V are codewords in $\mathcal{M}$, to avoid superposition between them we must impose $\mu_L(V, O) \ge 2 \times 2 + 1 = 5$. On the other hand, if we suppose $\mu_L(V, O) \ge 6$, then for W such that $\mu_L(W, O) = 3$ we get $\mu_L(V, W) \ge 3$.

Following the same idea used in the characterization of the words which are distant three units from O, we conclude that $V \in \mathcal{T}$ is of one and only one of the types: $[\pm 5]$, $[\pm 4, \pm 1]$, $[\pm 3, \pm 2]$, $[\pm 3, \pm 1^2]$, $[\pm 2^2, \pm 1]$, $[\pm 2, \pm 1^3]$ and $[\pm 1^5]$. We will denote the subsets of $\mathcal{T}$ containing codewords of each one of these types by, respectively, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$. Furthermore, we set $a = |\mathcal{A}|$, $b = |\mathcal{B}|$, $c = |\mathcal{C}|$, $d = |\mathcal{D}|$, $e = |\mathcal{E}|$, $f = |\mathcal{F}|$ and $g = |\mathcal{G}|$, where $|\mathcal{A}|$ denotes the cardinality of the set $\mathcal{A}$ and so on.

Consider

$$\mathcal{I} = \{+1, +2, \dots, +n, -1, -2, \dots, -n\}$$

the **set of signed coordinates**. Let $W, V \in \mathbb{Z}^n$, with $W = (w_1, \dots, w_n)$ and $V = (v_1, \dots, v_n)$. If $iw_{|i|} > 0$ for $i \in \mathcal{I}$, then i and $w_{|i|}$ have the same sign. If $iw_{|i|} > 0$ and $iv_{|i|} > 0$, with $i \in \mathcal{I}$, then the $|i| - th$ coordinates of W and V have the same sign and we say that W and V are sign equivalent in the $|i| - th$ coordinate.

Let $\mathcal{H} \subset \mathbb{Z}^n$. For $i_1, i_2, \dots, i_p \in \mathcal{I}$, with $p \le n$ and $|i_1|, |i_2|, \dots, |i_p|$ pairwise distinct, $\mathcal{H}_{i_1 i_2 \dots i_p}$ will denote the following set:

$$\{W \in \mathcal{H} \;:\; i_1 w_{|i_1|} > 0 \;\wedge\; i_2 w_{|i_2|} > 0 \;\wedge \dots \wedge\; i_p w_{|i_p|} > 0\}.$$

Given a positive integer number k and $i \in \mathcal{I}$, $\mathcal{H}_i^{(k)}$ will denote:

$$\{W \in \mathcal{H} \;:\; iw_{|i|} > 0 \;\wedge\; |w_{|i|}| = k\}.$$

These sets are called **index subsets** of $\mathcal{H}$. We note that, it makes no sense to consider $\mathcal{H}_{ij}$ for $i = j$ or $i = -j$, so, in the rest of the document, when we write $\mathcal{H}_{i_1 i_2 \dots i_p}$, with $\mathcal{H} \subset \mathbb{Z}^n$ and $i_1, i_2, \dots, i_p \in \mathcal{I}$, we assume $|i_1|, |i_2|, \dots, |i_p|$ pairwise distinct.

Consider $W \in \mathcal{G}$. Since the codewords of $\mathcal{G}$ are of type $[\pm 1^5]$, there are $i, j, k, l, m \in \mathcal{I}$ such that $W \in \mathcal{G}_{ijklm}$, where $iw_{|i|}, jw_{|j|}, kw_{|k|}, lw_{|l|}, mw_{|m|} > 0$ and $|w_{|i|}| = |w_{|j|}| = |w_{|k|}| = |w_{|l|}| = |w_{|m|}| = 1$. In this case i, j, k, l and m characterize the **index distribution** of $W \in \mathcal{G}$. If we consider $W \in \mathcal{F}$, since the codewords of $\mathcal{F}$ are of type $[\pm 2, \pm 1^3]$, there exist $i, j, k, l \in \mathcal{I}$ so that $W \in \mathcal{F}_{ijkl}$, more precisely, $W \in \mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)} \cap \mathcal{F}_k^{(1)} \cap \mathcal{F}_l^{(1)}$, where $iw_{|i|}, jw_{|j|}, kw_{|k|}, lw_{|l|} > 0$, $|w_{|i|}| = 2$ and $|w_{|j|}| = |w_{|k|}| = |w_{|l|}| = 1$, being characterized the **index value distribution** of W.

3 PL(n, 2) codes

In this section some necessary conditions for the existence of PL(n, 2) codes, for $n \geq 7$, are given.

Let $\mathcal{M} \subset \mathbb{Z}^n$ be a PL(n, 2) code, with $n \geq 7$. Suppose that $O = (0, \ldots, 0)$ is a codeword of $\mathcal{M}$. Assume that $\mathcal{T} \subset \mathcal{M}$ is the set of the codewords which cover all the words $W \in \mathbb{Z}^n$ satisfying $\mu_L(W, O) = 3$. We have characterized in the previous section a partition of $\mathcal{T}$ formed by the sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, composed, respectively, by codewords of types $[\pm 5]$, $[\pm 4, \pm 1]$, $[\pm 3, \pm 2]$, $[\pm 3, \pm 1^2]$, $[\pm 2^2, \pm 1]$, $[\pm 2, \pm 1^3]$ and $[\pm 1^5]$.

We note that, the words of types:

- $[\pm 3]$ must be covered by codewords of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$;
- $[\pm 2, \pm 1]$ must be covered by codewords of $\mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$;
- $[\pm 1^3]$ must be covered by codewords of $\mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$.

Let $W \in \mathbb{Z}^n$ such that $W = (w_1, \ldots, w_n)$ and $\mu_L(W, O) = 3$. Suppose that W is a word of type $[\pm 2, \pm 1]$. Thus, there are $i, j \in \mathcal{I}$, with $|i| \neq |j|$, such that, $iw_{|i|}, jw_{|j|} > 0$, $|w_{|i|}| = 2$ and $|w_{|j|}| = 1$. In these conditions we must impose, for instance, $|\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| \leq 1$, otherwise, there are $V, V' \in \mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}$, with $V \neq V'$, covering the same word W, contradicting the definition of PL(n, 2) code. In fact, supposing $V \in \mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)} \cap \mathcal{D}_k^{(1)}$ and $V' \in \mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)} \cap \mathcal{D}_l^{(1)}$, we would have $\mu_L(V, W) = |v_{|i|} - w_{|i|}| + |v_{|k|} - w_{|k|}| = 2$ and $\mu_L(V', W) = |v'_{|i|} - w_{|i|}| + |v'_{|l|} - w_{|l|}| = 2$. Having in view the word W similar conditions can be deduced to another sets of codewords, such as $|(\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_j)| \leq 1$.

Taking into account the words of each one of the types $[\pm 3]$, $[\pm 2, \pm 1]$ and $[\pm 1^3]$, and considering the sets of codewords that can cover them, we get the following lemmas.

Lemma 3.1. *For each $i \in \mathcal{I}$, $|\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 1$.*

Proof. For each $i \in \mathcal{I}$ there exists a word $W \in \mathbb{Z}^n$ of type $[\pm 3]$, with $W = (w_1, \ldots, w_n)$, satisfying $iw_{|i|} > 0$ and $|w_{|i|}| = 3$. This word W must be covered by a codeword $V \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$, in particular, $V \in \mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}$. Thus, we conclude that $|\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| \geq 1$. If, by contradiction, we assume $|\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| \geq 2$, then there are two distinct codewords V and V' in $\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}$ satisfying $\mu_L(V, W) \leq 2$ and $\mu_L(V', W) \leq 2$, which contradicts the definition of PL(n, 2) code. □

Lemma 3.2. *For each $i, j \in \mathcal{I}$, with $|i| \neq |j|$,*

$$|\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}| + |\mathcal{C}_i \cap \mathcal{C}_j| + |\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| + |\mathcal{E}_i^{(2)} \cap \mathcal{E}_j| + |\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}| = 1.$$

Proof. For each $i, j \in \mathcal{I}$, with $|i| \neq |j|$, there exists a word $W \in \mathbb{Z}^n$ of type $[\pm 2, \pm 1]$, with $W = (w_1, \ldots, w_n)$, satisfying $iw_{|i|}, jw_{|j|} > 0$, $|w_{|i|}| = 2$ and $|w_{|j|}| = 1$. This word must be covered by a codeword $V \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$, in particular, $V \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}) \cup (\mathcal{C}_i \cap \mathcal{C}_j) \cup (\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_j) \cup (\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)})$. Consequently, taking into account that $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ are disjoint sets,

$$|\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}| + |\mathcal{C}_i \cap \mathcal{C}_j| + |\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| + |\mathcal{E}_i^{(2)} \cap \mathcal{E}_j| + |\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}| \geq 1.$$

If, by contradiction, we suppose

$$|\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}| + |\mathcal{C}_i \cap \mathcal{C}_j| + |\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| + |\mathcal{E}_i^{(2)} \cap \mathcal{E}_j| + |\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}| \geq 2,$$

then, there are distinct codewords V and V' satisfying

$$V, V' \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}) \cup (\mathcal{C}_i \cap \mathcal{C}_j) \cup (\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_j) \cup (\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}).$$

Consequently, $\mu_L(V, W) \leq 2$ and $\mu_L(V', W) \leq 2$, which contradicts the definition of perfect 2-error correcting code. □

Lemma 3.3. *For each $i, j, k \in \mathcal{I}$, with $|i|$, $|j|$ and $|k|$ pairwise distinct,*

$$|\mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}| = 1.$$

Proof. For each $i, j, k \in \mathcal{I}$, with $|i|$, $|j|$ and $|k|$ pairwise distinct, there is a word $W \in \mathbb{Z}^n$ of type $[\pm 1^3]$, with $W = (w_1, \dots, w_n)$, such that, $iw_{|i|}, jw_{|j|}, kw_{|k|} > 0$ and $|w_{|i|}| = |w_{|j|}| = |w_{|k|}| = 1$. This word must be covered by a codeword $V \in \mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}$, therefore $|\mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}| \ge 1$. If, by contradiction, we suppose that $|\mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}| \ge 2$, then there are distinct codewords $V, V' \in \mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}$ and, consequently, $\mu_L(V, W) \le 2$ and $\mu_L(V', W) \le 2$, contradicting the definition of PL$(n, 2)$ code. □

Taking into account the number of words of each one of the types $[\pm 3]$, $[\pm 2, \pm 1]$ and $[\pm 1^3]$, and considering the type of codewords which cover them, Horak has deduced in [12] the following proposition involving the parameters $a = |\mathcal{A}|$, $b = |\mathcal{B}|$, $c = |\mathcal{C}|$, $d = |\mathcal{D}|$, $e = |\mathcal{E}|$, $f = |\mathcal{F}|$ and $g = |\mathcal{G}|$.

Proposition 3.4. *The parameters a, b, c, d, e, f and g satisfy the system of equations*

$$\begin{cases} a + b + c + d = 2n \\ b + 2c + 2d + 4e + 3f = 8\binom{n}{2} \\ d + e + 4f + 10g = 8\binom{n}{3}. \end{cases}$$

There exist many nonnegative integer solutions for this system of equations. However, we are interested in determining "good" solutions, that is, solutions which do not contradict the definition of perfect 2-error correcting Lee code.

We may relate the cardinality of each set $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ with the cardinality of their index subsets. Taking into account, for instance, the set $\mathcal{G}$, since the codewords of $\mathcal{G}$ are of type $[\pm 1^5]$, we get

$$g = \frac{1}{5} \sum_{i \in \mathcal{I}} |\mathcal{G}_i|.$$

Besides, for $i \in \mathcal{I}$,

$$|\mathcal{G}_i| = \frac{1}{4} \sum_{j \in \mathcal{I} \setminus \{i, -i\}} |\mathcal{G}_{ij}|.$$

Analogous equalities for the other subsets of $\mathcal{T}$ may be derived.

The analysis of the solutions for the system of equations presented in Proposition 3.4 will be focused essentially in the study of the cardinality of the index subsets of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$ and $\mathcal{G}$.

Looking at the words of type $[\pm 1^3]$, Horak proved in [12] the following proposition in which a relation between the cardinality of index subsets of $\mathcal{D}, \mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ is given.

Proposition 3.5. *For each $i, j \in \mathcal{I}$, $|i| \ne |j|$,*

$$|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| + 2|\mathcal{F}_{ij}| + 3|\mathcal{G}_{ij}| = 2(n - 2).$$

4 Conditions for the existence of PL(7, 2) codes

In this section we concentrate our attention on the search of necessary conditions for the existence of PL$(7, 2)$ codes.

Let us suppose that $\mathcal{M} \subset \mathbb{Z}^7$ is a PL$(7, 2)$ code, with $O = (0, \dots, 0)$ a codeword of $\mathcal{M}$. By Proposition 3.4, the parameters a, b, c, d, e, f and g satisfy:

$$\begin{cases} a + b + c + d = 14 \\ b + 2c + 2d + 4e + 3f = 168 \\ d + e + 4f + 10g = 280. \end{cases}$$

As we have said before, there are many nonnegative integer solutions for this system of equations, however we are only interested in those which do not contradict the definition of a perfect 2-error correcting Lee code. Since, $g = |\mathcal{G}|$ is the variable with highest coefficient in the system and the codewords of $\mathcal{G}$ are the ones which have more nonzero coordinates, a particular attention to the set $\mathcal{G}$, more precisely, to the subsets $\mathcal{G}_i$, for $i \in \mathcal{I}$, will be given.

In [16], the following theorem which restricts the variation of $|\mathcal{G}_i|$, for any $i \in \mathcal{I}$, was established.

Theorem 4.1. *For each $i \in \mathcal{I}$, $3 \le |\mathcal{G}_i| \le 8$.*

This theorem restricts the variation of g, in fact, since

$$g = \frac{1}{5} \sum_{i \in \mathcal{I}} |\mathcal{G}_i|,$$

taking into account that $3 \le |\mathcal{G}_i| \le 8$ for all $i \in \mathcal{I}$ and that $|\mathcal{I}| = 14$, we conclude that the solutions which do not contradict the definition of $\mathrm{PL}(7, 2)$ code must satisfy

$$9 \le g \le 22.$$

Our strategy to prove the non-existence of $\mathrm{PL}(7, 2)$ codes relies on restricting more and more the variation of $|\mathcal{G}_i|$, for any $i \in \mathcal{I}$, more precisely, limiting more and more the variation of g.

In the following subsection we prove that $|\mathcal{G}_i| \neq 8$ for all $i \in \mathcal{I}$.

4.1 Proof of $|\mathcal{G}_i| \neq 8$ for any $i \in \mathcal{I}$

We will prove that $|\mathcal{G}_i| \neq 8$ for any $i \in \mathcal{I}$ by contradiction. Let us suppose that there exists $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 8$. Thus, since

$$|\mathcal{G}_i| = \frac{1}{4} \sum_{\omega \in \mathcal{I} \setminus \{i, -i\}} |\mathcal{G}_{i\omega}|,$$

we get

$$8 = \frac{1}{4} \sum_{\omega \in \mathcal{I} \setminus \{i, -i\}} |\mathcal{G}_{i\omega}|.$$

Consequently,

$$\sum_{\omega \in \mathcal{I} \setminus \{i, -i\}} |\mathcal{G}_{i\omega}| = 32. \tag{1}$$

From Proposition 3.5 it follows that $|\mathcal{G}_{i\omega}| \le 3$ for all $\omega \in \mathcal{I} \setminus \{i, -i\}$. Particular attention will be given to the elements $\omega \in \mathcal{I} \setminus \{i, -i\}$ such that $|\mathcal{G}_{i\omega}| = 3$ or $|\mathcal{G}_{i\omega}| = 2$.

Throughout this subsection $\mathcal{J}$ and $\mathcal{K}$ will denote the following sets:

$$\mathcal{J} = \{j \in \mathcal{I} \setminus \{i, -i\} : |\mathcal{G}_{ij}| = 3\}$$

and

$$\mathcal{K} = \{k \in \mathcal{I} \setminus \{i, -i\} : |\mathcal{G}_{ik}| = 2\}.$$

We begin by characterizing partially the index distribution of the codewords $W_1, \ldots, W_8 \in \mathcal{G}_i$.

Proposition 4.2. *If $|\mathcal{G}_i| = 8$, $i \in \mathcal{I}$, then $\mathcal{I} \setminus \{i, -i\} = \mathcal{J} \cup \mathcal{K}$, with $|\mathcal{J}| = 8$ and $|\mathcal{K}| = 4$. The partial index distribution of the codewords $W_1, \ldots, W_8 \in \mathcal{G}_i$ satisfies:*

Table 1

W_1	i	k_1	x	y
W_2	i	k_2	x	$-y$
W_3	i	k_3	x	
W_4	i	k_4	$-x$	y
W_5	i	k_5	$-x$	$-y$
W_6	i	k_6	$-x$	
W_7	i	k_7	y	
W_8	i	k_8	$-y$	

where $x, -x, y, -y \in \mathcal{J}$ and $k_1, \ldots, k_8 \in \mathcal{K}$. Consequently, for all $W \in \mathcal{G}_i$ there exists a unique element $k \in \mathcal{K}$ such that $W \in \mathcal{G}_{ik}$.

Proof. Let $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 8$. In these conditions, (1) is satisfied. By Proposition 3.5, for any $\omega \in \mathcal{I}\backslash\{i, -i\}$ we get $|\mathcal{G}_{i\omega}| \le 3$. As $|\mathcal{I}\backslash\{i, -i\}| = 12$, taking into account (1) we conclude that there are, at least, eight elements $\omega \in \mathcal{I}\backslash\{i, -i\}$ satisfying $|\mathcal{G}_{i\omega}| = 3$. We have just concluded that $|\mathcal{J}| \ge 8$.

Let us consider

$$\mathcal{L} = \{l \in \mathcal{I}\backslash\{i, -i\} : |\mathcal{G}_{il}| \le 2\}.$$

Observing that, $\mathcal{J} \cup \mathcal{L} = \mathcal{I}\backslash\{i, -i\}$, $\mathcal{J} \cap \mathcal{L} = \varnothing$, $|\mathcal{I}\backslash\{i, -i\}| = 12$ and $|\mathcal{J}| \ge 8$, then $|\mathcal{L}| \le 4$. Thus, there are, at most, four distinct elements $j \in \mathcal{J}$ such that $-j \in \mathcal{L}$. Since $|\mathcal{J}| \ge 8$, there exist $x, y \in \mathcal{J}$, distinct, such that $-x, -y \in \mathcal{J}$. Then, let us consider $x, -x, y, -y \in \mathcal{J}$.

By definition of $\mathcal{J}$, $|\mathcal{G}_{ix}| = |\mathcal{G}_{i,-x}| = |\mathcal{G}_{iy}| = |\mathcal{G}_{i,-y}| = 3$. Taking into account Lemma 3.3, the partial index distribution of the codewords $W_1, \ldots, W_8 \in \mathcal{G}_i$ must satisfy the conditions presented in the Table 2, in which $W_1 \in \mathcal{G}_{ixy}$, $W_2 \in \mathcal{G}_{i,x,-y}$ and so on.

Table 2. Partial index distribution of the codewords of $\mathcal{G}_i$.

W_1	i	x	y
W_2	i	x	$-y$
W_3	i	x	
W_4	i	$-x$	y
W_5	i	$-x$	$-y$
W_6	i	$-x$	
W_7	i	y	
W_8	i	$-y$	

Looking at $W_1 \in \mathcal{G}_{ixy}$, there are $\alpha, \beta \in \mathcal{I}\backslash\{i, -i, x, -x, y, -y\}$ such that $W_1 \in \mathcal{G}_{ixy\alpha\beta}$. Suppose that $\alpha, \beta \in \mathcal{J}$, that is, $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = 3$. Taking into account Lemma 3.3, $|\mathcal{G}_{ix\alpha}| = |\mathcal{G}_{iy\alpha}| = |\mathcal{G}_{ix\beta}| = |\mathcal{G}_{iy\beta}| = 1$. Besides, $\mathcal{G}_{ix\alpha} = \mathcal{G}_{iy\alpha} = \mathcal{G}_{ix\beta} = \mathcal{G}_{iy\beta} = \{W_1\}$. Since $|\mathcal{G}_{i\alpha}| = 3$, taking into account Table 2 and Lemma 3.3, $\mathcal{G}_{i\alpha}\backslash\{W_1\} \subset \{W_5, W_6, W_8\}$ and $\mathcal{G}_{i\beta}\backslash\{W_1\} \subset \{W_5, W_6, W_8\}$. As $|\mathcal{G}_{i\alpha}\backslash\{W_1\}| = |\mathcal{G}_{i\beta}\backslash\{W_1\}| = 2$, there exists $W \in \{W_5, W_6, W_8\}$ such that $W \in \mathcal{G}_{i\alpha\beta}$, which contradicts Lemma 3.3 since $W, W_1 \in \mathcal{G}_{i\alpha\beta}$. Therefore, there exists $l_1 \in \mathcal{L}$ so that $W_1 \in \mathcal{G}_{ixyl_1}$. Similarly, there are $l_2, l_4, l_5 \in \mathcal{L}$ such that $W_2 \in \mathcal{G}_{i,x,-y,l_2}$, $W_4 \in \mathcal{G}_{i,-x,y,l_4}$ and $W_5 \in \mathcal{G}_{i,-x,-y,l_5}$.

Let us consider $W_3 \in \mathcal{G}_{ix}$. Having in view $W_1, W_2 \in \mathcal{G}_{ix}$ and Lemma 3.3, there are $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i, -i, x, -x, y, -y\}$ so that $W_3 \in \mathcal{G}_{ix\alpha\beta\gamma}$. Assume that $\{\alpha, \beta, \gamma\} \subset \mathcal{J}$. Then, $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = 3$. Accordingly, considering Lemma 3.3, we get $|\mathcal{G}_{ix\alpha}| = |\mathcal{G}_{ix\beta}| = |\mathcal{G}_{ix\gamma}| = 1$ and, as a consequence, $\mathcal{G}_{ix\alpha} = \mathcal{G}_{ix\beta} = \mathcal{G}_{ix\gamma} = \{W_3\}$. Taking into account Table 2 and Lemma 3.3, we obtain: $\mathcal{G}_{i\alpha}\backslash\{W_3\} \subset \{W_4, \ldots, W_8\}$; $\mathcal{G}_{i\beta}\backslash\{W_3\} \subset \{W_4, \ldots, W_8\}$; $\mathcal{G}_{i\gamma}\backslash\{W_3\} \subset \{W_4, \ldots, W_8\}$. Since $|\mathcal{G}_{i\alpha}\backslash\{W_3\}| = |\mathcal{G}_{i\beta}\backslash\{W_3\}| = |\mathcal{G}_{i\gamma}\backslash\{W_3\}| = 2$ and $|\{W_4, \ldots, W_8\}| = 5$, there exists $W \in \{W_4, \ldots, W_8\}$ such that $W \in \mathcal{G}_{i\varepsilon\theta}$ for $\varepsilon, \theta \in \{\alpha, \beta, \gamma\}$, which contradicts Lemma 3.3 since $W, W_3 \in \mathcal{G}_{i\varepsilon\theta}$. Thus, there exists $l_3 \in \mathcal{L}$ such that $W_3 \in \mathcal{G}_{ixl_3}$. Likewise, there are $l_6, l_7, l_8 \in \mathcal{L}$ such that $W_6 \in \mathcal{G}_{i,-x,l_6}$, $W_7 \in \mathcal{G}_{iyl_7}$ and $W_8 \in \mathcal{G}_{i,-y,l_8}$.

Therefore, for all $W \in \mathcal{G}_i$ there exists $l \in \mathcal{L}$ such that $W \in \mathcal{G}_{il}$.

By definition of $\mathcal{L}$, $|\mathcal{G}_{il}| \leq 2$ for all $l \in \mathcal{L}$. We have concluded before that $|\mathcal{L}| \leq 4$. Since for any $W \in \mathcal{G}_i$ there exists $l \in \mathcal{L}$ such that $W \in \mathcal{G}_{il}$ and $|\mathcal{G}_i| = 8$, we must impose $|\mathcal{L}| = 4$ and $|\mathcal{G}_{il}| = 2$ for any $l \in \mathcal{L}$. That is, $\mathcal{K} = \{k \in \mathcal{I}\backslash\{i,-i\} : |\mathcal{G}_{ik}| = 2\}$ is such that $|\mathcal{K}| = 4$. Consequently, for each $W \in \mathcal{G}_i$ there exists a unique element $k \in \mathcal{K}$ such that $W \in \mathcal{G}_{ik}$. Furthermore, $|\mathcal{J}| = 8$, $\mathcal{I}\backslash\{i,-i\} = \mathcal{J} \cup \mathcal{K}$ and the partial index distribution of the codewords of $\mathcal{G}_i$ satisfies the conditions which are given in the statement of this proposition. □

The following result characterizes in more detail the set $\mathcal{K}$ and, consequently, the set $\mathcal{J}$.

Proposition 4.3. *If $k \in \mathcal{K}$, then $-k \in \mathcal{K}$.*

Proof. We are assuming $|\mathcal{G}_i| = 8$ for $i \in \mathcal{I}$. The partial index distribution of the codewords $W_1, \ldots, W_8 \in \mathcal{G}_i$ satisfies the conditions enunciated in Proposition 4.2. We recall that, from this proposition it follows that $\mathcal{I}\backslash\{i,-i\} = \mathcal{J} \cup \mathcal{K}$, with $|\mathcal{J}| = 8$ and $|\mathcal{K}| = 4$. Furthermore, $\{x, -x, y, -y\} \subset \mathcal{J}$ and $\{k_1, \ldots, k_8\} = \mathcal{K}$.

Let us consider $\mathcal{N} = \mathcal{J}\backslash\{x, -x, y, -y\} = \{\alpha, \beta, \gamma, \delta\}$. We note that,

$$\mathcal{I}\backslash\{i,-i\} = \{k_1, \ldots, k_8\} \cup \{x, -x, y, -y\} \cup \{\alpha, \beta, \gamma, \delta\}.$$

By Proposition 4.2, for each $W \in \mathcal{G}_i$ there exists a unique element $k \in \mathcal{K}$ such that $W \in \mathcal{G}_{ik}$. On the other hand, since $|\mathcal{G}_{ij}| = 3$ for all $j \in \mathcal{J}$, we have identified all codewords of $\mathcal{G}_{ix}$, $\mathcal{G}_{i,-x}$, $\mathcal{G}_{iy}$ and $\mathcal{G}_{i,-y}$. Thus, to characterize completely the index distribution of all codewords of $\mathcal{G}_i$ we must fill in with elements of $\mathcal{N}$ the empty entries of the table presented in Proposition 4.2.

Consider $W_1, W_2, W_3 \in \mathcal{G}_{ix}$, see table in Proposition 4.2. Taking into account Lemma 3.3, the index distribution of the codewords of $\mathcal{G}_{ix}$ must satisfy the conditions in Table 3.

Table 3. Partial index distribution of the codewords of $\mathcal{G}_i$.

W_1	i	k_1	x	y	α
W_2	i	k_2	x	$-y$	β
W_3	i	k_3	x	γ	δ
W_4	i	k_4	$-x$	y	
W_5	i	k_5	$-x$	$-y$	
W_6	i	k_6	$-x$		
W_7	i	k_7	y		
W_8	i	k_8	$-y$		

Let us now consider the codeword $W_4 \in \mathcal{G}_{i,k_4,-x,y}$. Having in mind Lemma 3.3 we conclude that $W_4 \notin \mathcal{G}_\alpha$, otherwise we would get $W_1, W_4 \in \mathcal{G}_{iy\alpha}$. Suppose that $W_4 \in \mathcal{G}_\beta$. In these conditions, $W_4, W_2 \in \mathcal{G}_{i\beta}$, with $W_4 \in \mathcal{G}_{i,k_4,-x,y,\beta}$ and $W_2 \in \mathcal{G}_{i,k_2,x,-y,\beta}$. Since $|\mathcal{G}_{i\beta}| = 3$ ($\beta \in \mathcal{J}$), there exists $W \in \mathcal{G}_i\backslash\{W_1, W_2, W_3, W_4\}$ such that $W \in \mathcal{G}_{i\beta}$. By Table 3 we verify that $W \in \mathcal{G}_{i,\beta,-x} \cup \mathcal{G}_{i\beta y} \cup \mathcal{G}_{i,\beta,-y}$. Consequently, taking into account W_2 and W_4, $|\mathcal{G}_{i\beta z}| \geq 2$ for some $z \in \{-x, y, -y\}$, contradicting Lemma 3.3.

Therefore, $W_4 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$. By a similar reasoning, we are led to the conclusion that $W_5 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$.

We are assuming $W_3 \in \mathcal{G}_{ik_3x\gamma\delta}$. As $k_3 \in \mathcal{K}$, by definition of $\mathcal{K}$ we get $|\mathcal{G}_{ik_3}| = 2$. Thus, there exists $k \in \{k_1,...,k_8\}\backslash\{k_3\}$ such that $k = k_3$. We note that, $k_3 \neq k_1, k_2$, otherwise Lemma 3.3 is contradicted. Since $W_4, W_5 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$, taking into account Lemma 3.3 we conclude that $k_3 \neq k_4, k_5$. Therefore, $k \in \{k_6, k_7, k_8\}$. If $k_3 = k_7$, then Lemma 3.3 forces $W_7 \in \mathcal{G}_{ik_7y\alpha\beta}$, which is a contradiction, since $W_1, W_7 \in \mathcal{G}_{iy\alpha}$. Then, $k_3 \neq k_7$. By a similar reasoning we may conclude that $k_3 \neq k_8$. Consequently, $k_3 = k_6$ and, applying once again Lemma 3.3, we must impose $W_6 \in \mathcal{G}_{i,k_3,-x,\alpha,\beta}$.

Note that $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = 3$. Since $W_4, W_5 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$, we must obligate $W_7, W_8 \in \mathcal{G}_\alpha \cup \mathcal{G}_\beta$. Considering W_1 and W_2, Lemma 3.3 leads us to conclude that $W_7 \in \mathcal{G}_\beta$ and $W_8 \in \mathcal{G}_\alpha$.

Accordingly, the partial index distribution of the codewords of $\mathcal{G}_i$ satisfies:

Table 4. Partial index distribution of the codewords of $\mathcal{G}_i$.

W_1	i	k_1	x	y	α
W_2	i	k_2	x	$-y$	β
W_3	i	k_3	x	γ	δ
W_4	i	k_4	$-x$	y	
W_5	i	k_5	$-x$	$-y$	
W_6	i	k_3	$-x$	α	β
W_7	i	k_7	y	β	
W_8	i	k_8	$-y$	α	

Note that, as $|\mathcal{G}_{i\gamma}| = |\mathcal{G}_{i\delta}| = 3$, the four empty entries of this table must be filled in with γ and δ. Thus, $W_4, W_5, W_7, W_8 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$.

Consider the elements of $\mathcal{K}$. By the analysis of the entries of the previous table, to avoid the contradiction of Lemma 3.3, one should have $k_1 = k_5$, $k_2 = k_4$ and $k_7 = k_8$. That is, $\mathcal{K} = \{k_1, k_2, k_3, k_7\}$ and the codewords of $\mathcal{G}_i$ are characterize as it is presented in Table 5.

Table 5. Partial index distribution of the codewords of $\mathcal{G}_i$.

W_1	i	k_1	x	y	α
W_2	i	k_2	x	$-y$	β
W_3	i	k_3	x	γ	δ
W_4	i	k_2	$-x$	y	
W_5	i	k_1	$-x$	$-y$	
W_6	i	k_3	$-x$	α	β
W_7	i	k_7	y	β	
W_8	i	k_7	$-y$	α	

We intend to show that if $k \in \mathcal{K}$, then $-k \in \mathcal{K}$. Let us focus our attention on $k_3 \in \mathcal{K}$. We have concluded before that $W_3, W_6 \in \mathcal{G}_{ik_3}$, with $W_3 \in \mathcal{G}_{ik_3x\gamma\delta}$ and $W_6 \in \mathcal{G}_{i,k_3,-x,\alpha,\beta}$. In these conditions, $-k_3 \in \mathcal{I}\backslash(\{i, -i, x, -x, y, -y\} \cup \mathcal{N})$. That is, $-k_3 \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$. Since $\mathcal{I} = \{i, -i\} \cup \mathcal{J} \cup \mathcal{K}$, then $-k_3 \in \mathcal{K}$.

Looking at the codewords $W_7, W_8 \in \mathcal{G}_{ik_7}$, we get $W_7 \in \mathcal{G}_\gamma$ and $W_8 \in \mathcal{G}_\delta$, or, $W_7 \in \mathcal{G}_\delta$ and $W_8 \in \mathcal{G}_\gamma$. In both cases $-k_7 \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$, accordingly $-k_7 \in \mathcal{K}$.

Now, $\mathcal{K} = \{k_1, k_2, k_3, k_7\}$ and $-k_3, -k_7 \in \mathcal{K}$. Either $k_3 \neq -k_7$ or $k_3 = -k_7$.

If $k_3 \neq -k_7$, then $-k \in \mathcal{K}$ for all $k \in \mathcal{K}$.

If $k_3 = -k_7$ and $k_1 = -k_2$, then $-k \in \mathcal{K}$ for all $k \in \mathcal{K}$.

Assume that $k_3 = -k_7$ and $k_1 \neq -k_2$. By this assumption it follows that $-k_1, -k_2 \in \mathcal{N} = \{\alpha, \beta, \gamma, \delta\}$. Thus, there are $\varepsilon_1, \varepsilon_2 \in \mathcal{N}$ so that $-k_1 = \varepsilon_1$, $-k_2 = \varepsilon_2$ and the remaining elements of $\mathcal{N}$, ε_3 and ε_4, satisfy $\varepsilon_3 = -\varepsilon_4$. As $W_1 \in \mathcal{G}_{ik_1xy\alpha}$, then $-k_1 \in \{\beta, \gamma, \delta\}$. On the other hand, since $W_2 \in \mathcal{G}_{i,k_2,x,-y,\beta}$, then $-k_2 \in \{\alpha, \gamma, \delta\}$. We note that, as $k_1 \neq k_2$, then $-k_1 \neq -k_2$.

If $-k_1 = \beta$ and $-k_2 = \alpha$, then $\gamma = -\delta$, which is a contradiction since $W_3 \in \mathcal{G}_{ik_3x\gamma\delta}$.

If $-k_1 = \beta$ and $-k_2 = \gamma$, then $\alpha = -\delta$. Analyzing Table 5 and taking into account that $W_4 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$, we conclude that $W_4 \in \mathcal{G}_{i,k_2,-x,y,\delta}$. Consequently, having in mind Lemma 3.3, $W_5 \in \mathcal{G}_{i,k_1,-x,-y,\gamma}$, $W_7 \in \mathcal{G}_{ik_7y\beta\gamma}$ and $W_8 \in \mathcal{G}_{i,k_7,-y,\alpha,\delta}$, which is not possible since we are supposing $\alpha = -\delta$.

If $-k_1 = \beta$ and $-k_2 = \delta$, then $\alpha = -\gamma$. Consequently, $W_8 \in \mathcal{G}_{i,k_7,-y,\alpha,\delta}$, $W_7 \in \mathcal{G}_{ik_7y\beta\gamma}$ and $W_4 \in \mathcal{G}_{i,k_2,-x,y,\delta}$. We get a contradiction since, by hypothesis, $-k_2 = \delta$.

Combining all possibilities for $-k_1 \in \{\beta, \gamma, \delta\}$ and $-k_2 \in \{\alpha, \gamma, \delta\}$, by a similar reasoning we get always a contradiction. Therefore, $-k \in \mathcal{K}$ for all $k \in \mathcal{K}$. □

From Proposition 4.2 we get $\mathcal{I}\backslash\{i, -i\} = \mathcal{J} \cup \mathcal{K}$. We have just seen that, if $k \in \mathcal{K}$ then $-k \in \mathcal{K}$. So, if $j \in \mathcal{J}$ then $-j \in \mathcal{J}$.

Until this moment we have focused our attention on the characterization of the codewords of $\mathcal{G}_i$. The two following propositions arise from the analysis of other type of codewords, in particular, codewords of $\mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$.

Proposition 4.4. *If* $|\mathcal{G}_i| = 8$, $i \in \mathcal{I}$, *then* $|\mathcal{F}_i| = 0$.

Proof. Let $|\mathcal{G}_i| = 8$ for $i \in \mathcal{I}$. Suppose, by contradiction, that $|\mathcal{F}_i| > 0$. Let $U \in \mathcal{F}_i$. Since the codewords of $\mathcal{F}$ are of type $[\pm 2, \pm 1^3]$, there exist $u_1, u_2, u_3 \in \mathcal{I} \backslash \{i, -i\}$, with $|u_1|, |u_2|$ and $|u_3|$ pairwise distinct, such that $U \in \mathcal{F}_{iu_1u_2u_3}$.

By Proposition 4.2, $\mathcal{I} \backslash \{i, -i\} = \mathcal{J} \cup \mathcal{K}$, therefore $u_1, u_2, u_3 \in \mathcal{J} \cup \mathcal{K}$. Recall that $|\mathcal{G}_{ij}| = 3$ for any $j \in \mathcal{J}$. Then, by Proposition 3.5 one has $|\mathcal{F}_{ij}| = 0$ for all $j \in \mathcal{J}$. Consequently, $u_1, u_2, u_3 \in \mathcal{K}$. From Proposition 4.2 it follows that $|\mathcal{K}| = 4$ and, taking into account Proposition 4.3, $-k \in \mathcal{K}$ for all $k \in \mathcal{K}$. Thus, is not possible to have $u_1, u_2, u_3 \in \mathcal{K}$ satisfying $|u_1|, |u_2|$ and $|u_3|$ pairwise distinct, contradicting our assumption. □

Proposition 4.5. *For all* $j \in \mathcal{J}$, $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$. *For all* $k \in \mathcal{K}$, $|\mathcal{D}_{ik} \cup \mathcal{E}_{ik}| = 4$. *Furthermore, if* $k \in \mathcal{K}$, *the codewords* $U_1, U_2, U_3, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$ *are such that* $U_1 \in \mathcal{D}_{iku_1} \cup \mathcal{E}_{iku_1}$, $U_2 \in \mathcal{D}_{iku_2} \cup \mathcal{E}_{iku_2}$, $U_3 \in \mathcal{D}_{iku_3} \cup \mathcal{E}_{iku_3}$ *and* $U_4 \in \mathcal{D}_{iku_4} \cup \mathcal{E}_{iku_4}$, *with* $u_1, u_2 \in \mathcal{J}$, $u_1 \neq u_2$, *and* $u_3, u_4 \in \mathcal{K} \backslash \{k, -k\}$, *with* $u_3 = -u_4$.

Proof. From Proposition 3.5 we get

$$|\mathcal{D}_{i\omega} \cup \mathcal{E}_{i\omega}| + 2|\mathcal{F}_{i\omega}| + 3|\mathcal{G}_{i\omega}| = 10 \tag{2}$$

for all $\omega \in \mathcal{I} \backslash \{i, -i\}$. By Proposition 4.4 we know that $|\mathcal{F}_i| = 0$ and, consequently, $|\mathcal{F}_{i\omega}| = 0$ for all $\omega \in \mathcal{I} \backslash \{i, -i\}$. As $|\mathcal{G}_{ij}| = 3$ for any $j \in \mathcal{J}$, from (2) we obtain $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$ for all $j \in \mathcal{J}$. Considering again (2), we conclude that $|\mathcal{D}_{ik} \cup \mathcal{E}_{ik}| = 4$ for each $k \in \mathcal{K}$, since $|\mathcal{G}_{ik}| = 2$ for all $k \in \mathcal{K}$.

Let $k \in \mathcal{K}$. Then, there exist $V_1, V_2 \in \mathcal{G}_{ik}$ and $U_1, \ldots, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$. We note that, the codewords of $\mathcal{D}$ are of type $[\pm 3, \pm 1^2]$ and the codewords of $\mathcal{E}$ are of type $[\pm 2^2, \pm 1]$. Thus, there are $v_1, \ldots, v_6, u_1, \ldots, u_4$ in $\mathcal{I} \backslash \{i, -i, k, -k\}$ such that:

Table 6. Index distribution of the codewords of $\mathcal{G}_{ik} \cup \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$.

V_1	i	k	v_1	v_2	v_3
V_2	i	k	v_4	v_5	v_6

U_1	i	k	u_1
U_2	i	k	u_2
U_3	i	k	u_3
U_4	i	k	u_4

It should be pointed out that, by Lemma 3.3, $v_1, \ldots, v_6, u_1, \ldots, u_4$ must be pairwise distinct. Therefore, $\{v_1, \ldots, v_6, u_1, \ldots, u_4\} = \mathcal{I} \backslash \{i, -i, k, -k\}$. By Proposition 4.2, $\mathcal{I} \backslash \{i, -i\} = \mathcal{J} \cup \mathcal{K}$, with $|\mathcal{J}| = 8$ and $|\mathcal{K}| = 4$. Furthermore, from Proposition 4.3, $-k \in \mathcal{K}$. Then, $\{v_1, \ldots, v_6, u_1, \ldots, u_4\} = \mathcal{J} \cup \mathcal{K} \backslash \{k, -k\}$. Since $V_1, V_2 \in \mathcal{G}_{ik}$, with $k \in \mathcal{K}$, taking into account Proposition 4.2 we must impose $\{v_1, \ldots, v_6\} \subset \mathcal{J}$. Consequently, without loss of generality, $u_1, u_2 \in \mathcal{J}$ and $u_3, u_4 \in \mathcal{K} \backslash \{k, -k\}$. Considering Proposition 4.3 we conclude that $u_3 = -u_4$. □

We are now able to establish the main result of this paper.

Theorem 4.6. *For any* $i \in \mathcal{I}$, $|\mathcal{G}_i| \neq 8$.

Proof. By contradiction, consider $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 8$.

From Proposition 4.2 we have $|\mathcal{K}| = 4$, so let k be an element of $\mathcal{K}$. By Proposition 4.5, there exist $U_1, \ldots, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$ whose index distribution satisfies the conditions presented in Table 7, where $u, -u \in \mathcal{K} \backslash \{k, -k\}$ and $j_1, j_2 \in \mathcal{J}$, with $j_1 \neq j_2$. We note that, in these conditions, $\mathcal{K} = \{k, -k, u, -u\}$.

Table 7. Index distribution of the codewords of $\mathcal{D}_{ik} \cup \mathcal{E}_{ik}$.

U_1	i	k	u
U_2	i	k	$-u$
U_3	i	k	j_1
U_4	i	k	j_2

Let us denote by $\mathcal{H}$ the set of words of type $[\pm 2, \pm 1]$. Consider the words $P_1, P_2 \in \mathcal{H}$ such that $P_1 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_1}^{(1)}$ and $P_2 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_2}^{(1)}$. The index distribution of the codewords of $\mathcal{D}_{ik} \cup \mathcal{E}_{ik}$ and the index value distribution of the words P_1 and P_2 are represented in the following table:

Table 8. Index distribution of $U_1, \dots, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$ and index value distribution of $P_1, P_2 \in \mathcal{H}_i$.

	i	k	u	$-u$	j_1	j_2
U_1	x	x	x			
U_2	x	x		x		
U_3	x	x			x	
U_4	x	x				x
P_1	± 2				± 1	
P_2	± 2					± 1

By definition of perfect 2-error correcting Lee code, for each $P \in \{P_1, P_2\}$ there exists a unique codeword $V \in \mathcal{T}$ such that $\mu_L(P, V) \leq 2$. Taking into account the type of words of $\mathcal{H}$ as well as the fact of $|\mathcal{F}_i| = 0$ (see Proposition 4.4), each word $P_q \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_q}$, with $j_q \in \mathcal{I} \backslash \{i, -i\}$, is covered by a unique codeword

$$V_q \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_q}^{(1)}) \cup \mathcal{C}_{ij_q} \cup (\mathcal{D}_i^{(3)} \cap \mathcal{D}_{j_q}^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_q}). \tag{3}$$

Thus, we may consider U_3 and U_4 as possible codewords to cover P_1 and P_2, respectively.

Suppose that P_1 is covered by U_3 and P_2 is covered by U_4. Then, we must impose

$$U_3 \in (\mathcal{D}_i^{(3)} \cap \mathcal{D}_k^{(1)} \cap \mathcal{D}_{j_1}^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_k \cap \mathcal{E}_{j_1})$$

and

$$U_4 \in (\mathcal{D}_i^{(3)} \cap \mathcal{D}_k^{(1)} \cap \mathcal{D}_{j_2}^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_k \cap \mathcal{E}_{j_2}),$$

which contradicts Lemma 3.2, since $U_3, U_4 \in (\mathcal{D}_i^{(3)} \cap \mathcal{D}_k^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_k)$. Therefore, either P_1 is not covered by U_3 or P_2 is not covered by U_4.

Without loss of generality, let us assume that P_1 is not covered by U_3. Note that, $U_3 \in \mathcal{D}_{ikj_1} \cup \mathcal{E}_{ikj_1}$. As $j_1 \in \mathcal{J}$, by Proposition 4.5 we get $|\mathcal{D}_{ij_1} \cup \mathcal{E}_{ij_1}| = 1$. Consequently, $\mathcal{D}_{ij_1} \cup \mathcal{E}_{ij_1} = \{U_3\}$. Since we are assuming that U_3 does not cover P_1, considering (3), P_1 is covered by a codeword V_1 satisfying $V_1 \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}) \cup \mathcal{C}_{ij_1}$.

Next, we will analyze, separately, the hypotheses:

1) $V_1 \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$;
2) $V_1 \in \mathcal{C}_{ij_1}$.

(1) Assume that P_1 is covered by $V_1 \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$.

Assuming that P_1 is covered by $V_1 \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$, by Lemma 3.1 we conclude $|\mathcal{B}_i^{(4)} \backslash \{V_1\} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 0$. Consequently, if $U \in \{U_1, \dots, U_4\}$ is such that $U \in \mathcal{D}$, then $U \in \mathcal{D}_i^{(1)}$. Furthermore, P_2 must be covered by

$$V_2 \in (\mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_2}^{(3)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_2}).$$

If $V_2 \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_2}$, since $j_2 \in \mathcal{J}$ we conclude, by Proposition 4.5, that $V_2 = U_4$. Having in mind U_1, U_2 and U_3, see Table 8, if $U \in \{U_1, U_2, U_3\}$ is such that $U \in \mathcal{E}$, then $U \in \mathcal{E}_i^{(1)}$, otherwise, $U, U_4 \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_k$, contradicting Lemma 3.2. Therefore, since we have concluded before that $\{U_1, U_2, U_3\} \cap \mathcal{D}_i^{(3)} = \varnothing$, we get $U_1, U_2, U_3 \in \mathcal{D}_i^{(1)} \cup \mathcal{E}_i^{(1)}$. Taking into account the index distribution of U_1 and U_2, we must have $U_1 \in \mathcal{D}_u^{(3)}$ or $U_2 \in \mathcal{D}_{-u}^{(3)}$, otherwise we get $U_1, U_2 \in (\mathcal{D}_i^{(1)} \cap \mathcal{D}_k^{(3)}) \cup (\mathcal{E}_i^{(1)} \cap \mathcal{E}_k^{(2)})$, contradicting, once again, Lemma 3.2.

If $V_2 \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_2}^{(3)}$, to avoid the contradiction of Lemma 3.2 we must impose $U_4 \in \mathcal{D}_k^{(3)}$. Consequently, considering again Lemma 3.2, $U_1, U_2, U_3 \in \mathcal{D}_k^{(1)} \cup \mathcal{E}_k^{(1)}$. We recall that, we have seen before that $\{U_1, U_2, U_3\} \cap \mathcal{D}_i^{(3)} = \varnothing$. Thus, in these conditions, $U_1 \in \mathcal{D}_u^{(3)}$ or $U_2 \in \mathcal{D}_{-u}^{(3)}$, otherwise, $U_1, U_2 \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_k^{(1)}$, contradicting again Lemma 3.2.

Therefore, in both cases, supposing $V_2 \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_2}$ or $V_2 \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_2}^{(3)}$, we conclude that $U_1 \in \mathcal{D}_u^{(3)}$ or $U_2 \in \mathcal{D}_{-u}^{(3)}$.

Suppose, without loss of generality, that $U_1 \in \mathcal{D}_u^{(3)}$. As $u \in \mathcal{K}$, by Proposition 4.5 there are $U_5, U_6 \in \mathcal{D}_{iu} \cup \mathcal{E}_{iu}$ satisfying $U_5 \in \mathcal{D}_{iuj_3} \cup \mathcal{E}_{iuj_3}$ and $U_6 \in \mathcal{D}_{iuj_4} \cup \mathcal{E}_{iuj_4}$, with $j_3, j_4 \in \mathcal{J}$ distinct. Note that, $j_1, \ldots, j_4 \in \mathcal{J}$ are pairwise distinct, since by Proposition 4.5 we have $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$ for all $j \in \mathcal{J}$.

Let us consider $P_3 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_3}^{(1)}$ and $P_4 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_4}^{(1)}$. Table 9 summarizes the conditions that the index distribution, and, in some cases, the index value distribution, of the codewords and words described until now, must satisfy.

Table 9. Index conditions on $\mathcal{B}_i \cup \mathcal{D}_i \cup \mathcal{E}_i$ and on 4 words of type $[\pm 2, \pm 1]$.

	i	k	u	$-u$	j_1	j_2	j_3	j_4
U_1	±1	±1	±3					
U_2	x	x		x				
U_3	x	x			x			
U_4	x	x				x		
P_1	±2				±1			
P_2	±2					±1		
V_1	±4				±1			
U_5	x		x				x	
U_6	x		x					x
P_3	±2						±1	
P_4	±2							±1

Taking into account the words P_3 and P_4 we may conclude, as we have concluded before for P_1 and P_2, that either P_3 is not covered by U_5 or P_4 is not covered by U_6. In fact, if U_5 covers P_3 and U_6 covers P_4, then $U_5, U_6 \in (\mathcal{D}_i^{(3)} \cap \mathcal{D}_u^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_u)$, contradicting Lemma 3.2. Let us assume, without loss of generality, that P_3 is not covered by U_5. By Proposition 4.5 it follows that $|\mathcal{D}_{ij_3} \cup \mathcal{E}_{ij_3}| = 1$. Consequently, $\mathcal{D}_{ij_3} \cup \mathcal{E}_{ij_3} = \{U_5\}$. As a consequence of the assumption $V_1 \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$ we get $|\mathcal{B}_i^{(4)} \backslash \{V_1\} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 0$. Thus, under these conditions and taking into account (3), P_3 must be covered by a codeword V_3 satisfying $V_3 \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_3}^{(3)}$. Consequently, $U_5 \in \mathcal{D}_u^{(3)}$, otherwise, $U_5 \in (\mathcal{D}_i^{(1)} \cap \mathcal{D}_{j_3}^{(3)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_3}) \cup (\mathcal{E}_i \cap \mathcal{E}_{j_3}^{(2)})$ and contradicts with the codeword V_3 Lemma 3.2. However, $U_1, U_5 \in \mathcal{D}_u^{(3)}$, contradicting Lemma 3.1.

Accordingly, P_1 can not be covered by the codeword $V_1 \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$.

2**) Assume that** P_1 **is covered by** $V_1 \in \mathcal{C}_{ij_1}$**.**

Since $V_1 \in \mathcal{C}$, then V_1 is a codeword of type $[\pm 3, \pm 2]$. According with what is being supposed, $V_1 \in \mathcal{C}_i^{(3)} \cap \mathcal{C}_{j_1}^{(2)}$ or $V_1 \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_1}^{(3)}$. Consider $U_3 \in \mathcal{D}_{ikj_1} \cup \mathcal{E}_{ikj_1}$. In order to have Lemma 3.2 fulfilled we must force $U_3 \in \mathcal{D}_i^{(1)} \cap \mathcal{D}_k^{(3)} \cap \mathcal{D}_{j_1}^{(1)}$. Schematically, we get Table 10.

Taking into account U_3, by Lemma 3.2 we must have $U_1, U_2, U_4 \in \mathcal{D}_k^{(1)} \cup \mathcal{E}_k^{(1)}$. Besides, $U_1 \in \mathcal{D}_u^{(3)}$ or $U_2 \in \mathcal{D}_{-u}^{(3)}$, otherwise, $U_1, U_2 \in (\mathcal{D}_i^{(3)} \cap \mathcal{D}_k^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_k^{(1)})$, contradicting Lemma 3.2.

Let us assume, without loss of generality, that $U_1 \in \mathcal{D}_u^{(3)}$.

Table 10. Index distribution on $\mathcal{C}_i \cup \mathcal{D}_i \cup \mathcal{E}_i$ and on 2 words of type $[\pm 2, \pm 1]$.

	i	k	u	$-u$	j_1	j_2
U_1	x	x	x			
U_2	x	x		x		
U_3	± 1	± 3			± 1	
U_4	x	x				x
P_1	± 2				± 1	
P_2	± 2					± 1
V_1	x				x	

Proceeding as in the previous case, we will consider $U_5 \in \mathcal{D}_{iuj_3} \cup \mathcal{E}_{iuj_3}$ and $U_6 \in \mathcal{D}_{iuj_4} \cup \mathcal{E}_{iuj_4}$, with $j_3, j_4 \in \mathcal{J}$ and distinct. We will consider also $P_3 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_3}^{(1)}$ and $P_4 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_4}^{(1)}$. Gathering the information obtained so far, one has the index distribution presented in Table 11.

Table 11. Index distribution on $\mathcal{C}_i \cup \mathcal{D}_i \cup \mathcal{E}_i$ and on 4 words of type $[\pm 2, \pm 1]$.

	i	k	u	$-u$	j_1	j_2	j_3	j_4
U_1	± 1	± 1	± 3					
U_2	x	x		x				
U_3	± 1	± 3			± 1			
U_4	x	x				x		
P_1	± 2				± 1			
P_2	± 2					± 1		
V_1	x				x			
U_5	x		x				x	
U_6	x		x					x
P_3	± 2						± 1	
P_4	± 2							± 1

As seen in the previous case, either U_5 does not cover P_3 or U_6 does not cover P_4. Assume, without loss of generality, that P_3 is not covered by U_5. By Proposition 4.5 we get $\mathcal{D}_{ij_3} \cup \mathcal{E}_{ij_3} = \{U_5\}$. Therefore, considering (3), P_3 must be covered by a codeword $V_3 \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_3}^{(1)}) \cup \mathcal{C}_{ij_3}$. If $V_3 \in \mathcal{C}_{ij_3}$, then, by Lemma 3.2, we must impose $U_5 \in \mathcal{D}_u^{(3)}$ and, consequently, $|\mathcal{D}_u^{(3)}| \ge 2$, contradicting Lemma 3.1. Accordingly, $V_3 \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_3}^{(1)}$.

Taking into account Lemma 3.1, $|\mathcal{B}_i^{(4)} \backslash \{V_3\} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 0$. Thus, by (3) we may conclude that P_4 must be covered by a codeword

$$V_4 \in (\mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_4}^{(3)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_4}).$$

Note that, if $V_4 \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_4}^{(3)}$, then, by Lemma 3.2, $U_6 \in \mathcal{D}_u^{(3)}$ implying $|\mathcal{D}_u^{(3)}| \ge 2$ and contradicting Lemma 3.1. Thus, $V_4 \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_4}$. By Proposition 4.5, $|\mathcal{D}_{ij_4} \cup \mathcal{E}_{ij_4}| = 1$ leading to $\mathcal{D}_{ij_4} \cup \mathcal{E}_{ij_4} = \{U_6\}$ and, consequently, $V_4 = U_6$. Since $U_1 \in \mathcal{D}_i^{(1)} \cap \mathcal{D}_k^{(1)} \cap \mathcal{D}_u^{(3)}$, taking into account Lemma 3.2, we must force $U_6 \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_u^{(1)} \cap \mathcal{E}_{j_4}^{(2)}$. The index distribution, and, in some cases the index value distribution, of the codewords and words which we are dealing with are presented in Table 12.
Let us now focus our attention on $-u \in \mathcal{K}$. By Proposition 4.5, there are codewords $U_7, U_8 \in \mathcal{D}_{i,-u} \cup \mathcal{E}_{i,-u}$, so that, $U_7 \in \mathcal{D}_{i,-u,j_5} \cup \mathcal{E}_{i,-u,j_5}$ and $U_8 \in \mathcal{D}_{i,-u,j_6} \cup \mathcal{E}_{i,-u,j_6}$, with $j_5, j_6 \in \mathcal{J}$ distinct. Note that, by Proposition 4.5, $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$ for all $j \in \mathcal{J}$, and so $j_1, \ldots, j_6$ are pairwise distinct. Taking into account the existence of the words $P_5 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_5}^{(1)}$ and $P_6 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_6}^{(1)}$, we obtain the index distribution presented schematically in Table 13.
By a similar reasoning to the one done with the words $P_1, P_2, P_3, P_4 \in \mathcal{H}_i^{(2)}$, we conclude that either P_5 is not covered by U_7 or P_6 is not covered by U_8. Let us assume, without loss of generality, that U_7 does not cover P_5. Then, considering (3) we are lead to conclude that P_5 must be covered by a codeword

$$P_5 \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_5}^{(1)}) \cup (\mathcal{C}_{ij_5}).$$

Table 12. Index distribution on $\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{D}_i \cup \mathcal{E}_i$ and on 4 words of type $[\pm 2, \pm 1]$.

	i	k	u	$-u$	j_1	j_2	j_3	j_4
U_1	±1	±1	±3					
U_2	x	x		x				
U_3	±1	±3			±1			
U_4	x	x				x		
P_1	±2				±1			
P_2	±2					±1		
V_1	x				x			
U_5	x		x				x	
U_6	±2		±1					±2
P_3	±2						±1	
P_4	±2							±1
V_3	±4						±1	

Table 13. Index distribution on $\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{D}_i \cup \mathcal{E}_i$ and on 6 words of type $[\pm 2, \pm 1]$.

	i	k	u	$-u$	j_1	j_2	j_3	j_4	j_5	j_6
U_1	±1	±1	±3							
U_2	x	x		x						
U_3	±1	±3			±1					
U_4	x	x				x				
P_1	±2				±1					
P_2	±2					±1				
V_1	x				x					
U_5	x		x				x			
U_6	±2		±1					±2		
P_3	±2						±1			
P_4	±2							±1		
V_3	±4						±1			
U_7	x			x					x	
U_8	x			x						x
P_5	±2								±1	
P_6	±2									±1

As $V_3 \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_3}^{(1)}$, by Lemma 3.1, $P_5 \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_5}^{(3)}$. Consequently, taking into account Lemma 3.2, we must force $U_7 \in \mathcal{D}_{-u}^{(3)}$.

Focus our attention on the codeword $U_2 \in \mathcal{D}_{i,k,-u} \cup \mathcal{E}_{i,k,-u}$. Having in mind the index value distribution of the codewords V_3, U_3 and U_7 and considering Lemma 3.1, we conclude that $U_2 \in \mathcal{E}_i$. Consequently, either $U_2 \in \mathcal{E}_i \cap \mathcal{E}_k^{(2)}$ or $U_2 \in \mathcal{E}_i \cap \mathcal{E}_{-u}^{(2)}$. If $U_2 \in \mathcal{E}_i \cap \mathcal{E}_k^{(2)}$, then the index value distribution of U_2 and U_3 contradicts Lemma 3.2. If $U_2 \in \mathcal{E}_i \cap \mathcal{E}_{-u}^{(2)}$, the index value distribution of U_2 and U_7 contradicts also Lemma 3.2.

In both hypotheses, P_1 covered by $V_1 \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$ or P_1 covered by $V_1 \in \mathcal{C}_{ij_1}$, we get a contradiction. □

We have proved in [16] that for each $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 8$. From last theorem it follows immediately:

Corollary 4.7. *For any $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 7$.*

Since $g = |\mathcal{G}| = \frac{1}{5}\sum_{i\in\mathcal{I}} |\mathcal{G}_i|$, the required solutions for the system of equations presented in Proposition 3.4 must satisfy $9 \leq g \leq 19$. As we have said before, our strategy to prove the non-existence of $PL(7, 2)$ codes consists in getting a minimum range for the variation of $|\mathcal{G}_i|$, with $i \in \mathcal{I}$, and consequently to reduce the number of solutions for the referred system of equations.

We have already started working on the analysis of other values for $|\mathcal{G}_i|$, with $i \in \mathcal{I}$, which brings increased difficulties, imposing new strategies and techniques. It seems that our intuition on the new strategy to be applied (from now on) for the proving of the non-existence of $PL(7, 2)$ codes will be successful.

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