

Open Mathematics

Research Article

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A greedy algorithm for interval greedoids

<https://doi.org/10.1515/math-2018-0026>

Received December 31, 2016; accepted February 8, 2018.

Abstract: We show that the greedy algorithm provided in this paper works for interval greedoids with positive weights under some conditions, and also characterize an exchangeable system to be an interval greedoid with the assistance of the greedy algorithm.

Keywords: Interval greedoid, Exchangeable system, Greedy algorithm, Positive weight

MSC: 90C27, 05B35

1 Introduction and Preliminaries

Let $\mathcal{F}$ (called *feasible sets*) be a set system on E (i.e. a non-empty family $\mathcal{F} \subseteq 2^E$ where 2^E is the set of all subsets of a finite set E). We can suppose $\cup \mathcal{F} = E$ in this paper, since $E \setminus \cup \mathcal{F} \neq \emptyset$ will bring $x \in E \setminus \cup \mathcal{F}$ to own nothing in $\mathcal{F}$ which is not interesting for studying. Actually, such supposition is also done in [1].

Let $\omega : E \rightarrow \mathbb{R}^+$ be a weighting on E . Abbreviating $\omega(X) = \sum_{x \in X} \omega(x)$, especially $\omega(\emptyset) = 0$, we want to find an $A \in \mathcal{F}$ satisfying $\omega(A) = \max\{\omega(X) \mid X \in \mathcal{F}\}$. We call this problem $(\mathcal{F}, \omega)$. An element of $\mathcal{F}$ is *optimal* if it has the maximal weight. The *greedy algorithm* for $(E, \mathcal{F})$ attempts to solve the above problem. In fact, Helman et al. [2] point that obtaining an exact characterization of the class of problems for which the greedy algorithm returns an optimal solution has been an open problem. The process of greedy algorithm (cf.[3,p.14]) is as follows.

- (1) Set $X = \emptyset$.
- (2) Set $T = \{x \in E \setminus X \mid X \cup x \in \mathcal{F}\}$,
If $T = \emptyset$, stop;
If $T \neq \emptyset$, choose $x \in T$ such that $\omega(x) \geq \omega(y)$ for all $y \in T$.
- (3) Set $X = X \cup x$ and go to (2).

Björner et al. indicate [1] that greedoids were invented around 1980 by Korte and Lovász. The relative definitions to greedoids are reviewed as follows.

Definition 1.1 ([1,3]). Let $\mathcal{F}$ be a set system on E .

- (1) A *greedoid* is a pair $(E, \mathcal{F})$, where $\mathcal{F}$ satisfies the following conditions:
 - (G1) For every non-empty $X \in \mathcal{F}$, there is an $x \in X$ such that $X \setminus \{x\} \in \mathcal{F}$. (*accessible*)
 - (G2) For $X, Y \in \mathcal{F}$ such that $|X| > |Y|$, there is an $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{F}$. (*exchangeable*)
- (2) A greedoid $(E, \mathcal{F})$ has the *interval property* (or to be an *interval greedoid*) if $A \subseteq B \subseteq C$, $A, B, C \in \mathcal{F}$ and $x \in E \setminus C$, then $A \cup \{x\} \in \mathcal{F}$ and $C \cup \{x\} \in \mathcal{F}$ imply $B \cup \{x\} \in \mathcal{F}$.
- (3) A maximal element in $(\mathcal{F}, \subseteq)$ is called a *basis*.
- (4) A *loop* in $(E, \mathcal{F})$ is an element $x \in E$ that is contained in no basis.
- (5) A language $\mathcal{L}$ over E is a non-empty set $\mathcal{L} \subseteq E^*$ (the free monoid of all words over the alphabet E) of words

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over the alphabet E ; it is called simple if every word in $\mathcal{L}$ is simple (i.e. it does not contain any letter more than once).

A greedoid language over E is a pair $(E, \mathcal{L})$, where $\mathcal{L}$ is a simple language $\mathcal{L} \subseteq E_s^*$ (the set of simple words in E^*) satisfying the following conditions:

(L1) If $\alpha = \beta\gamma$ and $\alpha \in \mathcal{L}$, then $\beta \in \mathcal{L}$. (hereditary)

(L2) If $\alpha, \beta \in \mathcal{L}$ and $|\alpha| > |\beta|$, then α contains a letter x such that $\beta x \in \mathcal{L}$. (exchange).

Björner et al. indicate [1] that greedoids were originally developed to give a unified approach to the optimality of various greedy algorithms known in combinatorial optimization. Such algorithms can be loosely characterized as having locally optimal strategy and no backtracking. Nowadays, researchers provide different greedy algorithms to characterize the different kinds of greedoids (cf. [1,2,3,4]). Helman et al. [2] characterize greedy structures. [2, Theorem 1] is the best and main result in [2]. That is, let $(E, \mathcal{F})$ be an accessible set system. Then $(E, \mathcal{F})$ has an optimal greedy basis for every positive weighted linear function if and only if $(E, \mathcal{F})$ is a matroid embedding (cf. [2, Definition 2], i.e. A matroid embedding is an accessible set system which is extensible, closure-congruent, and the hereditary closure of which is a matroid). Below [2, Definition 2], Helman et al. say that $(S, \mathcal{C}_1)$ in [2, Example 1] is a matroid embedding, yet not a greedoid. Combining [2, Example 2] and the definition of a matroid embedding, $(S, \mathcal{C}_2)$ in [2, Example 2] is a greedoid, yet not a matroid embedding. In other words, [2, Theorem 1] does not characterize a greedoid structure with greedy algorithm. Korte et al. say [4, p.358, Theorem 14.7]: Let $(E, \mathcal{F})$ be a greedoid. The greedy algorithm finds a set $F \in \mathcal{F}$ of maximum weight for each modular weight function if and only if $(E, \mathcal{F})$ has the so-called strong exchange axiom (see [2], [3, p.160], [4, p.358], or say: For all $A \in \mathcal{F}$, B is a maximal in $\mathcal{F}$ and $A \subseteq B$. If $x \in E \setminus B$ with $A \cup \{x\} \in \mathcal{F}$, then there exists a $y \in B \setminus A$ such that $A \cup \{y\} \in \mathcal{F}$ and $(B \setminus \{y\}) \cup \{x\} \in \mathcal{F}$). Actually, Korte et al. [3, p.160, Theorem 2.2] is the same result as [4, p.358, Theorem 14.7]. However, from [3], it is easily seen that a greedoid can not be ensured to satisfy the strong exchange axiom. Hence, we may be asserted that [3, p.160, Theorem 2.2] or [4, p.358, Theorem 14.7] is not a characterization for all of greedoids with greedy algorithm, but only a characterization for a part of class of greedoids. In addition, among the known characterizations relative to greedoids with greedy algorithm, we think [1, Theorem 8.5.2] (the same as [3, p.157, Theorem 1.4]) to be better, that is: suppose $(E, \mathcal{L})$ is a simple hereditary language, then $(E, \mathcal{L})$ is a greedoid if and only if greedy algorithm gives an optimal solution for every compatible objective function on $\mathcal{L}$. In the characterizations using Definition 1.1 and the greedy algorithms for a greedoid $(E, \mathcal{F})$ proved in [1, Theorem 8.5.2] and [3, p.157, Theorem 1.4], $\mathcal{F}$ must be hereditary (i.e. $X \subseteq Y, Y \in \mathcal{F} \Rightarrow X \in \mathcal{F}$).

Now returning to our question: under what conditions on a greedoid, can every linear function be optimized by the greedy algorithm? Up to now, we do not find an answer for all of greedoids. Though we do not find out the solution to the open problem for all of greedoids, using the research methods in [1,2,3,4] for reference, we can pay our attention to some special class of greedoids to look for the answer. By [1,3], an interval greedoid $(E, \mathcal{F}_0)$ does not ask $\mathcal{F}_0$ to be hereditary or satisfy strong exchange axiom. The authors describe [1] that the ‘interval property’ characterizes a very large class of greedoids and interval greedoids behave better than general greedoids in many respects. In some types of study, the interval property has to be assumed to obtain meaningful results [1,3,5,6]. Hence, this paper will focus on interval greedoids in hope to find the answer for the open problem.

We may find from Definition 1.1 that for a greedoid $(E, \mathcal{F})$, if $\mathcal{F}$ is hereditary, then $(E, \mathcal{F})$ is interval. In addition, it is necessary to generalize the results in [1, Theorem 8.5.2] and [3, p.157, Theorem 1.4] for interval greedoids. This is done in this paper.

Lemma 1.2. Let $\mathcal{F} \subseteq 2^E$ and $x \in E$ be a loop. Then $\mathcal{F}$ is a set system on $E \setminus \{x\}$.

Proof. Suppose that a loop x is contained in a $X \in \mathcal{F}$. Then, there is a basis B_X satisfying $X \subseteq B_X$ according to Definition 1.1(3). This follows $x \in B_X$, a contrary to the loop of x . Therefore, we demonstrate that $\mathcal{F}$ is defined on $E \setminus \{x\}$. $\square$

By Lemma 1.2, this paper only considers the set systems with no loops.

Lemma 1.3 ([3, p.47]). *For a given set system $\mathcal{F}$ on E , the property (G2) holds if and only if for any $A \subseteq E$, all bases of A have the same cardinality.*

According to Lemma 1.3, we can state that in a set system such that (G2), then $X \subseteq E$ is a basis of $A \subseteq E$ if and only if $X \in \mathcal{F}$, $X \subseteq A$ satisfies $|X| = \max_{Y \in \mathcal{F}, Y \subseteq A} |Y|$.

2 Main results

We give some notions for a set system $(E, \mathcal{F})$:

(1) $\mathcal{F}^{(k)} = \{X \in \mathcal{F} \mid |X| = k\}$;

(2) $\mathcal{F}|_A = \{X \mid X \subseteq A, X \in \mathcal{F}\}$ for any $A \in \mathcal{F}$;

(3) $n = \max_{X \in \mathcal{F}} |X|$;

(4) Let $\omega : E \rightarrow \mathbb{R}^+$ be a positive weight function (i.e. $\omega(x) > 0$ for any $x \in E$). For $X \subseteq E$, define $\omega_X : X \rightarrow \mathbb{R}^+$ as $\omega_X(x) = \omega(x)$ for any $x \in X$.

We know that, generally, the solution of the greedy algorithm in Section 1 is not optimal. The already existing greedy algorithms for greedoids (see [1, 3, 4]) are satisfied (or say, characterized) by some different classes of greedoids. In order to search out a characterization of a type of greedy algorithms for some class of interval greedoids, we provide a type of greedy algorithm (i.e. Algorithm 1) as follows. After that, we will demonstrate under what conditions for a set system, Algorithm 1 has an optimal solution. We also find under what conditions Algorithm 1 characterizes an interval greedoid.

Algorithm 1 (Interval Greedy Algorithm). *Input: $\mathcal{F}$, a set system on E ; $\omega : E \rightarrow \mathbb{R}^+$, a positive weight function; $n, \max_{X \in \mathcal{F}} |X|$.*

Output: $\bar{S}$, the greedy solution.

1 Set $\bar{S} = \emptyset, j = 0$.

2 If $j < n - 1$, then go to 3.

If $j = n - 1$, then go to 4.

If $j \geq n$, then $\bar{S} := \bar{S}$, stop.

3 Set $D_{j+1} = \{e \mid \text{there exist } S_{j+1} \in \mathcal{F}^{(j+1)} \text{ and } \bar{S} \subset S_{j+1} \text{ such that } e \in E \setminus S_{j+1} \text{ and } S_{j+1} \cup \{e\} \in \mathcal{F}\}$, and $G_j = \{e \in E \setminus \bar{S} \mid \bar{S} \cup \{e\} \in \mathcal{F}\}$.

3.1 If $D_{j+1} \cap G_j \neq \emptyset$, then choose $e_{j+1} \in D_{j+1} \cap G_j$ such that $\omega(e_{j+1}) = \max_{e \in D_{j+1} \cap G_j} \omega(e)$, and set $\bar{S} := \bar{S} \cup \{e_{j+1}\}$, $j := j + 1$, go to 2.

3.2 If $D_{j+1} \cap G_j = \emptyset$, then $\bar{S} := \bar{S}$, and $j := j + 1$, go to 2.

4 Set $G_j = \{e \in E \setminus \bar{S} \mid \bar{S} \cup \{e\} \in \mathcal{F}\}$.

4.1 If $G_j \neq \emptyset$, then choose $e_{j+1} \in G_j$ such that $\omega(e_{j+1}) = \max_{e \in G_j} \omega(e)$, and set $\bar{S} := \bar{S} \cup \{e_{j+1}\}$, $j := j + 1$, go to 2.

4.2 If $G_j = \emptyset$, then $\bar{S} := \bar{S}$ and $j := j + 1$, go to 2.

We say the greedy algorithm works if $\omega(\bar{S}) \geq \omega(A)$ for $\forall A \in \mathcal{F}$. In the process of Algorithm 1, we can use $\bar{S}_{t+1}$ to stand for the solution when the cyclic variable j is $t \leq n - 1$.

Example 2.1. Let $E_1 = \{a_1, a_2, a_3, a_4\}$ and $\mathcal{F}_1 = \{\emptyset, \{a_1\}, \{a_4\}, \{a_1, a_2\}, \{a_1, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_3, a_4\}, \{a_1, a_2, a_3, a_4\}\}$. We can easily check that $\mathcal{F}_1$ satisfies (G1) and (G2) in Definition 1.1 (1).

Let $A = \emptyset, B = \{a_1, a_2\}$ and $C = \{a_1, a_2, a_3\}$. We easily find $A \subset B \subset C$. For $a_4 \in E_1 \setminus C$, we obtain $A \cup \{a_4\} = \{a_4\} \in \mathcal{F}_1, C \cup \{a_4\} = \{a_1, a_2, a_3, a_4\} \in \mathcal{F}_1$ and $B \cup \{a_4\} = \{a_1, a_2, a_4\} \notin \mathcal{F}_1$. Using Definition 1.1 (2), $(E_1, \mathcal{F}_1)$ is not an interval greedoid.

Define $\omega_1 : E_1 \rightarrow \mathbb{R}^+$ as $\omega_1(a_1) = 5, \omega_1(a_2) = 4, \omega_1(a_3) = 3, \omega_1(a_4) = 2$. Then, we can demonstrate that $\{a_1, a_2, a_3, a_4\}$ is an optimal set. Applying Algorithm 1 on $(\mathcal{F}_1, \omega_1)$, we look for the solution $\bar{S}$ of Algorithm 1 as follows: There is $n = 4$.

When $j = 0$, there are $S_0 = \emptyset, D_1 = \{a_1, a_2, a_4\}, G_0 = \{a_1, a_4\}$, and so $\bar{S}_1 = \{a_1\}$.

When $j = 1$, there are $D_2 = \{a_3\}, G_1 = \{a_2, a_4\}$, and so $D_2 \cap G_1 = \emptyset$. Thus, we attain $\bar{S}_2 = \bar{S}_1 = \{a_1\}$.

When $j = 2$, there are $D_3 = \{a_2, a_4\}, G_2 = \{a_2, a_4\}$, and so $\bar{S}_3 = \{a_1, a_2\}$.

When $j = 3$, there is $j = n - 1$. We find $G_3 = \{a_3\}$. Hence, there is $\bar{S}_4 = \{a_1, a_2, a_3\}$.

When $j = 4$, there is $j \geq n$. So, we obtain $\bar{S} = \bar{S}_4$.

Actually, $\omega_1(\{a_1, a_2, a_3\}) = 12 < 14 = \omega_1(\{a_1, a_2, a_3, a_4\})$ indicates that $\bar{S}$ is not optimal.

Remark 2.2. After analyzing Example 2.1, we attain some properties as follows.

(1) If we hope Algorithm 1 to work for $(E, \mathcal{F})$, then $(E, \mathcal{F})$ should be an interval greedoid.

(2) There are $\{a_1\}, \{a_4\} \in \mathcal{F}_1$ and $\{a_2\}, \{a_3\} \notin \mathcal{F}_1$ holds. Hence, we can ask $\{x\} \in \mathcal{F}$ for any $x \in E$ if we hope Algorithm 1 to work for a set system $\mathcal{F}$ with any positive weight function ω .

Example 2.3. Let $E_2 = \{a_1, a_2, a_3, a_4\}$ and $\mathcal{F}_2 = \{\emptyset, \{a_1\}, \{a_4\}, \{a_1, a_2\}, \{a_1, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_3, a_4\}\}$. We easily check up $\mathcal{F}_2$ to satisfy (G1) and (G2) and the interval property. Hence, $(E_2, \mathcal{F}_2)$ is an interval greedoid.

Define $\omega_2 : E_2 \rightarrow \mathbb{R}^+$ as $\omega_2(a_1) = 5, \omega_2(a_2) = 4, \omega_2(a_3) = 3, \omega_2(a_4) = 2$. Then, we can demonstrate that $\{a_1, a_2, a_3\}$ is an optimal set. Applying Algorithm 1 on $(\mathcal{F}_2, \omega_2)$, we look for the solution $\bar{S}$ of Algorithm 1 as follows: There is $n = 3$.

When $j = 0$, there are $S_0 = \emptyset, D_1 = \{a_1, a_2, a_4\}, G_0 = \{a_1, a_4\}$, and so $\bar{S}_1 = \{a_1\}$.

When $j = 1$, there are $D_2 = \{a_3\}, G_1 = \{a_2, a_4\}$, and so $D_2 \cap G_1 = \emptyset$. Thus, we attain $\bar{S}_2 = \bar{S}_1 = \{a_1\}$.

When $j = 2$, there is $j = n - 1$. We find $G_2 = \{a_2, a_4\}$. Hence, there is $\bar{S}_3 = \{a_1, a_2\}$.

When $j = 3$, there is $j \geq n$. So, we obtain $\bar{S} = \bar{S}_3$.

Actually, $\omega_2(\{a_1, a_2, a_3\}) = 12 > 9 = \omega_2(\{a_1, a_2\})$ indicates that $\bar{S}$ is not optimal.

Remark 2.4. In Example 2.3, $B_1 = \{a_1, a_2, a_3\}$ and $B_2 = \{a_1, a_3, a_4\}$ are two bases of $\mathcal{F}_2$ such that $B_1 \setminus \{a_3\} = \{a_1, a_2\}, B_2 \setminus \{a_3\} = \{a_1, a_4\} \in \mathcal{F}_2$. But $\omega_2(a_1) > \omega_2(a_2) > \omega_2(a_3) > \omega_2(a_4)$ and $\mathcal{F}^{(2)} = \{\{a_1, a_2\}, \{a_1, a_4\}\}$ imply that $(E_2, \mathcal{F}_2)$ does not satisfy the following condition:

(G3) Let X, Y be bases in $\mathcal{F}$ and $S = Y \setminus \{y_0\} \in \mathcal{F}$ for some $y_0 \in Y$. Then there is $x_0 \in X \setminus S$ such that $S \cup \{x_0\} \in \mathcal{F}$ and $\omega(x_0) = \max_{x \in X} \omega(x)$.

Hence, we should ask if $\mathcal{F}$ with ω satisfy (G3) if we hope Algorithm 1 to work for $(\mathcal{F}, \omega)$ though $(E, \mathcal{F})$ is an interval greedoid.

Lemma 2.5. Let $\mathcal{F} \subseteq 2^E$ with $\emptyset \in \mathcal{F}$ and $\omega : E \rightarrow \mathbb{R}^+$ be a positive weight function.

(1) If $\mathcal{F}$ satisfies (G2), then there is $\mathcal{F}^{(k)} \neq \emptyset$ for any $k = 0, 1, \dots, n$.

(2) An optimal set in $(\mathcal{F}, \omega)$ is a basis.

Proof. (1) Using Lemma 1.3 and Definition 1.1 (3), all of bases in $\mathcal{F}$ have the cardinality n . Let $B \in \mathcal{F}$ be a basis. From Björner et al. [1, 8.2.A], we know that $\emptyset \in \mathcal{F}$ and (G2) together define greedoids as well as (G1) and (G2). Considering (G1) on B , we may easily obtain $\mathcal{F}^{(k)} \neq \emptyset, (k = 0, 1, \dots, n)$.

(2) Since an optimal set S satisfies $\omega(S) \geq \omega(B)$ for any basis B of $\mathcal{F}$. If S is not a basis, then $S \subset B_S$ holds for some basis B_S according to Definition 1.1(3). Thus, there is $\omega(S) < \omega(B_S)$ since ω is positive, a contradiction with $\omega(S) \geq \omega(B)$. Hence S is a basis. $\square$

Theorem 2.6. Let $\mathcal{F} \subseteq 2^E$ satisfy $\emptyset \in \mathcal{F}$ and $\{x\} \in \mathcal{F}$ for any $x \in E$. Let $\omega : E \rightarrow \mathbb{R}^+$ be a positive weight function. If $(E, \mathcal{F})$ is an interval greedoid satisfying the condition (G3), then Algorithm 1 works for $(\mathcal{F}, \omega)$.

Proof. Step 1. Let $\mathcal{F}_k = \{X \mid X \in \mathcal{F}, |X| \leq k\}$ ($k \geq 1$). We prove the following statements.

(st1) $\emptyset \in \mathcal{F}_k$, $\{x\} \in \mathcal{F}_k$ for any $x \in E$.

(st2) $(E, \mathcal{F}_k)$ is an interval greedoid.

$\{x\} \in \mathcal{F}$ and $|\{x\}| = 1 \leq k$ for any $x \in E$ imply $\{x\} \in \mathcal{F}_k$ for any $x \in E$. $\emptyset \in \mathcal{F}$ and $|\emptyset| = 0 \leq k$ together means $\emptyset \in \mathcal{F}_k$. Hence, we can say that $\mathcal{F}_k$ is a set system with no loops. According to $\mathcal{F}_k \subseteq \mathcal{F}$ and $\mathcal{F}$ satisfying both of (G1) and (G2), we easily obtain that $\mathcal{F}_k$ satisfies both of (G1) and (G2).

Let $A \subseteq B \subseteq C$, $A, B, C \in \mathcal{F}_k$ and $a \in E_k \setminus C$ satisfy $A \cup \{a\} \in \mathcal{F}_k$ and $C \cup \{a\} \in \mathcal{F}_k$. Using $\mathcal{F}_k \subseteq \mathcal{F}$ and the interval property of $\mathcal{F}$, we obtain $B \cup \{a\} \in \mathcal{F}$. $C \cup \{a\} \in \mathcal{F}_k$ follows $|C \cup \{a\}| \leq k$. Combining with $B \cup \{a\} \subseteq C \cup \{a\}$, we decide $|B \cup \{a\}| \leq k$. Hence, there is $B \cup \{a\} \in \mathcal{F}_k$.

Therefore, $(E, \mathcal{F}_k)$ is an interval greedoid.

Step 2. We will prove that Algorithm 1 works for $(\mathcal{F}, \omega)$ by induction on n .

If $n = 0$. This means $\mathcal{F} = \{\emptyset\}$. Hence, the needed result follows.

If $n = 1$. By Lemma 2.5 (1) and Definition 1.1 (2), there are $\mathcal{F}^{(0)} \neq \emptyset$ and $\mathcal{F}^{(1)} \neq \emptyset$.

Then, in the process of Algorithm 1, when $j = 0 = 1 - 1 = n - 1$, according to $\bar{S}_0 = \emptyset$, there is $G_0 = \{e \in E \setminus \emptyset \mid \emptyset \cup \{e\} \in \mathcal{F}\} = E$ since $\{x\} \in \mathcal{F}$ for any $x \in E$. Hence, $G_0 \neq \emptyset$ holds. Choose $\omega(e_{0+1}) = \omega(e_1) = \max_{e \in G_0} \omega(e)$, and put $\bar{S}_1 = \emptyset \cup \{e_1\} = \{e_1\}$ and $j := 0 + 1 = 1$. When $j = 1$, then $j \geq 1 = n$ follows the process of Algorithm 1 to stop. Therefore, the solution of Algorithm 1 is optimal. That is to say, Algorithm 1 works for $(\mathcal{F}, \omega)$.

Suppose that if $n \leq m - 1$, then the needed result is correct. Now, let $n = m$.

Since $(E, \mathcal{F})$ is an interval greedoid, $\mathcal{F}$ satisfies (G1) and (G2). Combining Lemma 1.3, there is $m = |B|$ for any basis B of $\mathcal{F}$. Utilizing Lemma 2.5 (1), we obtain $\mathcal{F}^{(k)} \neq \emptyset$ where $k = 0, 1, \dots, m$.

Let $\bar{S}$ be the solution of Algorithm 1 for $(\mathcal{F}, \omega)$. During the process of Algorithm 1, when $j < m - 1$, according to the interval property, $\emptyset \cup \{e_0\} \in \mathcal{F}$ for any $e_0 \in E$, $\emptyset \subseteq \bar{S}_j \subseteq S_{j+1}$, $S_{j+1} \cup \{e\} \in \mathcal{F}$ for any $e \in D_{j+1}$, and the definitions of D_{j+1} and G_j , there is $\bar{S}_j \cup \{e\} \in \mathcal{F}$ for any $e \in D_{j+1}$, and so $D_{j+1} \cap G_j \neq \emptyset$. Considering Lemma 2.5 (1), we arrive at $\mathcal{F}^{(m)} \neq \emptyset$. So, there is $G_{m-1} = \{e \in E \setminus \bar{S}_{m-1} \mid \bar{S}_{m-1} \cup \{e\} \in \mathcal{F}\} \neq \emptyset$. Therefore, we can demonstrate that $\bar{S}$ is a basis. Hence, $|\bar{S}| = m$ holds.

Since $\bar{S}$ is accessible according to the process of Algorithm 1 for the interval greedoid $(E, \mathcal{F})$ and ω , there is $\bar{S} = \bar{S}_m$. Using Step 1 and the inductive supposition, Algorithm 1 works for $(E, \mathcal{F}_{m-1})$. That is to say, the solution $\bar{S}^{m-1}$ of Algorithm 1 for $(E, \mathcal{F}_{m-1})$ satisfies $\omega(\bar{S}^{m-1}) \geq \omega(X)$ for any $X \in \mathcal{F}_{m-1}$. Combining the process of Algorithm 1, we confirm $\bar{S}^{m-1} = \bar{S}_{m-1}$.

Considering $G_{m-1} = \{e \in E \setminus \bar{S}_{m-1} \mid \bar{S}_{m-1} \cup \{e\} \in \mathcal{F}\}$ and $\mathcal{F}^{(m)} \neq \emptyset$, we obtain $\bar{S}_m = \bar{S}_{m-1} \cup \{e_m\}$ where $\omega(e_m) = \max_{e \in G_{m-1}} \omega(e)$. Moreover, $\bar{S}_m$ is the solution of Algorithm 1 for $(\mathcal{F}, \omega)$.

Let B be a basis of $\mathcal{F}$. We easily find $|B| = |\bar{S}_{m-1}| + 1$. Thus, $\bar{S}_{m-1} \cup \{b\} \in \mathcal{F}$ holds for some $b \in B \setminus \bar{S}_{m-1}$ according to (G2) satisfied by $\mathcal{F}$. Thus, $\bar{S}_{m-1} \cup \{b\}$ is a basis of $\mathcal{F}$. Using (G3), there is $\bar{S}_{m-1} \cup \{b_0\} \in \mathcal{F}$ where $\omega(b_0) = \max_{x \in B \setminus \bar{S}_{m-1}} \omega(x)$.

On the other hand, for any $x \in B$, if $B \setminus \{x\} \in \mathcal{F}$, then $\omega(B \setminus \{x\}) \leq \omega(\bar{S}^{m-1}) = \omega(\bar{S}_{m-1})$ in view of the inductive supposition. Since $(B \setminus \{x\}) \cup \{x\}$ is a basis, there is $\omega((B \setminus \{x\}) \cup \{x\}) \leq \omega(\bar{S}_{m-1} \cup \{b_0\}) \leq \omega(\bar{S}_{m-1} \cup \{e_m\}) = \omega(\bar{S}_m)$ in virtue of (G3) and the process of searching $\bar{S}_m$.

Therefore, $\omega(\bar{S}_m) \geq \omega(B)$ holds for any basis B in $\mathcal{F}$. Furthermore, $\omega(\bar{S}_m) \geq \omega(X)$ is correct for any $X \in \mathcal{F}$ since X must be contained in a basis and ω is positive. Thus, $\bar{S}_m$ is optimal.

Summing up, Algorithm 1 works for $(\mathcal{F}, \omega)$. □

Remark 2.7. Example 2.1 shows that $(E_1, \mathcal{F}_1)$ is not an interval greedoid. In addition, for $\{a_1\} \subset \{a_1, a_2\}$ and $a_3 \in E_1 \setminus \{a_1, a_2\}$, there are $\{a_1, a_2\} \cup \{a_3\} = \{a_1, a_2, a_3\} \in \mathcal{F}_1$ and $\{a_1\} \cup \{a_3\} = \{a_1, a_3\} \notin \mathcal{F}_1$. That is to say, $\mathcal{F}$ does not satisfy the semi-interval property (that is, if $Y \subseteq Z$, $Y, Z \in \mathcal{F}$, $a \in E \setminus Z$, then $Z \cup \{a\} \in \mathcal{F} \Rightarrow Y \cup \{a\} \in \mathcal{F}$).

It is more interesting that the converse of Theorem 2.6 is also true under some pre-conditions.

Theorem 2.8. Let $\mathcal{F}$ be a set system on E with $\emptyset \in \mathcal{F}$. If for any positive weight function $\omega : E \rightarrow \mathbb{R}^+$, there are the following statements:

(s1) Algorithm 1 works for $(\mathcal{F}|_A, \omega_A)$ for any $A \in \mathcal{F}$.

(s2) $\mathcal{F}$ has semi-interval property.

(s3) $\mathcal{F}$ satisfies (G2).

Then, $(E, \mathcal{F})$ is an interval greedoid.

Proof. Step 1. To prove: $\mathcal{F}$ is accessible.

Let $A \in \mathcal{F}$. Since A is the basis of $\mathcal{F}|_A$ and ω is positive, there is $\omega(A) = \max_{X \in \mathcal{F}|_A} \omega(X)$. Let $\bar{S}_A$ be the solution of Algorithm 1 for $(\mathcal{F}|_A, \omega_A)$. Consider the process of Algorithm 1 and $A \in \mathcal{F}$, we can decide $\bar{S}_A \subseteq A$. Hence, we obtain $\omega_A(\bar{S}_A) \leq \omega_A(A)$ since ω_A is positive. Furthermore, since ω_A is positive, we find $S_A \subset A \Leftrightarrow \omega_A(\bar{S}_A) < \omega_A(A)$. By (s1), we confirm $\omega_A(\bar{S}_A) = \max_{X \in \mathcal{F}|_A} \omega_A(X)$. Summing up the above results, we may follow $\bar{S}_A = A$.

From $\emptyset \in \mathcal{F}$ and the process of Algorithm 1 for $(\mathcal{F}|_A, \omega_A)$, we may assert that $\bar{S}_A$ is accessible. We will prove this assertion as follows.

Let $m = |\bar{S}_A|$, that is, $m = |A|$ since $\bar{S}_A = A$.

If $m = 0$. Then $\bar{S}_A = \emptyset$. So, $\bar{S}_A$ is accessible.

If $m = 1$. Then $A = \{a\}$ and $\bar{S}_0 = \emptyset$.

When $j = 0 = m - 1 = 1 - 1$. There is $G_j = G_0 = \{a\}$. Hence, $G_j \neq \emptyset$ and $\omega(e_{0+1}) = \omega(a)$ and $\bar{S}_1 = \emptyset \cup \{a\} = \{a\}$.

When $j = 0 + 1 = 1 = m$, then stop, and $\bar{S}_A = \{a\}$. We easily obtain $S_A \setminus \{a\} = \emptyset \in \mathcal{F}$. Hence, $\bar{S}_A$ is accessible.

Suppose that if $m \leq k - 1$, then $\bar{S}_A$ is accessible. Now, let $m = k > 1$.

If for every $j < k - 1$, there is $D_{j+1} \cap G_j \neq \emptyset$ in the process of Algorithm 1, then according to the definitions of D_{j+1} and G_j , there are $\bar{S}_0 = \emptyset \subset \bar{S}_1 \subset \bar{S}_2 \subset \dots \subset \bar{S}_{j+1}$ and $\bar{S}_{i+1} = \bar{S}_i \cup \{e_{i+1}\}$ for some $e_{i+1} \in A \setminus \bar{S}_i$, ($i = 0, \dots, j + 1$), we can state that $\bar{S}_j$ is accessible ($j = 0, \dots, k - 1$).

If for some $j_0 < k - 1$, there is $D_{j_0+1} \cap G_{j_0} = \emptyset$ in the process of Algorithm 1, then $\bar{S}_{j_0+1} = \bar{S}_{j_0}$. This follows $|\bar{S}_{j_0+1}| < j_0 + 1$. Furthermore, we obtain $|\bar{S}_k| < k$ according to the process of Algorithm 1. Thus, we attain $\bar{S}_A$ (that is $\bar{S}_k$) satisfying $|\bar{S}_A| < k = |A| = |\bar{S}_A|$, a contradiction. Hence, for every $j < k - 1$, there is $D_{j+1} \cap G_j \neq \emptyset$.

So, $\bar{S}_j$ is accessible ($j = 0, 1, \dots, k - 1$).

Let $j = k - 1$. If $G_{k-1} \neq \emptyset$, then we obtain $\bar{S}_k = \bar{S}_{k-1} \cup \{e_k\}$. Thus, we may easily find $\bar{S}_k$ to be accessible since $\bar{S}_k \setminus e_{k-1} = \bar{S}_{k-1} \in \mathcal{F}$ and the above discussion.

If $G_{k-1} = \emptyset$, then $\bar{S}_k = \bar{S}_{k-1}$. Thus, there is $|\bar{S}_k| = |\bar{S}_{k-1}| = k - 1$. On the other hand, $\bar{S}_k = \bar{S}_m = \bar{S}_A = A$ hints $|\bar{S}_k| = |A| = m = k$. This is a contradiction to $|\bar{S}_k| = k - 1$. Therefore, we confirm $G_k \neq \emptyset$.

Adding up the above discussion with the induction, we can state that $\bar{S}_A$, that is A , is accessible.

According to the arbitrariness of A , we attain that $\mathcal{F}$ is accessible.

Step 2. To prove: $\mathcal{F}$ satisfies the interval property.

Let $X, Y, Z \in \mathcal{F}$ satisfy $X \subseteq Y \subseteq Z$. Let $a \in E \setminus Z$ satisfy $X \cup \{a\}, Z \cup \{a\} \in \mathcal{F}$. Using the statement (s2), there is $Y \cup \{a\} \in \mathcal{F}$. Therefore, $\mathcal{F}$ satisfies the interval property.

Step 3. $\mathcal{F}$ is exchangeable according to the statement (s3).

Step 4. Combining Steps 1, 2 and 3 with Definition 1.1, $(E, \mathcal{F})$ is an interval greedoid. $\square$

Example 2.9. Let $E_3 = \{a_1, a_2, a_3, a_4\}$ and $\mathcal{F}_3 = \{\emptyset, \{a_1\}, \{a_4\}, \{a_1, a_2\}, \{a_3, a_4\}\}$. There is $|\{a_3, a_4\}| = 2 = |\{a_1\}| + 1$, but no element $a \in \{a_3, a_4\} \setminus \{a_1\}$ satisfies $\{a_1\} \cup \{a\} \in \mathcal{F}_3$. Define $\omega_3 : E_3 \rightarrow \mathbb{R}^+$ as $\omega_3(a_1) = 5, \omega_3(a_2) = 1, \omega_3(a_3) = \omega_3(a_4) = 4$. The solution of Algorithm 1 for $(\mathcal{F}_3, \omega_3)$ is $\{a_1\}$. But, $\omega_3(\{a_1\}) = 5 < \omega_3(\{a_3, a_4\}) = 8$ implies Algorithm 1 not to work for $(\mathcal{F}_3, \omega_3)$.

Example 2.9 shows that if we want Algorithm 1 to work for $(\mathcal{F}, \omega)$, then $\mathcal{F}$ should satisfy (G2). Combining Theorems 2.6 and 2.8, we give the following characterization for interval greedoids.

Theorem 2.10. *Let $\mathcal{F}$ be an exchangeable set system on E with $\emptyset \in \mathcal{F}$ and $\{x\} \in \mathcal{F}$ for any $x \in E$. $\mathcal{F}_A$ satisfies (G3) for any $A \in \mathcal{F}$. Then $\mathcal{F}$ is the set of feasible sets of an interval greedoid on E if and only if for all positive weight functions $\omega : E \rightarrow \mathbb{R}^+$, $\mathcal{F}$ satisfies the statements (s1) and (s2).*

Proof. ($\Rightarrow$) We easily prove $(A, \mathcal{F}|_A)$ to be an interval greedoid for any $A \in \mathcal{F}$. Combining Theorem 2.1 and $\mathcal{F}|_A$ satisfying (G3), we obtain the correctness of (s1). $\emptyset, \{x\} \in \mathcal{F}$ and the interval of $\mathcal{F}$ follow the correctness of (s2).

($\Leftarrow$) Using Theorem 2.8, all of needed results are straightforward. $\square$

Next, we will compare our results with some known results for greedoids.

(I) To compare our results with [1, Theorem 8.5.2] (or say [3, p.157, Theorem 1.4]).

(1) In [1, Theorem 8.5.2] and [3, p.157, Theorem 1.4], the authors give a kind of greedy algorithm to characterize a greedoid $(E, \mathcal{F}_{13})$, where $\mathcal{F}_{13}$ is asked to be hereditary. In other words, if a greedoid $(E, \mathcal{F})$ does not satisfy the hereditary property for $\mathcal{F}$, then the characterizations with greedy algorithms in [1, Theorem 8.5.2] and [3, p.157, Theorem 1.4] will not be successful.

(2) Let $\mathcal{F}$ be a set system on E satisfying the hereditary. Then, we easily find that $\mathcal{F}$ has the following properties:

- $\emptyset, \{x\} \in \mathcal{F}$ holds for any $x \in E$; $\mathcal{F}$ satisfies the condition (G3).
- Let $Y \subseteq Z$, $Y, Z \in \mathcal{F}$ and $a \in E \setminus Z$. If $Z \cup \{a\} \in \mathcal{F}$ is correct, then $Y \cup \{a\} \in \mathcal{F}$ holds according to $Y \cup \{a\} \subseteq Z \cup \{a\}$ and the hereditary property of $\mathcal{F}$. This implies that every hereditary sets system satisfies the semi-interval property.

(3) Considered items (1) and (2), we know that the characterization for greedoids with the greedy algorithm provided in [1, Theorem 8.5.2] and [3, p.157, Theorem 1.4] are really effective only for some of interval greedoids and not for the other kinds of greedoids.

(4) Evidently, the given conditions in Theorems 2.6, 2.8 and 2.10 do not ask $\mathcal{F}$ to be hereditary. Combining the above three items, we can say that for a hereditary set system $\mathcal{F}$, Theorem 2.6 and Theorem 2.8 are satisfied by much more greedoids than that in [1, Theorem 8.5.2] (or say, [3, p.157, Theorem 1.4]) respectively.

Moreover, the characterization (i.e. Theorem 2.3) proposed in this paper for interval greedoids generalize the results in [1, Theorem 8.5.2] and [3, p.157, Theorem 1.4] respectively.

Therefore, Algorithm 1 generalizes the greedy algorithm for [1, Theorem 8.5.2] and [3, p.157, Theorem 1.4].

(II) To compare our results with [4, p.358, Theorem 14.7].

It is well known that a greedoid is perhaps not satisfying the strong exchange axiom. In other words, not every greedoid has strong exchange axiom, though any greedoid is exchangeable. We also know that an interval greedoid can not be ensured to satisfy strong exchange axiom. Thus, we can state that Theorem 2.10 is a characterization of a greedy algorithm for some class of interval greedoids. [4, p.358, Theorem 14.7] can not substitute for the results in this paper. Therefore, Algorithm 1 is a new algorithm and not covered by the algorithm for [4, p.358, Theorem 14.7].

More generalized characterization for greedoids with greedy algorithms will be studied in the future. We also hope to give more answers to the open problem stated in Section 1.

Acknowledgement: This research is supported by NNSF of China (61572011) and NSF of Hebei Province (A2018201117).

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