

Open Mathematics

Research Article

Zahid Raza, Agha Kashif*, and Imran Anwar

On algebraic characterization of SSC of the Jahangir's graph $\mathcal{J}_{n,m}$

<https://doi.org/10.1515/math-2018-0025>

Received July 11, 2016; accepted December 18, 2017.

Abstract: In this paper, some algebraic and combinatorial characterizations of the spanning simplicial complex $\Delta_s(\mathcal{J}_{n,m})$ of the Jahangir's graph $\mathcal{J}_{n,m}$ are explored. We show that $\Delta_s(\mathcal{J}_{n,m})$ is pure, present the formula for f -vectors associated to it and hence deduce a recipe for computing the Hilbert series of the Face ring $k[\Delta_s(\mathcal{J}_{n,m})]$. Finally, we show that the face ring of $\Delta_s(\mathcal{J}_{n,m})$ is Cohen-Macaulay and give some open scopes of the current work.

Keywords: Simplicial complexes, Spanning trees, Face ring, Hilbert series, f -vectors, Cohen Macaulay

MSC: 13P10, 13H10, 13F20, 13C14

1 Introduction

The concept of spanning simplicial complex (SSC) associated with the edge set of a simple finite connected graph is introduced by Anwar, Raza and Kashif in [1]. They revealed some important algebraic properties of SSC of a unicyclic graph. Kashif, Raza and Anwar further established the theory and explored algebraic characterizations of some more general classes of n -cyclic graphs in [10, 11]. The problem of finding the SSC for a general simple finite connected graph is not an easy task to handle. Recently in [15] Zhu, Shi and Geng discussed the SSC of another class n -cyclic graphs with a common edge.

In this article, we discuss some algebraic and combinatorial properties of the spanning simplicial complex $\Delta_s(\mathcal{J}_{n,m})$ of a certain class of *cyclic graphs*, $\mathcal{J}_{n,m}$. For simplicity, we fixed $n = 2$ in our results. Here, $\mathcal{J}_{n,m}$ is the class of Jahangir's graph defined in [12] as follows:

The Jahangir's graph $\mathcal{J}_{n,m}$, for $m \geq 3$, is a graph on $nm + 1$ vertices i.e., a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm} .

More explicitly, it consists of a cycle C_{nm} which is further divided into m consecutive cycles C_i of equal length such that all these cycles have one vertex common and every pair of consecutive cycles has exactly one edge common. For example the graph $\mathcal{J}_{2,3}$ is given in Figure 1. We fix the edge set of $\mathcal{J}_{2,m}$ as follows:

$$E = \{e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23}, \dots, e_{m1}, e_{m2}, e_{m3}\}. \quad (1)$$

Here, $\{e_{k1}, e_{k2}, e_{k3}, e_{(k+1)1}\}$ is the edge set of the cycle C_k for $k \in \{1, 2, \dots, m-1\}$ and $\{e_{m1}, e_{m2}, e_{m3}, e_{11}\}$ is the edge set of cycle C_m . Also e_{k1} always represents the common edge between C_{k-1} and C_k for $k \in \{1, 2, \dots, m-1\}$ and e_{11} is the common edge between the cycle C_m and C_1 .

Zahid Raza: University of Sharjah, College of Sciences, Department of Mathematics, United Arab Emirates,
E-mail: zraza@sharjah.ac.ae

***Corresponding Author: Agha Kashif:** University of Management and Technology, Lahore, Pakistan,
E-mail: kashif.khan@umt.edu.pk, aghakashifkhan@hotmail.com

Imran Anwar: Abdus Salam School of Mathematical Sciences, Government College University, Lahore, Pakistan,
E-mail: iimrananwar@gmail.com

2 Preliminaries

In this section, we give some background and preliminaries of the topic and define some important notions to make this paper self-contained. However, for more details of the notions we refer the reader to [3–7, 13, 14].

Definition 2.1. A spanning tree of a simple connected finite graph $G(V, E)$ is a subtree of G that contains every vertex of G .

We represent the collection of all edge-sets of the spanning trees of G by $s(G)$, in other words;

$$s(G) := \{E(T_i) \subset E, \text{ where } T_i \text{ is a spanning tree of } G\}.$$

Lemma 2.2. Let $G = (V, E)$ be a simple finite connected graph containing m cycles. Then its spanning tree contains exactly $|E| - m$ edges.

Proof. A spanning tree of a graph is its spanning subgraph containing no cycles and no disconnection. If G is a unicyclic graph then deletion of one edge from it results in a spanning tree. If more than one edge is removed from the cycle in G then a disconnection is obtained which is not a spanning tree. Therefore, spanning tree has exactly $|E| - 1$ edges.

If G has m disjoint cycles in it i.e. cycles sharing no common edges, then its spanning tree is obtained by removing exactly m edges from it, one from each of its cycle. Therefore, its spanning tree has $|E| - m$ edges in it.

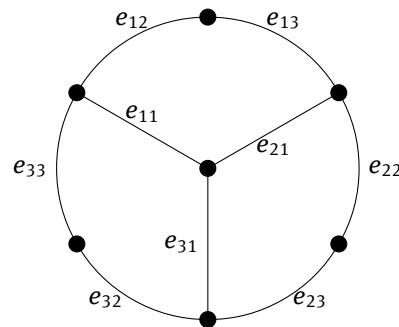
If any two cycles of G share one or more common edges and remaining are disjoint cycles, then one edge is needed to be removed from each cycle of G to obtain a spanning tree. However, if a common edge between two cycles is removed then exactly one edge from non common edges must be removed of the resulting big cycle. Therefore, its spanning tree has $|E| - m$ edges in it. This can be extended to any number of cycles in G sharing common edges. This completes the proof. \square

Applying Lemma 2.2, we can obtain the spanning tree of the Jahangir's graph $\mathcal{J}_{2,m}$ by removing exactly m edges from it keeping in view the following:

- Not more than one edge can be removed from the non common edges of any cycle.
- If a common edge between two or more consecutive cycles is removed then exactly one edge must be removed from the resulting big cycle.
- Not all common edges can be removed simultaneously.

This method is referred as the *cutting-down method*. For example, by using the *cutting-down method* for the graph $\mathcal{J}_{2,3}$ given in Fig. 1 we obtain:

$$\begin{aligned} s(\mathcal{J}_{2,3}) = \{ & \{e_{11}, e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\}, \{e_{11}, e_{21}, e_{31}, e_{12}, e_{22}, e_{33}\}, \{e_{11}, e_{21}, e_{31}, e_{12}, e_{23}, \\ & e_{32}\}, \{e_{11}, e_{21}, e_{31}, e_{12}, e_{23}, e_{33}\}, \{e_{11}, e_{21}, e_{31}, e_{13}, e_{22}, e_{32}\}, \{e_{11}, e_{21}, e_{31}, e_{13}, e_{22}, e_{33}\}, \{e_{11}, \\ & e_{21}, e_{31}, e_{13}, e_{23}, e_{32}\}, \{e_{11}, e_{21}, e_{31}, e_{13}, e_{23}, e_{33}\}, \{e_{21}, e_{31}, e_{32}, e_{33}, e_{12}, e_{22}\}, \{e_{21}, e_{31}, e_{32}, e_{33}, \\ & e_{12}, e_{23}\}, \{e_{21}, e_{31}, e_{32}, e_{33}, e_{13}, e_{22}\}, \{e_{21}, e_{31}, e_{32}, e_{33}, e_{13}, e_{23}\}, \{e_{21}, e_{31}, e_{12}, e_{13}, e_{33}, e_{22}\}, \\ & \{e_{21}, e_{31}, e_{12}, e_{13}, e_{33}, e_{23}\}, \{e_{21}, e_{31}, e_{12}, e_{13}, e_{32}, e_{22}\}, \{e_{21}, e_{31}, e_{12}, e_{13}, e_{32}, e_{23}\}, \{e_{11}, e_{31}, e_{12}, \\ & e_{13}, e_{22}, e_{32}\}, \{e_{11}, e_{31}, e_{12}, e_{13}, e_{22}, e_{33}\}, \{e_{11}, e_{31}, e_{12}, e_{13}, e_{23}, e_{32}\}, \{e_{11}, e_{31}, e_{12}, e_{13}, e_{23}, e_{33}\}, \\ & \{e_{11}, e_{31}, e_{22}, e_{23}, e_{13}, e_{32}\}, \{e_{11}, e_{31}, e_{22}, e_{23}, e_{13}, e_{33}\}, \{e_{11}, e_{31}, e_{22}, e_{23}, e_{12}, e_{32}\}, \{e_{11}, e_{31}, e_{22}, \\ & e_{23}, e_{12}, e_{33}\}, \{e_{11}, e_{21}, e_{23}, e_{22}, e_{32}, e_{12}\}, \{e_{11}, e_{21}, e_{23}, e_{22}, e_{32}, e_{13}\}, \{e_{11}, e_{21}, e_{23}, e_{22}, e_{33}, e_{12}\}, \\ & \{e_{11}, e_{21}, e_{23}, e_{22}, e_{33}, e_{13}\}, \{e_{11}, e_{21}, e_{32}, e_{33}, e_{22}, e_{12}\}, \{e_{11}, e_{21}, e_{32}, e_{33}, e_{22}, e_{13}\}, \{e_{11}, e_{21}, e_{32}, \\ & e_{33}, e_{23}, e_{12}\}, \{e_{11}, e_{21}, e_{32}, e_{33}, e_{23}, e_{13}\}, \{e_{11}, e_{13}, e_{22}, e_{23}, e_{32}, e_{33}\}, \{e_{11}, e_{12}, e_{22}, e_{23}, e_{32}, e_{33}\}, \\ & \{e_{11}, e_{12}, e_{13}, e_{23}, e_{32}, e_{33}\}, \{e_{11}, e_{12}, e_{13}, e_{22}, e_{32}, e_{33}\}, \{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33}\}, \{e_{11}, e_{12}, e_{13}, \\ & e_{22}, e_{23}, e_{32}\}, \{e_{21}, e_{13}, e_{22}, e_{23}, e_{32}, e_{33}\}, \{e_{21}, e_{12}, e_{22}, e_{23}, e_{32}, e_{33}\}, \{e_{21}, e_{12}, e_{13}, e_{23}, e_{32}, e_{33}\}, \\ & \{e_{21}, e_{12}, e_{13}, e_{22}, e_{32}, e_{33}\}, \{e_{21}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33}\}, \{e_{21}, e_{12}, e_{13}, e_{22}, e_{23}, e_{32}\}, \{e_{31}, e_{13}, e_{22}, \\ & e_{23}, e_{32}, e_{33}\}, \{e_{31}, e_{12}, e_{22}, e_{23}, e_{32}, e_{33}\}, \{e_{31}, e_{12}, e_{13}, e_{23}, e_{32}, e_{33}\}, \{e_{31}, e_{12}, e_{13}, e_{22}, e_{32}, e_{33}\}, \\ & \{e_{31}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33}\}, \{e_{31}, e_{12}, e_{13}, e_{22}, e_{23}, e_{32}\} \}. \end{aligned}$$

Fig. 1. The graph $\mathcal{J}_{2,3}$ 

Definition 2.3. A simplicial complex Δ over a finite set $[n] = \{1, 2, \dots, n\}$ is a collection of subsets of $[n]$, with the property that $\{i\} \in \Delta$ for all $i \in [n]$, and if $F \in \Delta$ then Δ will contain all the subsets of F (including the empty set). An element of Δ is called a face of Δ , and the dimension of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F . The maximal faces of Δ under inclusion are called facets of Δ . The dimension of the simplicial complex Δ is :

$$\dim \Delta = \max\{\dim F | F \in \Delta\}.$$

We denote the simplicial complex Δ with facets $\{F_1, \dots, F_q\}$ by

$$\Delta = \langle F_1, \dots, F_q \rangle$$

Definition 2.4. For a simplicial complex Δ having dimension d , its f -vector is a $d + 1$ -tuple, defined as:

$$f(\Delta) = (f_0, f_1, \dots, f_d)$$

where f_i denotes the number of i -dimensional faces of Δ .

Definition 2.5 (Spanning Simplicial Complex). Let $G(V, E)$ be a simple finite connected graph and $s(G) = \{E_1, E_2, \dots, E_t\}$ be the edge-sets of all possible spanning trees of $G(V, E)$, then we defined (in [1]) a simplicial complex $\Delta_s(G)$ on E such that the facets of $\Delta_s(G)$ are precisely the elements of $s(G)$, we call $\Delta_s(G)$ as the spanning simplicial complex of $G(V, E)$. In other words;

$$\Delta_s(G) = \langle E_1, E_2, \dots, E_t \rangle.$$

For example, the spanning simplicial complex of the graph $\mathcal{J}_{2,3}$ given in Fig. 1 is:

$$\begin{aligned} \Delta_s(\mathcal{J}_{2,3}) = \{ & \{e_{11}, e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\}, \{e_{11}, e_{21}, e_{31}, e_{12}, e_{22}, e_{33}\}, \{e_{11}, e_{21}, e_{31}, e_{12}, e_{23}, \\ & e_{32}\}, \{e_{11}, e_{21}, e_{31}, e_{12}, e_{23}, e_{33}\}, \{e_{11}, e_{21}, e_{31}, e_{13}, e_{22}, e_{32}\}, \{e_{11}, e_{21}, e_{31}, e_{13}, e_{22}, e_{33}\}, \{e_{11}, \\ & e_{21}, e_{31}, e_{13}, e_{23}, e_{32}\}, \{e_{11}, e_{21}, e_{31}, e_{13}, e_{23}, e_{33}\}, \{e_{21}, e_{31}, e_{32}, e_{33}, e_{12}, e_{22}\}, \{e_{21}, e_{31}, e_{32}, e_{33}, \\ & e_{12}, e_{23}\}, \{e_{21}, e_{31}, e_{32}, e_{33}, e_{13}, e_{22}\}, \{e_{21}, e_{31}, e_{32}, e_{33}, e_{13}, e_{23}\}, \{e_{21}, e_{31}, e_{12}, e_{13}, e_{33}, e_{22}\}, \\ & \{e_{21}, e_{31}, e_{12}, e_{13}, e_{33}, e_{23}\}, \{e_{21}, e_{31}, e_{12}, e_{13}, e_{32}, e_{22}\}, \{e_{21}, e_{31}, e_{12}, e_{13}, e_{32}, e_{23}\}, \{e_{11}, e_{31}, e_{12}, \\ & e_{13}, e_{22}, e_{32}\}, \{e_{11}, e_{31}, e_{12}, e_{13}, e_{22}, e_{33}\}, \{e_{11}, e_{31}, e_{12}, e_{13}, e_{23}, e_{32}\}, \{e_{11}, e_{31}, e_{12}, e_{13}, e_{23}, e_{33}\}, \\ & \{e_{11}, e_{31}, e_{22}, e_{23}, e_{13}, e_{32}\}, \{e_{11}, e_{31}, e_{22}, e_{23}, e_{13}, e_{33}\}, \{e_{11}, e_{31}, e_{22}, e_{23}, e_{12}, e_{32}\}, \{e_{11}, e_{31}, e_{22}, \\ & e_{23}, e_{12}, e_{33}\}, \{e_{11}, e_{21}, e_{23}, e_{22}, e_{32}, e_{12}\}, \{e_{11}, e_{21}, e_{23}, e_{22}, e_{32}, e_{13}\}, \{e_{11}, e_{21}, e_{23}, e_{22}, e_{33}, e_{12}\}, \\ & \{e_{11}, e_{21}, e_{23}, e_{22}, e_{33}, e_{13}\}, \{e_{11}, e_{21}, e_{32}, e_{33}, e_{22}, e_{12}\}, \{e_{11}, e_{21}, e_{32}, e_{33}, e_{22}, e_{13}\}, \{e_{11}, e_{21}, e_{32}, \\ & e_{33}, e_{23}, e_{12}\}, \{e_{11}, e_{21}, e_{32}, e_{33}, e_{23}, e_{13}\}, \{e_{11}, e_{13}, e_{22}, e_{23}, e_{32}, e_{33}\}, \{e_{11}, e_{12}, e_{22}, e_{23}, e_{32}, e_{33}\}, \\ & \{e_{11}, e_{12}, e_{13}, e_{23}, e_{32}, e_{33}\}, \{e_{11}, e_{12}, e_{13}, e_{22}, e_{32}, e_{33}\}, \{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33}\}, \{e_{11}, e_{12}, e_{13}, \\ & e_{22}, e_{23}, e_{32}\}, \{e_{21}, e_{13}, e_{22}, e_{23}, e_{32}, e_{33}\}, \{e_{21}, e_{12}, e_{22}, e_{23}, e_{32}, e_{33}\}, \{e_{21}, e_{12}, e_{13}, e_{23}, e_{32}, e_{33}\}, \\ & \{e_{21}, e_{12}, e_{13}, e_{22}, e_{32}, e_{33}\}, \{e_{21}, e_{12}, e_{13}, e_{22}, e_{23}, e_{32}\}, \{e_{31}, e_{13}, e_{22}, \\ & e_{23}, e_{32}, e_{33}\}, \{e_{31}, e_{12}, e_{22}, e_{23}, e_{32}, e_{33}\}, \{e_{31}, e_{12}, e_{13}, e_{23}, e_{32}, e_{33}\}, \{e_{31}, e_{12}, e_{13}, e_{22}, e_{32}, e_{33}\}, \\ & \{e_{31}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33}\}, \{e_{31}, e_{12}, e_{13}, e_{22}, e_{23}, e_{32}\} \}. \end{aligned}$$

3 Spanning trees of $\mathcal{J}_{2,m}$ and Face ring $\Delta_s(\mathcal{J}_{2,m})$

In this section, we give two lemmas which give an important characterization of the graph $\mathcal{J}_{2,m}$ and its spanning simplicial complex $s(\mathcal{J}_{2,m})$. We present a proposition which gives the f -vectors and the dimension of the $\mathcal{J}_{2,m}$. Finally, in Theorem 3.13 we give the formulation for the Hilbert series of the Face ring $k[\Delta_s(\mathcal{J}_{2,m})]$.

Definition 3.1. Let $C_{i_1}, C_{i_2}, \dots, C_{i_k}$ be consecutive cycles in the Jahangir's graph $\mathcal{J}_{2,m}$. Then the cycle obtained by deleting the common edges between the consecutive cycles $C_{i_1}, C_{i_2}, \dots, C_{i_k}$ is a new cycle of the Jahangir's graph $\mathcal{J}_{2,m}$ is denoted by C_{i_1, i_2, \dots, i_k} . The cardinality count of the number of edges in the cycle C_{i_1, i_2, \dots, i_k} is denoted by $\beta_{i_1, i_2, \dots, i_k} = |C_{i_1, i_2, \dots, i_k}|$.

The following lemma computes the total number of cycles in the Jahangir's graph $\mathcal{J}_{2,m}$ and the cardinality count of the edges in these cycles.

Lemma 3.2 (Characterization of $\mathcal{J}_{2,m}$). Let $\mathcal{J}_{2,m}$ be the graph with the edges E as is defined in eq. (1) and C_1, C_2, \dots, C_m be its m consecutive cycles of equal lengths, then the total number of cycles in the graph are

$$\tau = m^2$$

such that $\beta_{i_1, i_2, \dots, i_k} = 2(k+1)$.

Proof. The Jahangir's graph $\mathcal{J}_{2,m}$ contains more than just m consecutive cycles. The remaining cycles can be obtained by deleting the common edges between any number (included) of consecutive cycles and getting a cycle by their remaining edges. The cycle obtained in this way by adjoining consecutive cycles $C_{i_1}, C_{i_2}, \dots, C_{i_k}$ is denoted by C_{i_1, i_2, \dots, i_k} . Therefore, we get the following cycles

$$\begin{aligned} &C_{1,2}, C_{2,3}, \dots, C_{m-1,m}, C_{m,1}, C_{1,2,3}, \dots, C_{m-2,m-1,m}, C_{m-1,m,1}, \\ &C_{m,1,2}, \dots, C_{1,2,3,\dots,m}, C_{2,3,4,\dots,m,1}, C_{3,4,5,\dots,m,1,2}, C_{m,1,2,\dots,m-1}. \end{aligned}$$

Combining these with m cycles given we have total cycles in the graph $\mathcal{J}_{2,m}$,

$$C_{i_1, i_2, \dots, i_k} \quad i_j \in \{1, 2, \dots, m\} \text{ and } 1 \leq k \leq m,$$

such that $i_{j+1} = i_j + 1$ if $i_j \neq m$ and $i_{j+1} = 1$ if $i_j = m$.

Now for a fixed value of k , simple counting reveals that the total number of cycles C_{i_1, i_2, \dots, i_k} is m for $i_k < m$. Hence the total number of cycles in $\mathcal{J}_{2,m}$ is τ . Also it is clear from the construction above that C_{i_1, i_2, \dots, i_k} is obtained by deleting common edges between consecutive cycles $C_{i_1}, C_{i_2}, \dots, C_{i_k}$ which are $k-1$ in number. Therefore, the order of the cycle C_{i_1, i_2, \dots, i_k} is obtained by adding orders of all $C_{i_1}, C_{i_2}, \dots, C_{i_k}$ that is, $4k$ and subtracting $2(k-1)$ from it, since the common edges are being counted twice in sum. This implies

$$\beta_{i_1, i_2, \dots, i_k} = |C_{i_1, i_2, \dots, i_k}| = \sum_{t=1}^k |C_{i_t}| - 2(k-1) = 2(k+1). \quad \square$$

In the following results, we fix $C_{u_1, u_2, \dots, u_p}, C_{v_1, v_2, \dots, v_q}$ to represent any two cycles from the cycles

$$C_{i_1, i_2, \dots, i_k} \quad i_j \in \{1, 2, \dots, m\} \text{ and } 1 \leq k \leq m,$$

such that $i_{j+1} = i_j + 1$ if $i_j \neq m$ and $i_{j+1} = 1$ if $i_j = m$, of the graph $\mathcal{J}_{2,m}$. Also we fix the notation " $a \rightarrow b$ " if b immediately proceeds a i.e., the very next in order of preferences.

Proposition 3.3. Let $\mathcal{J}_{2,m}$ be the graph with the edges E as defined in eq. (1) such that $\{u_1, u_2, \dots, u_p\} \subseteq \{v_1, v_2, \dots, v_q\}$ then we have

$$|C_{u_1, u_2, \dots, u_p} \cap C_{v_1, v_2, \dots, v_q}| = \begin{cases} \beta_{u_1, u_2, \dots, u_p} - 2, & \{u_1, u_p\} \notin \{v_1, v_q\} \\ \beta_{u_1, u_2, \dots, u_p} - 1, & u_1 \in \{v_1, v_q\} \text{ \& } u_p \notin \{v_1, v_q\} \\ \beta_{u_1, u_2, \dots, u_p} - 1, & u_p \in \{v_1, v_q\} \text{ \& } u_1 \notin \{v_1, v_q\} \\ \beta_{u_1, u_2, \dots, u_p}, & u_1 = v_1 \text{ \& } u_p = v_q \text{ or } u_1 = v_q \text{ \& } u_p = v_1 \end{cases}$$

Proof. Since the cycles C_{u_1, u_2, \dots, u_p} and C_{v_1, v_2, \dots, v_q} are obtained by deleting the common edges between cycles $C_{u_1}, C_{u_2}, \dots, C_{u_p}$ and $C_{v_1}, C_{v_2}, \dots, C_{v_q}$ respectively, therefore, $\{u_1, u_p\} \notin \{v_1, v_q\}$ implies $\{u_1, u_2, \dots, u_p\} \subset \{v_1, v_2, \dots, v_q\}$. Hence, the intersection $C_{u_1, u_2, \dots, u_p} \cap C_{v_1, v_2, \dots, v_q}$ will contain only the non common edges of the cycle C_{u_1, u_2, \dots, u_p} excluding its two edges common with the cycles on its each end. This gives the order of intersection in this case $\beta_{u_1, u_2, \dots, u_p} - 2$. The remaining cases can be visualized in a similar manner. \square

Proposition 3.4. Let $\mathcal{J}_{2,m}$ be the graph with the edges E as defined in eq. (1) such that $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_\sigma\} \subseteq \{v_1, v_2, \dots, v_q\}$ and $\bar{u}_t \in \{u_1, u_2, \dots, u_p\}$ & $\bar{u}_{t-1} \rightarrow \bar{u}_t$ with $t \leq \sigma < p$ then we have

$$\left| C_{u_1, u_2, \dots, u_p} \cap C_{v_1, v_2, \dots, v_q} \right| = \begin{cases} \beta_{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_\sigma} - 1, & \bar{u}_1 = v_1 \text{ \& } v_q \rightarrow u_1 \\ \beta_{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_\sigma} - 2, & \bar{u}_1 = v_1 \text{ \& } v_q \not\rightarrow u_1 \\ \beta_{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_\sigma} - 1, & \bar{u}_\sigma = v_q \text{ \& } u_p \rightarrow v_1 \\ \beta_{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_\sigma} - 2, & \bar{u}_\sigma = v_q \text{ \& } u_p \not\rightarrow v_1 \end{cases}$$

Proof. Here, the cycles $C_{\bar{u}_1}, C_{\bar{u}_2}, \dots, C_{\bar{u}_\sigma}$ are amongst σ consecutive adjoining cycles of the cycle C_{u_1, u_2, \dots, u_p} which are also overlapping with the σ consecutive adjoining cycles of the cycle C_{v_1, v_2, \dots, v_q} . If the adjoining cycle $C_{\bar{u}_1}$ of the cycle C_{u_1, u_2, \dots, u_p} overlaps with the first adjoining cycle C_{v_1} of the cycle C_{v_1, v_2, \dots, v_q} and the adjoining cycles C_{v_q} and C_{u_1} are consecutive then by previous proposition the order of the intersection $C_{u_1, u_2, \dots, u_p} \cap C_{v_1, v_2, \dots, v_q}$ is indeed $\beta_{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_\sigma} - 1$. Similarly, if the adjoining cycles C_{v_q} and C_{u_1} are not consecutive then they will have no common edge and the use of proposition 3.3 gives the order of the intersection $C_{u_1, u_2, \dots, u_p} \cap C_{v_1, v_2, \dots, v_q}$ as $\beta_{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_\sigma} - 2$. Similar can be done for the remaining cases. \square

Remark 3.5. The case when there exists a $t_0 < \sigma < p$ such that $\bar{u}_{t_0-1} \not\rightarrow \bar{u}_{t_0}$ in above proposition i.e., when cycles $C_{\bar{u}_1}, C_{\bar{u}_2}, \dots, C_{\bar{u}_{t_0-1}}, C_{\bar{u}_{t_0}}, \dots, C_{\bar{u}_\sigma}$ are not amongst σ consecutive adjoining cycles of the cycle C_{u_1, u_2, \dots, u_p} , the order of the intersection $C_{u_1, u_2, \dots, u_p} \cap C_{v_1, v_2, \dots, v_q}$ can be calculated by applying proposition 3.4 on the overlapping portions.

Proposition 3.6. Let $\mathcal{J}_{2,m}$ be the graph with the edges E as defined in (1) such that $\{u_1, u_2, \dots, u_p\} \cap \{v_1, v_2, \dots, v_q\} = \varphi$ and $p \leq q$. Then we have

$$\left| C_{u_1, u_2, \dots, u_p} \cap C_{v_1, v_2, \dots, v_q} \right| = \begin{cases} 1, & u_p \rightarrow v_1 \text{ \& } v_q \not\rightarrow u_1 \\ 1, & u_p \not\rightarrow v_1 \text{ \& } v_q \rightarrow u_1 \\ 2, & u_p \rightarrow v_1 \text{ \& } v_q \rightarrow u_1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. In this case the adjoining cycles of C_{u_1, u_2, \dots, u_p} and C_{v_1, v_2, \dots, v_q} have no common cycle. However, if the adjoining cycle on one of the extreme ends of the cycle C_{u_1, u_2, \dots, u_p} is consecutive with the adjoining cycles on one of the extreme ends of the other cycle C_{v_1, v_2, \dots, v_q} then the intersection $C_{u_1, u_2, \dots, u_p} \cap C_{v_1, v_2, \dots, v_q}$ will have only one edge. The remaining cases are easy to see. \square

In the following three propositions we give some characterizations of $\mathcal{J}_{2,m}$. We fix $E(T_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)})$, where $j_\alpha \in \{1, 2, \dots, m\}$ and $i_\alpha \in \{1, 2, 3\}$, as a subset of $E(\mathcal{J}_{2,m})$.

Proposition 3.7. A subset $E(T_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)})$ of E with $j_\alpha i_\alpha \neq j_\alpha 1$ for all α will belong to $s(\mathcal{J}_{2,m})$ if and only if

$$E(T_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}) = E \setminus \{e_{1i_1}, e_{2i_2}, \dots, e_{mi_m}\}$$

Proof. $\mathcal{J}_{2,m}$ is a graph with cycles C_1, C_2, \dots, C_m and $e_{11}, e_{21}, \dots, e_{m1}$ are the common edges between the consecutive cycles. The cutting down process explains we need to remove exactly m edges, keeping the graph connected and no cycles and no isolated edge left and no isolated vertices left in the graph. Therefore, in order to obtain a spanning tree of $\mathcal{J}_{2,m}$ with none of common edges $e_{11}, e_{21}, \dots, e_{m1}$ to be removed, we need to remove exactly one edge from the non common edges from each cycle. This explains the proof of the proposition. \square

Proposition 3.8. A subset $E(T_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)})$ of E with $j_\alpha i_\alpha = j_\alpha 1$ for any α will belong to $s(\mathcal{J}_{2,m})$ if and only if

$$E(T_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}) = E \setminus \{e_{j_1 i_1}, e_{j_2 i_2}, \dots, e_{j_m i_m}\}$$

where, $\{e_{j_1 i_1}, e_{j_2 i_2}, \dots, e_{j_m i_m}\}$ will contain exactly one edge from $C_{(j_\alpha - 1)(j_\alpha)} \setminus \{e_{(j_\alpha - 1)1}, e_{(j_\alpha + 1)1}\}$ other than $e_{j_\alpha 1}$.

Proof. For a spanning tree of $\mathcal{J}_{2,m}$ such that exactly one common edge $e_{j_\alpha 1}$ is removed, we need to remove precisely $m - 1$ edges from the remaining edges using the cutting down process. However, we cannot remove more than one edge from the non common edges of the cycle $C_{(j_\alpha - 1)(j_\alpha)}$ (since this will result a disconnected graph. This explains the proof of the above case. \square

Proposition 3.9. A subset $E(T_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}) \subset E$, where $j_\alpha i_\alpha = j_\alpha 1$ for $\alpha \in \{r_1, r_2, \dots, r_\rho\} \subset \{1, 2, \dots, m\}$, will belong to $s(\mathcal{J}_{2,m})$ if and only if it satisfies any of the following:

1. if $e_{j_{r_1} 1}, e_{j_{r_2} 1}, \dots, e_{j_{r_\rho} 1}$ are common edges from consecutive cycles then

$$E(T_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}) = E \setminus \{e_{j_1 i_1}, e_{j_2 i_2}, \dots, e_{j_m i_m}\}$$

such that $\{e_{j_1 i_1}, e_{j_2 i_2}, \dots, e_{j_m i_m}\}$ will contain exactly one edge from $C_{j_{r_0} j_{r_1} \dots j_{r_\rho}}$ other than $e_{j_{r_1} 1}, e_{j_{(r_2)} 1}, \dots, e_{j_{r_\rho} 1}$, where $j_{r_0} \rightarrow j_{r_1}$.

2. if none of $e_{j_{r_1} 1}, e_{j_{r_2} 1}, \dots, e_{j_{r_\rho} 1}$ are common edges from consecutive cycles then

$$E(T_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}) = E \setminus \{e_{j_1 i_1}, e_{j_2 i_2}, \dots, e_{j_m i_m}\}$$

such that for each edge $e_{j_{r_t} 1}$ proposition 3.6 holds.

3. if some of $e_{j_{r_1} 1}, e_{j_{r_2} 1}, \dots, e_{j_{r_\rho} 1}$ are common edges from consecutive cycles then

$$E(T_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}) = E \setminus \{e_{j_1 i_1}, e_{j_2 i_2}, \dots, e_{j_m i_m}\}$$

such that proposition 3.9.1 is satisfied for the common edges of consecutive cycles and proposition 3.9.2 is satisfied for remaining common edges.

Proof. For the case 1, we need to obtain a spanning tree of $\mathcal{J}_{2,m}$ such that $|r_\rho - r_1|_m$ common edges must be removed from ρ consecutive cycles $C_{j_{r_1}}, C_{j_{r_2}}, \dots, C_{j_{r_\rho}}$. The remaining $m - |r_\rho - r_1|_m$ edges must be removed in such a way that exactly one edge is removed from the non common edges of the adjoining cycles $C_{j_{r_0}}, C_{j_{r_1}}, \dots, C_{j_{r_\rho}}$ and the remaining $m - |r_\rho - r_1|_m$ cycles of the graph $\mathcal{J}_{2,m}$. This concludes the case.

The remaining cases of the proposition can be visualised in a similar manner using the propositions 3.7 and 3.8. This completes the proof. \square

Remark 3.10. If we denote the disjoint classes of subsets of E discussed in propositions 3.7, 3.8 and 3.9 by $\mathcal{C}_{\mathcal{J}_1}, \mathcal{C}_{\mathcal{J}_2}, \mathcal{C}_{\mathcal{J}_{3a}}, \mathcal{C}_{\mathcal{J}_{3b}}, \mathcal{C}_{\mathcal{J}_{3c}}$ respectively, then, we can write $s(\mathcal{J}_{2,m})$ as follows:

$$s(\mathcal{J}_{2,m}) = \mathcal{C}_{\mathcal{J}_1} \cup \mathcal{C}_{\mathcal{J}_2} \cup \mathcal{C}_{\mathcal{J}_{3a}} \cup \mathcal{C}_{\mathcal{J}_{3b}} \cup \mathcal{C}_{\mathcal{J}_{3c}}.$$

In our next result, we give an important characterization of the f -vectors of $\Delta_s(\mathcal{J}_{2,m})$.

Proposition 3.11. Let $\Delta_s(\mathcal{J}_{2,m})$ be a spanning simplicial complex of the graph $\mathcal{J}_{2,m}$, then the $\dim(\Delta_s(\mathcal{J}_{2,m})) = 2m - 1$ with f -vector $f(\Delta_s(\mathcal{J}_{2,m})) = (f_0, f_1, \dots, f_{2m-1})$ and

$$f_i = \binom{3m}{i+1} + \sum_{t=1}^{\tau} (-1)^t \left[\sum_{\{i_1, i_2, \dots, i_t\} \in C_i^t} \left(3m - \sum_{s=1}^t \beta_{i_s} + \sum_{\{i_u, i_v\} \subseteq \{i_p\}_{p=1}^t} |C_{i_u} \cap C_{i_v}| \right) \right]$$

where $0 \leq i \leq 2m - 1$, $I = \{i_1 i_2 \dots i_k | i_j \in \{1, 2, \dots, m\} \text{ and } 1 \leq k \leq m \text{ such that } i_{j+1} = i_j + 1 \text{ if } i_j \neq m \text{ and } i_{j+1} = 1 \text{ if } i_j = m\}$ and $C_i^t = \{\text{Subsets of } I \text{ of cardinality } t\}$.

Proof. Let E be the edge set of $\mathcal{J}_{2,m}$ and $\mathcal{C}_{\mathcal{J}_1}, \mathcal{C}_{\mathcal{J}_2}, \mathcal{C}_{\mathcal{J}_{3a}}, \mathcal{C}_{\mathcal{J}_{3b}}, \mathcal{C}_{\mathcal{J}_{3c}}$ are disjoint classes of spanning trees of $\mathcal{J}_{2,m}$ then from propositions 3.7, 3.8, 3.9 and the remark 3.10 we have

$$s(\mathcal{J}_{2,m}) = \mathcal{C}_{\mathcal{J}_1} \cup \mathcal{C}_{\mathcal{J}_2} \cup \mathcal{C}_{\mathcal{J}_{3a}} \cup \mathcal{C}_{\mathcal{J}_{3b}} \cup \mathcal{C}_{\mathcal{J}_{3c}}.$$

Therefore, by definition 2.5 we can write $\Delta_s(\mathcal{J}_{2,m}) = \langle \mathcal{C}_{\mathcal{J}_1} \cup \mathcal{C}_{\mathcal{J}_2} \cup \mathcal{C}_{\mathcal{J}_{3a}} \cup \mathcal{C}_{\mathcal{J}_{3b}} \cup \mathcal{C}_{\mathcal{J}_{3c}} \rangle$. Since each facet $\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)} = E(T_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)})$ is obtained by deleting exactly m edges from the edge set of $\mathcal{J}_{2,m}$, keeping in view the propositions 3.7, 3.8 and 3.9, therefore dimension of each facet is the same i.e., $2m - 1$ (since $|\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}| = 2m$) and hence dimension of $\Delta_s(\mathcal{J}_{2,m})$ will be $2m - 1$.

Also it is clear from the definition of $\Delta_s(\mathcal{J}_{2,m})$ that it contains all those subsets of E which do not contain the given sets of cycles $\{e_{k1}, e_{k2}, e_{k3}, e_{(k+1)1}\}$ for $k \in \{1, 2, \dots, m-1\}$ and $\{e_{m1}, e_{m2}, e_{m3}, e_{11}\}$ in graph as well as any other cycle in the graph $\mathcal{J}_{2,m}$.

Now by lemma 3.2 the total cycles in the graph $\mathcal{J}_{2,m}$ are

$$C_{i_1, i_2, \dots, i_k} \quad i_j \in \{1, 2, \dots, m\} \text{ and } 1 \leq k \leq m,$$

such that $i_{j+1} = i_j + 1$ if $i_j \neq m$ and $i_{j+1} = 1$ if $i_j = m$, and their total number is τ . Let F be any subset of E of order $i + 1$ such that it does not contain any C_{i_1, i_2, \dots, i_k} $i_j \in \{1, 2, \dots, m\}$ and $1 \leq k \leq m$, in it. The total number of such F is indeed f_i for $0 \leq i \leq 2m - 1$. We use inclusion exclusion principle to find this number. Therefore, $f_i =$ Total number of subsets of E of order $i + 1$ not containing C_{i_1, i_2, \dots, i_k} $i_j \in \{1, 2, \dots, m\}$ and $1 \leq k \leq m$ such that $i_{j+1} = i_j + 1$ if $i_j \neq m$ and $i_{j+1} = 1$ if $i_j = m$.

Therefore, using these notations and applying Inclusion Exclusion Principle we can write, $f_i =$ (Total number of subsets of E of order $i + 1$) $- \sum_{\{i_1\} \in C_I^1} (\text{subset of } E \text{ of order } i + 1 \text{ containing } C_{i_s} \text{ for } s = 1) + \sum_{\{i_1, i_2\} \in C_I^2} (\text{subset of } E \text{ of order } i + 1 \text{ containing both } C_{i_s} \text{ for all } 1 \leq s \leq 2) - \dots + (-1)^\tau \sum_{\{i_1, i_2, \dots, i_\tau\} \in C_I^\tau} (\text{subset of } E \text{ of order } i + 1 \text{ simultaneously containing each } C_{i_s} \text{ for all } 1 \leq s \leq \tau).$

This implies

$$f_i = \binom{3m}{i+1} - \left[\sum_{\{i_1\} \in C_I^1} \binom{3m - \beta_{i_1}}{i+1 - \beta_{i_1}} \right] + \left[\sum_{\{i_1, i_2\} \in C_I^2} \left(3m - \sum_{s=1}^2 \beta_{i_s} + \sum_{\{i_u, i_v\} \subseteq \{i_p\}_{p=1}^2} |C_{i_u} \cap C_{i_v}| \right) \right] - \dots + (-1)^\tau \left[\sum_{\{i_1, i_2, \dots, i_\tau\} \in C_I^\tau} \left(3m - \sum_{s=1}^\tau \beta_{i_s} + \sum_{\{i_u, i_v\} \subseteq \{i_p\}_{p=1}^\tau} |C_{i_u} \cap C_{i_v}| \right) \right].$$

This implies

$$f_i = \binom{3m}{i+1} + \sum_{t=1}^\tau (-1)^t \left[\sum_{\{i_1, i_2, \dots, i_t\} \in C_I^t} \left(3m - \sum_{s=1}^t \beta_{i_s} + \sum_{\{i_u, i_v\} \subseteq \{i_p\}_{p=1}^t} |C_{i_u} \cap C_{i_v}| \right) \right]. \quad \square$$

Example 3.12. Let $\Delta_s(\mathcal{J}_{2,3})$ be a spanning simplicial complex of the Jahangir's graph $\mathcal{J}_{2,m}$ given in Figure 1, then the $\dim(\Delta_s(\mathcal{J}_{2,3})) = 5$ and $\tau = 3^2 = 9$. Therefore, f -vectors $f(\Delta_s(\mathcal{J}_{2,3})) = (f_0, f_1, \dots, f_5)$ and

$$f_i = \binom{9}{i+1} - \left[\sum_{\{i_1\} \in C_I^1} \binom{9 - \beta_{i_1}}{i+1 - \beta_{i_1}} \right] +$$

$$\left[\sum_{\{i_1, i_2\} \in C_I^2} \left(9 - \sum_{s=1}^2 \beta_{i_s} + \sum_{\{i_u, i_v\} \subseteq \{i_p\}_{p=1}^2} |C_{i_u} \cap C_{i_v}| \right) \right] \\ - \dots + (-1)^9 \left[\sum_{\{i_1, i_2, \dots, i_9\} \in C_I^9} \left(3m - \sum_{s=1}^9 \beta_{i_s} + \sum_{\{i_u, i_v\} \subseteq \{i_p\}_{p=1}^9} |C_{i_u} \cap C_{i_v}| \right) \right],$$

where $0 \leq i \leq 5$.

For a simplicial complex Δ over $[n]$, one would associate to it the Stanley-Reisner ideal, that is, the monomial ideal $I_{\mathcal{N}}(\Delta)$ in $S = k[x_1, x_2, \dots, x_n]$ generated by monomials corresponding to non-faces of this complex (here we are assigning one variable of the polynomial ring to each vertex of the complex). It is well known that the Face ring $k[\Delta] = S/I_{\mathcal{N}}(\Delta)$ is a standard graded algebra. We refer the readers to [7] and [14] for more details about graded algebra A , the Hilbert function $H(A, t)$ and the Hilbert series $H_t(A)$ of a graded algebra.

Our main result of this section is as follows;

Theorem 3.13. *Let $\Delta_s(\mathcal{J}_{2,m})$ be the spanning simplicial complex of $\mathcal{J}_{2,m}$, then the Hilbert series of the Face ring $k[\Delta_s(\mathcal{J}_{2,m})]$ is given by,*

$$H(k[\Delta_s(\mathcal{J}_{2,m})], t) = 1 + \sum_{i=0}^d \frac{\binom{n}{i+1} t^{i+1}}{(1-t)^{i+1}} + \sum_{i=0}^d \sum_{k=1}^{\tau} (-1)^k \left[\sum_{\{i_1, i_2, \dots, i_k\} \in C_I^k} \left(3m - \sum_{s=1}^k \beta_{i_s} + \sum_{\{i_u, i_v\} \subseteq \{i_p\}_{p=1}^k} |C_{i_u} \cap C_{i_v}| \right) \right] \frac{t^{i+1}}{(1-t)^{i+1}}$$

Proof. From [14], we know that if Δ is a simplicial complex of dimension d and $f(\Delta) = (f_0, f_1, \dots, f_d)$ its f -vector, then the Hilbert series of the face ring $k[\Delta]$ is given by

$$H(k[\Delta], t) = 1 + \sum_{i=0}^d \frac{f_i t^{i+1}}{(1-t)^{i+1}}.$$

By substituting the values of f_i 's from Proposition 3.11 in this above expression, we get the desired result. \square

4 Cohen-Macaulayness of the face ring of $\Delta_s(\mathcal{J}_{2,m})$

In this section, we present the Cohen-Macaulayness of the face ring of SSC $\Delta_s(\mathcal{J}_{2,m})$, using the notions and results from [2].

Definition 4.1 ([2]). *Let $I \subset S = k[x_1, x_2, \dots, x_n]$ be a monomial ideal. We say that I has linear residuals, if there exists an ordered minimal monomial system of generators $\{m_1, m_2, \dots, m_r\}$ of I such that $\text{Res}(I_i)$ is minimally generated by linear monomials for all $1 < i \leq r$, where $\text{Res}(I_i) = \{u_1, u_2, \dots, u_{i-1}\}$ such that $u_k = \frac{m_i}{\gcd(m_k, m_i)}$ for all $1 \leq k \leq i-1$.*

Theorem 4.2 ([2]). *Let Δ be a simplicial complex of dimension d over $[n]$. Then Δ will be a shellable if and only if $I_{\mathcal{F}}(\Delta)$ has linear residuals.*

Corollary 4.3 ([2]). *If the facet ideal $I_{\mathcal{F}}(\Delta)$ of a pure simplicial complex Δ over $[n]$ has linear residuals, then the face ring $k[\Delta]$ is Cohen Macaulay.*

Here, we present the main result of this section.

Theorem 4.4. *The face ring of $\Delta_s(\mathcal{J}_{2,m})$ is Cohen-Macaulay.*

Proof. By corollary 4.3, it is sufficient to show that $I_{\mathcal{F}}(\Delta_s(\mathcal{J}_{2,m}))$ has linear residuals in $S = k[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, \dots, x_{m1}, x_{m2}, x_{m3}]$. By propositions 3.7, 3.8, 3.9 and the remark 3.10, we have

$$s(\mathcal{J}_{2,m}) = \mathcal{C}_{\mathcal{J}_1} \cup \mathcal{C}_{\mathcal{J}_2} \cup \mathcal{C}_{\mathcal{J}_{3a}} \cup \mathcal{C}_{\mathcal{J}_{3b}} \cup \mathcal{C}_{\mathcal{J}_{3c}}.$$

Therefore,

$$\Delta_s(\mathcal{J}_{2,m}) = \left\langle \hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)} = E \setminus \{e_{j_1 i_1}, e_{j_2 i_2}, \dots, e_{j_m i_m}\} \mid \hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)} \in s(\mathcal{J}_{2,m}) \right\rangle$$

and hence we can write,

$$I_{\mathcal{F}}(\Delta_s(\mathcal{J}_{2,m})) = \left(x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}} \mid \hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)} \in s(\mathcal{J}_{2,m}) \right).$$

Here, $I_{\mathcal{F}}(\Delta_s(\mathcal{J}_{2,m}))$ is a pure monomial ideal of degree $2m - 1$ with $x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}}$ as the product of all variables in S except $x_{j_1 i_1}, x_{j_2 i_2}, \dots, x_{j_m i_m}$. Now we will show that $I_{\mathcal{F}}(\Delta_s(\mathcal{J}_{2,m}))$ has linear residuals with respect to the following orders in its monomials:

$$\begin{aligned} & \{x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}} \mid i_{r_1} \neq 1; 1 \leq r_1 \leq m \text{ \& } i_k = 1; k \neq r_1\}, \\ & \{x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}} \mid i_{r_1}, i_{r_2} \neq 1; 1 \leq r_1, r_2 \leq m \text{ \& } i_k = 1; k \neq r_1, r_2\}, \\ & \dots \\ & \{x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}} \mid i_{r_1}, i_{r_2}, \dots, i_{r_m} \neq 1; 1 \leq r_1, r_2, \dots, r_m \leq m\}. \end{aligned} \quad (2)$$

More explicitly, the monomials $\{x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}} \mid i_{r_1} \neq 1; 1 \leq r_1 \leq m \text{ \& } i_k = 1; k \neq r_1\}$ in the order 2, consists of monomials of the form $x_{\hat{E}_{(11, 21, \dots, (m-1)1, j_m i_m)}}$, $x_{\hat{E}_{(11, 21, \dots, j_{(m-1)} i_{(m-1)}, m1)}}$, \dots , $x_{\hat{E}_{(11, j_2 i_2, 31, \dots, (m-1)1, m1)}}$, $x_{\hat{E}_{(j_1 i_1, 21, \dots, (m-1)1, m1)}}$, where $i_k \in \{2, 3\}$ and $1 \leq j_k \leq m$. Similarly, other monomials in order 2. Let us put

$$\text{Res}(x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}}) = \left\{ \frac{x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}}}{\gcd(m_k, x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}})} \mid m_k \text{ precedes } x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}} \text{ wrt order 2} \right\}$$

For instance, for $r_1 = m$ in $\text{Res}(x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}})$ we have, $\text{Res}(x_{\hat{E}_{(11, 21, \dots, (m-1)1, j_m i_m)}}) = \left\{ \frac{x_{\hat{E}_{(11, 21, \dots, (m-1)1, j_m i_m)}}}{\gcd(m_k, x_{\hat{E}_{(11, 21, \dots, (m-1)1, j_m i_m)}})} \right\}$ where, m_k in this case are all the monomials in S of the form $x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}}$ where $i_m \neq m$ and $j_m = 2, 3$. Since all the monomials m_k differ from $x_{\hat{E}_{(11, 21, \dots, (m-1)1, j_m i_m)}}$ at only one position, therefore, $\text{Res}(x_{\hat{E}_{(11, 21, \dots, (m-1)1, j_m i_m)}})$ have all linear terms i.e., $\text{Res}(x_{\hat{E}_{(11, 21, \dots, (m-1)1, j_m i_m)}})$ is minimally generated by linear monomials.

Continuing the same process the order 2 of the monomials of $I_{\mathcal{F}}(\Delta_s(\mathcal{J}_{2,m}))$ guarantees that $\text{Res}(x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}})$ is minimally generated by linear monomials for all $x_{\hat{E}_{(j_1 i_1, j_2 i_2, \dots, j_m i_m)}} \in I_{\mathcal{F}}(\Delta_s(\mathcal{J}_{2,m}))$. Hence, $I_{\mathcal{F}}(\Delta_s(\mathcal{J}_{2,m}))$ has linear residuals, and by Corollary 4.3 $\Delta_s(\mathcal{J}_{2,m})$ is Cohen-Macaulay. \square

5 Conclusions and Scopes

We conclude this paper with some perspectives for further study as well as some constraints related to our work.

- The results given in this paper can be naturally extended for any integer $n \geq 2$.
- The scope of SSC of a graph can be explored for some other classes of graphs like the wheel graph W_n etc. However, since finding spanning trees of a general graph is a NP-hard problem, therefore the results given here are not easily extendable for a general class of graph.
- In view of the work done in [8, 9], we intend to find some perspectives for the SSC in studying sensor networks.

Acknowledgement: The authors are grateful to the reviewers and editor for their valuable suggestions to improve the manuscript. ZR is partially supported by the research Grant (1602144025 – P), from University of Sharjah, Sharjah, UAE.

References

- [1] Anwar I., Raza Z., Kashif A., Spanning simplicial complexes of uni-cyclic graphs, *Algebra Colloquium*, 2015, 22-4, 707-710.
- [2] Anwar I., Kosar Z., Nazir S., An efficient algebraic criterion for shellability, submitted for publication, preprint available at <https://arxiv.org/abs/1705.09537>.
- [3] Bruns W., Herzog J., *Cohen Macaulay rings*, Vol.39, Cambridge studies in advanced mathematics, 1998.
- [4] Faridi S., The facet ideal of a simplicial complex, *Manuscripta Mathematica*, 2002, 109, 159-174.
- [5] Faridi S., Simplicial tree are sequentially Cohen-Macaulay, *J. Pure and Applied Algebra*, 2004, 190, 121-136.
- [6] Harary F., *Graph theory*, MA: Addison-Wesley, 1994.
- [7] Herzog J., Hibi T., *Monomial Algebra*, Springer-Verlag, New York Inc, 2009.
- [8] Imbesi M., Barbiera M.L., Vertex covers and sensor networks, submitted for publication, preprint available online at <http://arxiv.org/math/1211.6555v1>, 2012.
- [9] Imbesi M., Barbiera M.L., Vertex covers in graphs with loops, submitted for publication, preprint available online at <http://arxiv.org/math/1210.8198v1>, 2012.
- [10] Kashif A., Raza Z., Anwar I., On the algebraic study of spanning simplicial complex of r-cycles graphs $G_{n,r}$, *ARS Combinatoria*, 2014, 115, 89-99.
- [11] Kashif A., Raza Z., Anwar I., Algebraic characterization of the SSC $\Delta_s(\mathcal{G}_{n,r}^1)$, to appear in *JCMCC*, 2018.
- [12] Lourdusamy A., Jeyaseelan S.S., Mathivanan T., On pebbling jahangir graph, *Gen. Math. Notes*, 2011, 5-2, 42-49.
- [13] Miller E., Sturmfels B., *Combinatorial commutative algebra*, Springer-Verlag, New York Inc., 2005.
- [14] Villarreal R.H., *Monomial algebras*, Dekker, New York, 2001.
- [15] Zhu G., Shi F., Geng Y., Spanning simplicial complexes of n -cyclic graphs with a common edge, *International Electronic Journal of Algebra*, 2014, 15, 132-144.