

Open Mathematics

Research Article

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On some fixed point results for (s, p, α) -contractive mappings in b -metric-like spaces and applications to integral equations

<https://doi.org/10.1515/math-2018-0024>

Received November 15, 2017; accepted January 31, 2018.

Abstract: In this work, we introduce the notions of (s, p, α) -quasi-contractions and (s, p) -weak contractions and deduce some fixed point results concerning such contractions, in the setting of b -metric-like spaces. Our results extend and generalize some recent known results in literature to more general metric spaces. Moreover, some examples and applications support the results.

Keywords: (s, p, α) -quasi-contraction, (s, p) -weak contraction, b -metric-like space, Fixed point

MSC: 47H10, 54H25

1 Introduction

Fixed point theory has received much attention due to its applications in pure mathematics and applied sciences. Recently, a number of generalizations of metric spaces were introduced and extensively studied. In 1989, Bakhtin [1] (and also Czerwik [2]) introduced the concept of b -metric spaces and presented contraction mappings in such metric spaces thus obtaining a generalization of Banach contraction principle. For fixed point theory in b -metric spaces, see [3] – [11] and the references therein.

Amini-Harandi [12] introduced the notion of metric-like spaces, in which the self distance of a point need not be equal to zero. Such spaces play an important role in topology and logical programming. In 2013, Alghamdi et al. [13] generalized the notion of a b -metric by introduction of the concept of a b -metric-like and proved some related fixed point results. Recently, many results on fixed points, of mappings under certain contractive conditions in such spaces have been obtained (see [11] – [29]).

Fixed point theory has been extended in various directions either by using generalized contractions, or by using more general spaces. Under these directions, in the first part of this paper, we introduce the concept of (s, p, α) -contractions and quasi-contractions and prove some fixed point results. In the second part, we generalize further this new class of contractions for self-mappings, introducing the class of (s, p) -weak contractions. Considering such more general, and much wider classes of contractions, the obtained results greatly extend and improve some classical and recent fixed point results in the existing literature.

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2 Preliminaries

Definition 2.1 ([12]). Let X be a nonempty set. A mapping $\sigma : X \times X \rightarrow [0, \infty)$ is called metric-like if the following conditions hold for all $x, y, z \in X$:

- $\sigma(x, y) = 0$ implies $x = y$,
- $\sigma(x, y) = \sigma(y, x)$
- $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is called a metric-like space.

Definition 2.2 ([13]). Let X be a nonempty set. A mapping $\sigma_b : X \times X \rightarrow [0, \infty)$ is called b -metric-like if the following conditions hold for some $s \geq 1$ and for all $x, y, z \in X$:

- $\sigma_b(x, y) = 0$ implies $x = y$,
- $\sigma_b(x, y) = \sigma_b(y, x)$
- $\sigma_b(x, y) \leq s[\sigma_b(x, z) + \sigma_b(z, y)]$.

The pair (X, σ_b) is called a b -metric-like space.

In a b -metric-like space (X, σ_b) , if $x, y \in X$ and $\sigma_b(x, y) = 0$, then $x = y$, but the converse need not be true, and $\sigma_b(x, x)$ may be positive for some $x \in X$.

Example 2.3. If $X = \mathbb{R}$, then $\sigma_b(x, y) = |x| + |y|$ defines a metric-like on X .

Example 2.4. Let $X = \mathbb{R}^+ \cup \{0\}$ and $\alpha > 0$ be any constant. Define the distance function $\sigma : X \times X \rightarrow [0, \infty)$ by $\sigma(x, y) = \alpha(x + y)$. Then, the pair (X, σ) is a metric-like space.

Example 2.5. If $X = \mathbb{R}^+ \cup \{0\}$, then $\sigma_b(x, y) = (x + y)^2$ defines a b -metric-like on X with parameter $s = 2$.

Definition 2.6 ([13]). Let (X, σ_b) be a b -metric-like space with parameter s , and let $\{x_n\}$ be any sequence in X and $x \in X$. Then

1. The sequence $\{x_n\}$ is said to be convergent to x if $\lim_{n \rightarrow \infty} \sigma_b(x_n, x) = \sigma_b(x, x)$;
2. The sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, σ_b) if $\lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m)$ exists and is finite;
3. (X, σ_b) is said to be a complete b -metric-like space if, for every Cauchy sequence $\{x_n\}$ in X , there exists an $x \in X$ such that $\lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) = \lim_{n \rightarrow \infty} \sigma_b(x_n, x) = \sigma_b(x, x)$.

The limit of a sequence in a b -metric-like space need not be unique.

Lemma 2.7 ([19]). Let (X, σ_b) be a b -metric-like space with parameter s , and $f : X \rightarrow X$ be a mapping. Suppose that f is continuous at $u \in X$. Then for all sequences $\{x_n\}$ in X such that $x_n \rightarrow u$, we have $fx_n \rightarrow fu$ that is

$$\lim_{n \rightarrow \infty} \sigma_b(fx_n, fu) = \sigma_b(fu, fu).$$

Lemma 2.8 ([15]). Let (X, σ_b) be a b -metric-like space with parameter $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are σ_b -convergent to x and y , respectively. Then we have

$$\begin{aligned} \frac{1}{s^2} \sigma_b(x, y) - \frac{1}{s} \sigma_b(x, x) - \sigma_b(y, y) &\leq \liminf_{n \rightarrow \infty} \sigma_b(x_n, y_n) \\ &\leq \limsup_{n \rightarrow \infty} \sigma_b(x_n, y_n) \leq s \sigma_b(x, x) + s^2 \sigma_b(y, y) + s^2 \sigma_b(x, y). \end{aligned}$$

In particular, if $\sigma_b(x, y) = 0$, then we have $\lim_{n \rightarrow \infty} \sigma_b(x_n, y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\begin{aligned} \frac{1}{s} \sigma_b(x, z) - \sigma_b(x, x) &\leq \liminf_{n \rightarrow \infty} \sigma_b(x_n, z) \\ &\leq \limsup_{n \rightarrow \infty} \sigma_b(x_n, z) \leq s \sigma_b(x, z) + s \sigma_b(x, x). \end{aligned}$$

In particular, if $\sigma_b(x, x) = 0$, then

$$\begin{aligned} \frac{1}{s} \sigma_b(x, z) &\leq \liminf_{n \rightarrow \infty} \sigma_b(x_n, z) \\ &\leq \limsup_{n \rightarrow \infty} \sigma_b(x_n, z) \leq s \sigma_b(x, z). \end{aligned}$$

The following result is useful.

Lemma 2.9. Let (X, σ_b) be a b -metric-like space with parameter $s \geq 1$. Then

1. If $\sigma_b(x, y) = 0$, then $\sigma_b(x, x) = \sigma_b(y, y) = 0$;
2. If (x_n) is a sequence such that $\lim_{n \rightarrow \infty} \sigma_b(x_n, x_{n+1}) = 0$, then we have

$$\lim_{n \rightarrow \infty} \sigma_b(x_n, x_n) = \lim_{n \rightarrow \infty} \sigma_b(x_{n+1}, x_{n+1}) = 0;$$

3. If $x \neq y$, then $\sigma_b(x, y) > 0$;

Proof. The proof is obvious. □

Lemma 2.10. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$ and let $\{x_n\}$ be a sequence such that

$$\lim_{n \rightarrow \infty} \sigma_b(x_n, x_{n+1}) = 0 \quad (1)$$

If for the sequence $\{x_n\}$, $\lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) \neq 0$, then there exist $\varepsilon > 0$ and sequences $\{m(k)\}_{k=1}^{\infty}$ and $\{n(k)\}_{k=1}^{\infty}$ of positive integers with $n_k > m_k > k$, such that $\sigma_b(x_{m_k}, x_{n_k}) \geq \varepsilon$, $\sigma_b(x_{m_k}, x_{n_k-1}) < \varepsilon$, $\varepsilon/s^2 \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s$, and $\varepsilon/s \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{n_k-1}, x_{m_k}) \leq \varepsilon$, $\varepsilon/s \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k-1}, x_{n_k}) \leq \varepsilon s^2$.

Proof. If $\lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) \neq 0$, then there exist an $\varepsilon > 0$ and sequences $\{m(k)\}_{k=1}^{\infty}$ and $\{n(k)\}_{k=1}^{\infty}$ of positive integers with $n_k > m_k > k$, such that n_k is the smallest index for which

$$n_k > m_k > k, \quad \sigma_b(x_{m_k}, x_{n_k}) \geq \varepsilon. \quad (2)$$

This means that

$$\sigma_b(x_{m_k}, x_{n_k-1}) < \varepsilon \quad (3)$$

From (2) and property of Definition 2.2, we have

$$\begin{aligned} \varepsilon &\leq \sigma_b(x_{m_k}, x_{n_k}) \leq s \sigma_b(x_{m_k}, x_{m_k-1}) + s \sigma_b(x_{m_k-1}, x_{n_k}) \\ &\leq s \sigma_b(x_{m_k}, x_{m_k-1}) + s^2 \sigma_b(x_{m_k-1}, x_{n_k-1}) + s^2 \sigma_b(x_{n_k-1}, x_{n_k}). \end{aligned} \quad (4)$$

Taking the upper limit as $k \rightarrow \infty$ in (4), using the assumption (1) and relations (2) and (3) we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k-1}, x_{n_k-1}). \quad (5)$$

By the triangular inequality, we have

$$\sigma_b(x_{m_k-1}, x_{n_k-1}) \leq s \sigma_b(x_{m_k-1}, x_{m_k}) + s \sigma_b(x_{m_k}, x_{n_k-1}),$$

so, taking the upper limit as $k \rightarrow \infty$ and using (1), we get

$$\limsup_{k \rightarrow \infty} \sigma_b(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s. \quad (6)$$

By (5) and (6) we have

$$\frac{\varepsilon}{s^2} \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s. \quad (7)$$

Also we have

$$\varepsilon \leq \sigma_b(x_{m_k}, x_{n_k}) \leq S\sigma_b(x_{m_k}, x_{m_{k-1}}) + S\sigma_b(x_{m_{k-1}}, x_{n_k}),$$

and, taking the upper limit as $k \rightarrow \infty$, we get

$$\frac{\varepsilon}{S} \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_{k-1}}, x_{n_k}). \quad (8)$$

Again

$$\varepsilon \leq \sigma_b(x_{m_k}, x_{n_k}) \leq S\sigma_b(x_{m_k}, x_{n_{k-1}}) + S\sigma_b(x_{n_{k-1}}, x_{n_k}).$$

Taking the upper limit as $k \rightarrow \infty$ and using (1), we get

$$\frac{\varepsilon}{S} \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{n_{k-1}}, x_{m_k}). \quad (9)$$

By (2) we have

$$\lim_{k \rightarrow \infty} \sigma_b(x_{n_{k-1}}, x_{m_k}) \leq \varepsilon. \quad (10)$$

Consequently,

$$\frac{\varepsilon}{S} \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{n_{k-1}}, x_{m_k}) \leq \varepsilon. \quad (11)$$

Also

$$\sigma_b(x_{m_{k-1}}, x_{n_k}) \leq S\sigma_b(x_{m_{k-1}}, x_{n_{k-1}}) + S\sigma_b(x_{n_{k-1}}, x_{n_k}).$$

Then from (7), (8) and (1) we have

$$\limsup_{k \rightarrow \infty} \sigma_b(x_{m_{k-1}}, x_{n_k}) \leq S \limsup_{k \rightarrow \infty} \sigma_b(x_{m_{k-1}}, x_{n_{k-1}}) \leq \varepsilon S^2.$$

Consequently,

$$\frac{\varepsilon}{S} \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_{k-1}}, x_{n_k}) \leq \varepsilon S^2. \quad (12)$$

This completes the proof. \square

3 Main results

In this section, we introduce the concept of generalized (s, p, α) -contractions and obtain some fixed point theorems for such class of contractions in the framework of b -metric-like spaces.

Definition 3.1. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$. If $f : X \rightarrow X$ is a self-mapping that satisfies:

$$S\sigma_b(fx, fy) \leq \alpha\sigma_b(x, y)$$

for some $\alpha \in [0, 1)$ and all $x, y \in X$, then f is called an (s, α) -Banach contraction.

Definition 3.2. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$. If $f : X \rightarrow X$ is a self-mapping that satisfies:

$$s^p \sigma_b(fx, fy) \leq \alpha\sigma_b(x, y)$$

for some constants $p \geq 1$ and $\alpha \in [0, 1)$ and for all $x, y \in X$, then f is called an (s, p, α) -Banach contraction.

We denote by Ψ, Φ the families of altering distance functions satisfying the following condition, respectively:

- $\Psi : [0, \infty) \rightarrow [0, \infty)$ is an increasing and continuous function and $\Psi(t) = 0$, iff $t = 0$,
- $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a lower semicontinuous function and $\Phi(t) = 0$, iff $t = 0$.

Based on the definition of Ćirić's quasi-contractions, we introduce the following definition in the setting of a b -metric-like space.

Definition 3.3. Let (X, σ_b) be a b -metric-like space with parameter $s \geq 1$. Let $\psi \in \Psi$, and let constants α, p be such that $0 \leq \alpha < 1$ and $p \geq 2$. A mapping $f : X \rightarrow X$ is said to be a (ψ, s, p, α) -quasicontraction mapping, if for all $x, y \in X$

$$\psi(2s^p \sigma_b(fx, fy)) \leq \alpha \psi \left(\max \{ \sigma_b(x, y), \sigma_b(x, fx), \sigma_b(y, fy), \sigma_b(x, fy), \sigma_b(y, fx) \} \right). \quad (13)$$

- Remark 3.4.**
1. It is obvious that by taking $\psi(t) = \frac{1}{2}t$ (or the identity mapping $\psi(t) = t$) the above notion reduces to an (s, p, α) -quasicontraction.
 2. Taking $\psi(t) = \frac{1}{2}t$ and the arbitrary constant $p = 2$ we obtain the definition of an (s, α) -quasi-contraction given in [30].
 3. If we take $s = 1$, it corresponds to the case of metric-like spaces.

Our first main result is as follows:

Theorem 3.5. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$, $f : X \rightarrow X$ be a given self-mapping. If f is an (ψ, s, p, α) -quasicontraction, then f has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X . We construct a Picard iteration sequence $\{x_n\}$ with initial point x_0 as usual:

$$x_1 = f(x_0), x_2 = f(x_1), \dots, x_{n+1} = f(x_n), \dots \text{ for } n \in \mathbb{N}.$$

If we assume $\sigma_b(x_{n_0}, x_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}$, then we have $x_{n_0+1} = x_{n_0}$ that is $x_{n_0} = x_{n_0+1} = f(x_{n_0})$. Hence, x_{n_0} is a fixed point of f and the proof is completed. From now on, we assume that for all $n \in \mathbb{N}$, $\sigma_b(x_n, x_{n+1}) > 0$ (that is $x_{n+1} \neq x_n$).

By condition (13), we have

$$\begin{aligned} \psi(2s \sigma_b(x_n, x_{n+1})) &= \psi(2s^p \sigma_b(x_n, x_{n+1})) \\ &= \psi(2s^p \sigma_b(fx_{n-1}, fx_n)) \\ &\leq \alpha \psi \left(\max \left\{ \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, fx_{n-1}), \sigma_b(x_n, fx_n), \right. \right. \\ &\quad \left. \left. \sigma_b(x_{n-1}, fx_n), \sigma_b(x_n, fx_{n-1}) \right\} \right) \\ &= \alpha \psi \left(\max \left\{ \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, x_n), \sigma_b(x_n, x_{n+1}), \right. \right. \\ &\quad \left. \left. \sigma_b(x_{n-1}, x_{n+1}), \sigma_b(x_n, x_n) \right\} \right) \\ &\leq \alpha \psi \left(\max \left\{ \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, x_n), \sigma_b(x_n, x_{n+1}), \right. \right. \\ &\quad \left. \left. s[\sigma_b(x_{n-1}, x_n) + \sigma_b(x_n, x_{n+1})], 2s \sigma_b(x_{n-1}, x_n) \right\} \right). \end{aligned} \quad (14)$$

If $\sigma_b(x_{n-1}, x_n) \leq \sigma_b(x_n, x_{n+1})$ for some $n \in \mathbb{N}$, then we find from inequality (14) that

$$\psi(2s \sigma_b(x_n, x_{n+1})) \leq \alpha \psi(2s \sigma_b(x_n, x_{n+1})) < \psi(2s \sigma_b(x_n, x_{n+1})).$$

By the properties of ψ the above inequality gives $\sigma_b(x_n, x_{n+1}) = 0$, which is a contradiction, since we have supposed $\sigma_b(x_n, x_{n+1}) > 0$. Hence, for all $n \in \mathbb{N}$

$$\sigma_b(x_n, x_{n+1}) < \sigma_b(x_{n-1}, x_n),$$

that is, the sequence $\{\sigma_b(x_n, x_{n+1})\}$ is decreasing and bounded below. Thus there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} \sigma_b(x_n, x_{n+1}) = r. \quad (15)$$

Let us prove that $r = 0$. If we suppose that $r > 0$, then applying the condition (14), we have

$$\psi(2s \sigma_b(x_n, x_{n+1})) \leq \psi(2s^p \sigma_b(x_n, x_{n+1})) \leq \alpha \psi(2s \sigma_b(x_{n-1}, x_n)). \quad (16)$$

Taking limit as $n \rightarrow \infty$ in (16), using (15), since $0 \leq \alpha < 1$ and by the properties of ψ , we get

$$\psi(2sr) \leq \alpha \psi(2sr),$$

which is a contradiction. Hence

$$\lim_{n \rightarrow \infty} \sigma_b(x_n, x_{n+1}) = 0. \quad (17)$$

In the next step, we claim that

$$\lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) = 0.$$

Suppose, on the contrary that $\lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) \neq 0$. Then by Lemma 2.10, there exist $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $n_k > m_k > k$, such that $\sigma_b(x_{m_k}, x_{n_k}) \geq \varepsilon$, $\sigma_b(x_{m_k}, x_{n_k-1}) < \varepsilon$ and

$$\begin{aligned} \frac{\varepsilon}{s^2} &\leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s, \\ \frac{\varepsilon}{s} &\leq \limsup_{k \rightarrow \infty} \sigma_b(x_{n_k-1}, x_{m_k}) \leq \varepsilon, \\ \frac{\varepsilon}{s} &\leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k-1}, x_{n_k}) \leq \varepsilon s^2. \end{aligned} \quad (18)$$

From the contractive condition (13), we have

$$\begin{aligned} \psi(2s^2 \sigma_b(x_{m_k}, x_{n_k})) &\leq \psi(2s^p \sigma_b(x_{m_k}, x_{n_k})) = \psi(2s^p \sigma_b(fx_{m_k-1}, fx_{n_k-1})) \\ &\leq \alpha \psi \left(\max \left\{ \sigma_b(x_{m_k-1}, x_{n_k-1}), \sigma_b(x_{m_k-1}, fx_{m_k-1}), \sigma_b(x_{n_k-1}, fx_{n_k-1}), \right. \right. \\ &\quad \left. \left. \sigma_b(x_{m_k-1}, fx_{n_k-1}), \sigma_b(x_{n_k-1}, fx_{m_k-1}) \right\} \right) \\ &= \alpha \psi \left(\max \left\{ \sigma_b(x_{m_k-1}, x_{n_k-1}), \sigma_b(x_{m_k-1}, x_{m_k}), \sigma_b(x_{n_k-1}, x_{n_k}), \right. \right. \\ &\quad \left. \left. \sigma_b(x_{m_k-1}, x_{n_k}), \sigma_b(x_{n_k-1}, x_{m_k}) \right\} \right). \end{aligned} \quad (19)$$

Taking the upper limit as $k \rightarrow \infty$ in (19) and using (17), (18), we obtain

$$\psi(2s^2 \varepsilon) \leq \alpha \psi \left(\max \left\{ \varepsilon s, 0, 0, \varepsilon s^2, \varepsilon \right\} \right) \leq \alpha \psi(2\varepsilon s^2),$$

which is a contradiction due to the properties of ψ and the assumption $\varepsilon > 0$. Hence the sequence $\{x_n\}$ is a Cauchy sequence in the complete b -metric-like space (X, σ_b) . So there is some $u \in X$ such that

$$\lim_{n \rightarrow \infty} \sigma_b(x_n, u) = \sigma_b(u, u) = \lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) = 0. \quad (20)$$

By continuity of f and Lemma 2.7, we have $fx_n \rightarrow fu$ that is $\lim_{n \rightarrow \infty} \sigma_b(x_n, fu) = \sigma_b(fu, fu)$.

On the other hand $\lim_{n \rightarrow \infty} \sigma_b(x_n, u) = 0 = \sigma_b(u, u)$ and so by Lemma 2.8

$$\frac{1}{s} \sigma_b(u, fu) \leq \lim_{n \rightarrow \infty} \sigma_b(x_n, fu) \leq s \sigma_b(u, fu).$$

This implies that

$$\frac{1}{s} \sigma_b(u, fu) \leq \sigma_b(fu, fu) \leq s \sigma_b(u, fu). \quad (21)$$

In view of the properties of ψ , constant $p \geq 2$, (20), (21) and using (13), we have

$$\begin{aligned} \psi(\sigma_b(u, fu)) &\leq \psi(s \sigma_b(fu, fu)) \leq \psi(2s^p \sigma_b(fu, fu)) \\ &\leq \alpha \psi \left(\max \left\{ \sigma_b(u, u), \sigma_b(u, fu), \sigma_b(u, fu), \sigma_b(u, fu), \sigma_b(u, fu) \right\} \right) \\ &= \alpha \psi(\sigma_b(u, fu)). \end{aligned} \quad (22)$$

From (22) and the properties of ψ , we get $\sigma_b(u, fu) = 0$, which implies $fu = u$. Hence u is a fixed point of f .

If the self-map f is not continuous then, we consider

$$\begin{aligned} \psi(2s^2 \sigma_b(x_{n+1}, fu)) &\leq \psi(2s^p \sigma_b(x_{n+1}, fu)) = \psi(2s^p \sigma_b(fx_n, fu)) \\ &\leq \alpha \psi \left(\max \left\{ \sigma_b(x_n, u), \sigma_b(x_n, fx_n), \sigma_b(u, fu), \right. \right. \\ &\quad \left. \left. \sigma_b(x_n, fu), \sigma_b(u, fx_n) \right\} \right) \\ &= \alpha \psi \left(\max \left\{ \sigma_b(x_n, u), \sigma_b(x_n, x_{n+1}), \sigma_b(u, fu), \right. \right. \\ &\quad \left. \left. \sigma_b(x_n, fu), \sigma_b(u, x_{n+1}) \right\} \right). \end{aligned}$$

By taking the upper limit as $n \rightarrow \infty$, using Lemmas 2.8 and 2.10, and the relation (17), we obtain

$$\psi(2s\sigma_b(u, fu)) = \psi\left(2s^2 \frac{1}{s} \sigma_b(u, fu)\right) \leq \psi\left(\limsup_{n \rightarrow \infty} 2s^2 \sigma_b(x_{n+1}, fu)\right) \leq \alpha \psi(2s\sigma_b(u, fu)).$$

From above inequality and the properties of ψ , we get $\sigma_b(u, fu) = 0$, which implies $fu = u$. Hence u is a fixed point of f .

Uniqueness: Let us suppose that u and v are two fixed points of f , i.e. $fu = u$ and $fv = v$. We will show that $u = v$. If not, by using condition (13), we have

$$\begin{aligned} \psi(2s^p \sigma_b(u, v)) &= \psi(2s^p \sigma_b(fu, fv)) \\ &\leq \alpha \psi(\max\{\sigma_b(u, v), \sigma_b(u, fu), \sigma_b(v, fv), \sigma_b(u, fv), \sigma_b(v, fu)\}) \\ &= \alpha \psi(\max\{\sigma_b(u, v), \sigma_b(u, u), \sigma_b(v, v), \sigma_b(u, v), \sigma_b(v, u)\}) \\ &\leq \alpha \psi(2s\sigma_b(u, v)). \end{aligned}$$

Since $0 \leq \alpha < 1$ and $p \geq 2$, the above inequality implies $\sigma_b(u, v) = 0$ which yields $u = v$. \square

The following example illustrates the theorem.

Example 3.6. Let $X = [0, 1]$ and $\sigma_b(x, y) = (x + y)^2$ for all $x, y \in X$. It is clear that σ_b is a b -metric-like on X with parameter $s = 2$ and (X, σ_b) is complete. Also, σ_b is not a metric-like or a b -metric on X . Define a self-mapping $f : X \rightarrow X$ by $fx = \frac{x}{6}$.

For all $x, y \in [0, 1]$, and the function $\psi(t) = 2t$, and constant $p = 2$, we have

$$\begin{aligned} \psi(2s^2 \sigma_b(fx, fy)) &= \psi\left(8\left(\frac{x}{6} + \frac{y}{6}\right)^2\right) = \psi\left(8\frac{(x+y)^2}{36}\right) = \frac{16}{36}(x+y)^2 \\ &= \frac{8}{36}2(x+y)^2 = \frac{8}{36}2\sigma_b(x, y) = \frac{8}{36}\psi(\sigma_b(x, y)) \leq \alpha \psi(\sigma_b(x, y)) \\ &\leq \alpha \psi(\max\{\sigma_b(x, y), \sigma_b(x, fx), \sigma_b(y, fy), \sigma_b(x, fy), \sigma_b(y, fx)\}). \end{aligned}$$

All conditions of Theorem 3.5 are satisfied and clearly $x = 0$ is a unique fixed point of f .

In particular, by taking $\psi(t) = \frac{1}{2}t$ in Theorem 3.5, we have the following result for a self-mapping (seen as a generalization of Ćirić type quasi-contraction).

Corollary 3.7. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$. If $f : X \rightarrow X$ is a self-mapping that satisfies:

$$s^p \sigma_b(fx, fy) \leq \alpha \max\{\sigma_b(x, y), \sigma_b(x, fx), \sigma_b(y, fy), \sigma_b(x, fy), \sigma_b(y, fx)\}$$

for some constants $\alpha \in [0, 1/2)$ and $p \geq 2$ all $x, y \in X$, then f has a unique fixed point in X .

The following is a version of Hardy-Rogers result in [31].

Corollary 3.8. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$. If $f : X \rightarrow X$ is a self-mapping and there exist $p \geq 2$ and constants $a_i \geq 0$, $i = 1, \dots, 5$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that

$$s^p \sigma_b(fx, fy) \leq \alpha_1 \sigma_b(x, y) + \alpha_2 \sigma_b(x, fx) + \alpha_3 \sigma_b(y, fy) + \alpha_4 \sigma_b(x, fy) + \alpha_5 \sigma_b(y, fx),$$

for all $x, y \in X$, then f has a unique fixed point in X .

Proof. This result can be considered as a consequence of Corollary 3.7, since we have

$$\begin{aligned} &\alpha_1 \sigma_b(x, y) + \alpha_2 \sigma_b(x, fx) + \alpha_3 \sigma_b(y, fy) + \alpha_4 \sigma_b(x, fy) + \alpha_5 \sigma_b(y, fx) \\ &\leq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \max\{\sigma_b(x, y), \sigma_b(x, fx), \sigma_b(y, fy), \sigma_b(x, fy), \sigma_b(y, fx)\} \\ &= \alpha \max\{\sigma_b(x, y), \sigma_b(x, fx), \sigma_b(y, fy), \sigma_b(x, fy), \sigma_b(y, fx)\}. \end{aligned}$$

\square

Remark 3.9. Theorem 3.5 generalizes Theorem 1.2 in [32]. Theorem 3.2 in [28] is a special case of Corollary 3.7 (and so also of Theorem 3.5) for choice constant $p = 2$. Also, Theorems 3.1 and 3.4 in [6] are special cases of our Theorem 3.5. In Corollary 3.8, by choosing the constants a_i in certain manner, we obtain certain classes of (s, p, α) -contractions.

The following corollaries are also consequences of Theorem 3.5, where self-maps satisfy contractive conditions given by rational expressions, and functions $\psi \in \Psi$, $\phi \in \Phi$ are used. To proceed with them, we denote by $M(x, y)$ the maximum of the set

$$\{\sigma_b(x, y), \sigma_b(x, fx), \sigma_b(y, fy), \sigma_b(x, fy), \sigma_b(y, fx)\}. \quad (23)$$

Corollary 3.10. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$ and $f : X \rightarrow X$ be a self-map. If there exist $\psi \in \Psi$, $0 \leq \alpha < \frac{1}{2}$ and $p \geq 2$, such that the condition

$$\psi(2s^p \sigma_b(fx, fy)) \leq \alpha \frac{\psi(M(x, y))}{1 + \psi(M(x, y))} \quad (24)$$

is satisfied for all $x, y \in X$, where $M(x, y)$ is defined as in (23), then f has a unique fixed point in X .

Proof. Taking into account that

$$\alpha \frac{\psi(M(x, y))}{1 + \psi(M(x, y))} = \alpha \frac{1}{1 + \psi(M(x, y))} \psi(M(x, y)) \leq \alpha \psi(M(x, y))$$

for all $x, y \in X$ and $0 \leq \alpha < \frac{1}{2}$, where $M(x, y)$ is defined as in (23), we get that condition (24) implies condition (13). As a consequence, Theorem 3.5 guarantees the existence of a unique fixed point of f . \square

Corollary 3.11. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$ and $f : X \rightarrow X$ be a self-map. If there exist $\psi \in \Psi$, $\phi \in \Phi$, $0 \leq \alpha < \frac{1}{2}$ and $p \geq 2$, such that the condition

$$\psi(2s^p \sigma_b(fx, fy)) \leq \alpha \frac{\psi(M(x, y))}{1 + \phi(M(x, y))} \quad (25)$$

is satisfied for all $x, y \in X$, where $M(x, y)$ is defined as in (23), then f has a unique fixed point in X .

Proof. The conclusion follows from Theorem 3.5, since the inequality (25) implies the inequality (13). \square

Corollary 3.12. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$ and $f : X \rightarrow X$ a self-map. If there exist $\psi \in \Psi$, $\phi \in \Phi$, $0 \leq \alpha < \frac{1}{2}$ and $p \geq 2$, such that the condition

$$\psi(2s^p \sigma_b(fx, fy)) \leq \alpha \frac{\psi(M(x, y))\phi(M(x, y))}{1 + \phi(M(x, y))} \quad (26)$$

is satisfied for all $x, y \in X$, where $M(x, y)$ is defined as in (23), then f has a unique fixed point in X .

Proof. The inequality (26) implies the inequality (13). Hence the conclusion follows from Theorem 3.5. \square

Corollary 3.13. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$ and $f : X \rightarrow X$ a self-map. If there exist $\psi \in \Psi$, $\phi \in \Phi$, $0 \leq \alpha < \frac{1}{2}$ and $p \geq 2$, such that the condition

$$\psi(2s^p \sigma_b(fx, fy)) \leq \alpha \frac{\psi(M(x, y)) - \phi(M(x, y))}{1 + \phi(M(x, y))} \quad (27)$$

is satisfied for all $x, y \in X$, where $M(x, y)$ is defined as in (23), then f has a unique fixed point in X .

Proof. Taking into account that ϕ is a lower semi continuous function with $\phi(t) = 0 \Leftrightarrow t = 0$, we have

$$\begin{aligned} \alpha \frac{\psi(M(x, y)) - \phi(M(x, y))}{1 + \phi(M(x, y))} &\leq \alpha \frac{\psi(M(x, y))}{1 + \phi(M(x, y))} \\ &\leq \alpha \frac{1}{1 + \phi(M(x, y))} \psi(M(x, y)) \leq \alpha \psi(M(x, y)) \end{aligned}$$

for all $x, y \in X$ and $0 \leq \alpha < \frac{1}{2}$, where $M(x, y)$ is defined as in (23). Hence inequality (27) implies inequality (13). Hence the conclusion follows from Theorem 3.5. \square

The basic result, related to the notion of weakly contractive maps, is due to Rhoades [33]. Further, this result has been generalized and extended by many authors to the notion of $(\psi - \varphi)$ -weakly contractive mappings. The aim of this part of the section is to extend and generalize the main classical result from [33] and other existing results in the literature on b -metric and metric-like spaces to the setup of b -metric-like spaces. Before presenting our results, we revise the weak contraction condition by introducing the notion of (s, p) -weak contraction.

Let (X, σ_b) be a b -metric-like space with parameter $s \geq 1$. For a self-mapping $f : X \rightarrow X$ we denote by $N(x, y)$ the following:

$$N(x, y) = \max\{\sigma_b(x, y), \sigma_b(x, fx), \sigma_b(y, fy), \frac{\sigma_b(x, fy) + \sigma_b(y, fx)}{4s}\} \quad (28)$$

for all $x, y \in X$.

Definition 3.14. Let (X, σ_b) be ab -metric-like space with parameter $s \geq 1$. A self-mapping $f : X \rightarrow X$ is called a generalized (s, p) -weak contraction, if there exist $\psi \in \Psi$ and a constant $p \geq 1$, such that

$$s^p \sigma_b(fx, fy) \leq N(x, y) - \phi(N(x, y)) \quad (29)$$

for all $x, y \in X$, where $N(x, y)$ is defined as in (28).

Remark 3.15. The above definition reduces to the definition of (s, p) -weak contraction if $N(x, y) = \sigma_b(x, y)$.

We now present the following result.

Theorem 3.16. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$. If $f : X \rightarrow X$ is a self-mapping that is a generalized (s, p) -weak contraction, then f has a unique fixed point in X .

Proof. Let x_0 be an arbitrary point in X . Define the iterative sequence $\{x_n\}$ as: $x_1 = f(x_0)$, $x_2 = f(x_1)$, \dots , $x_{n+1} = f(x_n)$, \dots for $n \in \mathbb{N}$.

If we assume that $\sigma_b(x_n, x_{n+1}) = 0$ for some $n \in \mathbb{N}$, then we have $x_{n+1} = x_n$ that is $x_n = x_{n+1} = f(x_n)$, so x_n is a fixed point of f and the proof is completed. From now on, we will assume that $\sigma_b(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$ (that is $x_{n+1} \neq x_n$). Using Definition of $N(x, y)$, we have

$$\begin{aligned} N(x_{n-1}, x_n) &= \max\left\{\sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, fx_{n-1}), \sigma_b(x_n, fx_n), \frac{\sigma_b(x_{n-1}, fx_n) + \sigma_b(x_n, fx_{n-1})}{4s}\right\} \\ &= \max\left\{\sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, x_n), \sigma_b(x_n, x_{n+1}), \frac{\sigma_b(x_{n-1}, x_{n+1}) + \sigma_b(x_n, x_n)}{4s}\right\} \\ &\leq \max\left\{\sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, x_n), \sigma_b(x_n, x_{n+1}), \frac{s[\sigma_b(x_{n-1}, x_n) + \sigma_b(x_n, x_{n+1})] + 2s\sigma_b(x_{n-1}, x_n)}{4s}\right\}. \end{aligned} \quad (30)$$

If we assume that for some $n \in \mathbb{N}$

$$\sigma_b(x_{n-1}, x_n) \leq \sigma_b(x_n, x_{n+1}),$$

then from the inequality (30), we get

$$N(x_{n-1}, x_n) \leq \sigma_b(x_n, x_{n+1}). \quad (31)$$

By the condition (29), we have

$$\begin{aligned} \sigma_b(x_n, x_{n+1}) &\leq s^p \sigma_b(x_n, x_{n+1}) \\ &= s^p \sigma_b(fx_{n-1}, fx_n) \\ &\leq N(x_{n-1}, x_n) - \phi(N(x_{n-1}, x_n)) \\ &\leq N(x_{n-1}, x_n). \end{aligned} \quad (32)$$

From (31) and (32), we have

$$N(x_{n-1}, x_n) = \sigma_b(x_n, x_{n+1}). \quad (33)$$

From (29), and using (33), we obtain

$$\begin{aligned} \sigma_b(x_n, x_{n+1}) &\leq s^p \sigma_b(x_n, x_{n+1}) \\ &= s^p \sigma_b(fx_{n-1}, fx_n) \\ &\leq N(x_{n-1}, x_n) - \phi(N(x_{n-1}, x_n)) \\ &= \sigma_b(x_n, x_{n+1}) - \phi(\sigma_b(x_n, x_{n+1})). \end{aligned} \quad (34)$$

The above inequality gives a contradiction, since we have assumed $\sigma_b(x_n, x_{n+1}) > 0$.

Hence, for all $n \in \mathbb{N}$, $\sigma_b(x_n, x_{n+1}) < \sigma_b(x_{n-1}, x_n)$, and the sequence $\{\sigma_b(x_n, x_{n+1})\}$ is decreasing and bounded below. So there exists $l \geq 0$ such that $\sigma_b(x_n, x_{n+1}) \rightarrow l$. Also

$$\lim_{n \rightarrow \infty} \sigma_b(x_n, x_{n+1}) = \lim_{n \rightarrow \infty} N(x_{n-1}, x_n) = l.$$

Since the function ϕ is lower semi continuous, we have

$$\phi(l) \leq \liminf_{n \rightarrow \infty} \phi(N(x_{n-1}, x_n)).$$

Let us prove that $l = 0$. If we suppose that $l > 0$, taking the limit in (34) we have

$$l \leq l - \phi(l),$$

that is a contradiction since $l > 0$. Thus $l = 0$.

Hence

$$\lim_{n \rightarrow \infty} \sigma_b(x_n, x_{n+1}) = \lim_{n \rightarrow \infty} N(x_{n-1}, x_n) = 0. \quad (35)$$

Next, we show that $\lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) = 0$. Suppose the contrary, that is, $\lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) \neq 0$. Then by Lemma 2.10, there exist $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$, such that

$$\sigma_b(x_{m_k}, x_{n_k}) \geq \varepsilon, \sigma_b(x_{m_k}, x_{n_k-1}) < \varepsilon$$

and

$$\begin{aligned} \frac{\varepsilon}{s^2} &\leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s, \\ \frac{\varepsilon}{s} &\leq \limsup_{k \rightarrow \infty} \sigma_b(x_{n_k-1}, x_{m_k}) \leq \varepsilon, \\ \frac{\varepsilon}{s} &\leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k-1}, x_{n_k}) \leq \varepsilon s^2 \end{aligned} \quad (36)$$

From the definition of $N(x, y)$, we have

$$\begin{aligned} N(x_{m_k-1}, x_{n_k-1}) &= \max \left\{ \frac{\sigma_b(x_{m_k-1}, x_{n_k-1}), \sigma_b(x_{m_k-1}, fx_{m_k-1}), \sigma_b(x_{n_k-1}, fx_{n_k-1})}{\sigma_b(x_{m_k-1}, fx_{n_k-1}) + \sigma_b(x_{n_k-1}, fx_{m_k-1})}, \right. \\ &\quad \left. \frac{\sigma_b(x_{m_k-1}, x_{n_k-1}), \sigma_b(x_{m_k-1}, x_{m_k}), \sigma_b(x_{n_k-1}, x_{n_k})}{\sigma_b(x_{m_k-1}, x_{n_k}) + \sigma_b(x_{n_k-1}, x_{m_k})} \right\} \\ &= \max \left\{ \frac{\sigma_b(x_{m_k-1}, x_{n_k-1})}{4s}, \frac{\sigma_b(x_{m_k-1}, x_{n_k}), \sigma_b(x_{n_k-1}, x_{m_k})}{4s} \right\}. \end{aligned} \quad (37)$$

Taking the upper limit as $k \rightarrow \infty$ in (37) and using (35) and (36), we get

$$\begin{aligned} \limsup_{k \rightarrow \infty} N(x_{m_k-1}, x_{n_k-1}) &= \limsup_{k \rightarrow \infty} \max \left\{ \frac{\sigma_b(x_{m_k-1}, x_{n_k-1}), \sigma_b(x_{m_k-1}, x_{m_k}), \sigma_b(x_{n_k-1}, x_{n_k})}{\sigma_b(x_{m_k-1}, x_{n_k}) + \sigma_b(x_{n_k-1}, x_{m_k})}, \right. \\ &\quad \left. \frac{\varepsilon s^2 + \varepsilon}{4s} \right\} \\ &\leq \max \left\{ \varepsilon s, 0, 0, \frac{\varepsilon s^2 + \varepsilon}{4s} \right\} \leq \varepsilon s. \end{aligned} \quad (38)$$

Also, as in Lemma 2.10, we can show that

$$\liminf_{k \rightarrow \infty} \sigma_b(x_{m_k-1}, x_{n_k-1}) \geq \frac{\varepsilon}{s^2}, \liminf_{k \rightarrow \infty} \sigma_b(x_{m_k-1}, x_{n_k}) \geq \frac{\varepsilon}{s}, \liminf_{k \rightarrow \infty} \sigma_b(x_{n_k-1}, x_{m_k}) \geq \frac{\varepsilon}{s},$$

and

$$\liminf_{k \rightarrow \infty} M(x_{m_k-1}, x_{n_k-1}) \geq \frac{\varepsilon}{2s^2}. \quad (39)$$

From the (s, p) -weak contractive condition, we have

$$\begin{aligned} s\sigma_b(x_{m_k}, x_{n_k}) &\leq s^p \sigma_b(fx_{m_k-1}, fx_{n_k-1}) \\ &\leq N(x_{m_k-1}, x_{n_k-1}) - \phi(N(x_{m_k-1}, x_{n_k-1})). \end{aligned} \quad (40)$$

Taking the upper limit in (40) and using (38) and (39), we obtain

$$\varepsilon s \leq \varepsilon s - \phi\left(\frac{\varepsilon}{2s^2}\right),$$

that is a contradiction since $\varepsilon > 0$. So $\lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) = 0$, and the sequence $\{x_n\}$ is a Cauchy sequence in the complete b -metric-like space (X, σ_b) . Thus, there is some $u \in X$, such that

$$\lim_{n \rightarrow \infty} \sigma_b(x_n, u) = \sigma_b(u, u) = \lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) = 0.$$

If f is a continuous mapping, similarly as in Theorem 3.5 we get that u is a fixed point of f .

If the self-map f is not continuous then we consider

$$\begin{aligned} N(x_n, u) &= \max \left\{ \frac{\sigma_b(x_n, u), \sigma_b(x_n, fx_n), \sigma_b(u, fu)}{\sigma_b(x_n, fu) + \sigma_b(u, fx_n)}, \frac{4s}{4s} \right\} \\ &= \max \left\{ \frac{\sigma_b(x_n, u), \sigma_b(x_n, x_{n+1}), \sigma_b(u, fu)}{\sigma_b(x_n, fu) + \sigma_b(u, x_{n+1})}, \frac{4s}{4s} \right\}. \end{aligned} \quad (41)$$

Taking the upper limit in (41) and using Lemma 2.8 and the result (35), we obtain

$$\lim_{n \rightarrow \infty} \sup N(x_n, u) \leq \max \left\{ 0, 0, b_d(u, fu), \frac{s\sigma_b(u, fu)}{4s} \right\} = \sigma_b(u, fu). \quad (42)$$

Now using the (s, p) -weak contractive condition, we have

$$\begin{aligned} s^p \sigma_b(x_{n+1}, fu) &= s^p \sigma_b(fx_n, fu) \\ &\leq N(x_n, u) - \phi(N(x_n, u)). \end{aligned} \quad (43)$$

Taking the upper limit in (43), and using Lemma 2.8 and result (42), it follows that

$$s^{p-1} \sigma_b(u, fu) = s^p \cdot \frac{1}{s} \sigma_b(u, fu) \leq \sigma_b(u, fu) - \phi(\sigma_b(u, fu)). \quad (44)$$

Hence, since $p \geq 1$, the inequality (44) implies $\sigma_b(u, fu) = 0$ and so $fu = u$.

Let us suppose that u and v , ($u \neq v$) are two fixed points of f where $fu = u$ and $fv = v$.

Firstly, since u is a fixed point of f , we have $\sigma_b(u, u) = 0$. From (s, p) -weak contractive condition, we have

$$\begin{aligned} s^p \sigma_b(u, u) &\leq s\sigma_b(fu, fu) \leq N(u, u) - \phi(N(u, u)) \\ &\leq b_d(u, u) - \phi(\sigma_b(u, u)), \end{aligned} \quad (45)$$

where

$$N(u, u) = \max \left\{ \sigma_b(u, u), \sigma_b(u, u), \sigma_b(u, u), \frac{\sigma_b(u, u) + \sigma_b(u, u)}{4s} \right\} = \sigma_b(u, u).$$

From the inequality (45) it follows that $\sigma_b(u, u) = 0$ (also $\sigma_b(v, v) = 0$).

Also, we have

$$\begin{aligned} s^p \sigma_b(u, v) &\leq s \sigma_b(fu, fv) \\ &\leq N(u, v) - \phi(N(u, v)) \\ &\leq \sigma_b(u, v) - \phi(\sigma_b(u, v)), \end{aligned} \quad (46)$$

where $N(u, v) = \sigma_b(u, v)$. The inequality (46) implies $\sigma_b(u, v) = 0$. Therefore $u = v$ and the fixed point is unique. \square

The following example illustrates the theorem.

Example 3.17. Let $X = [0, \infty)$ and $\sigma_b(x, y) = x^2 + y^2 + |x - y|^2$ for all $x, y \in X$. It is clear that σ_b is a b -metric-like on X , with parameter $s = 2$ and (X, σ_b) is complete. Also, σ_b is not a metric-like nor a b -metric (and nor a metric on X). Define the self-mapping $f : X \rightarrow X$ by $fx = \frac{\ln(1+x)}{4}$. For all $x, y \in X$, and the function $\phi(t) = \frac{3}{4}t$ and constant $p = 2$, we have

$$\begin{aligned} s^2 \sigma_b(fx, fy) &= 4 \left(f^2 x + f^2 y + |fx - fy|^2 \right) \\ &= 4 \left(\left(\frac{\ln(x+1)}{4} \right)^2 + \left(\frac{\ln(y+1)}{4} \right)^2 + \left| \frac{\ln(x+1)}{4} - \frac{\ln(y+1)}{4} \right|^2 \right) \\ &\leq 4 \left[\frac{x^2}{16} + \frac{y^2}{16} + \left| \frac{x}{4} - \frac{y}{4} \right|^2 \right] = \frac{1}{4} [x^2 + y^2 + |x - y|^2] \\ &= \frac{1}{4} \sigma_b(x, y) \leq \frac{1}{4} N(x, y) = N(x, y) - \frac{3}{4} N(x, y) \\ &= N(x, y) - \phi(N(x, y)). \end{aligned}$$

All of the conditions of Theorem 3.16 are satisfied and clearly $x = 0$ is a unique fixed point of f .

Corollary 3.18. Let (X, σ_b) be a complete b -metric-like space with parameter $s \geq 1$ and $f : X \rightarrow X$ be a self-mapping such that for some coefficient $p \geq 2$ and for all $x, y \in X$ it satisfies

$$s^p \sigma_b(fx, fy) \leq \alpha \max \left\{ \sigma_b(x, y), \sigma_b(x, fx), \sigma_b(y, fy), \frac{\sigma_b(x, fy) + \sigma_b(y, fx)}{4s} \right\}, \quad (47)$$

where $\alpha \in (0, 1)$. Then f has a unique fixed point.

Proof. In Theorem 3.16, taking $\phi(t) = (1 - \alpha)t$ for all $t \in [0, \infty)$, we get Corollary 3.18. \square

Remark 3.19. Since a b -metric-like space is a metric-like space when $s = 1$, so our results can be seen as a generalizations and extensions of several comparable results in metric-like spaces and b -metric spaces.

4 Application

In this section we will use Theorem 3.16 to show that there is a solution to the following integral equation:

$$x(t) = \int_0^T L(t, r, x(r)) dr \quad (48)$$

Let $X = C([0, T])$ be the set of real continuous functions defined on $[0, T]$ for $T > 0$.

We endow X with

$$\sigma_b(x, y) = \max_{t \in [0, 1]} (|x(t)| + |y(t)|)^m \text{ for all } x, y \in X,$$

where $m > 1$. It is evident that (X, σ_b) is a complete b -metric-like space with parameter $s = 2^{m-1}$.

Consider the mapping $f : X \rightarrow X$ given by $fx(t) = \int_0^T L(t, r, x(r)) dr$.

Theorem 4.1. Consider equation (48) and suppose that

1. $L : [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^+$, (that is $L(t, r, x(r)) \geq 0$) is continuous;
2. there exists a continuous $\gamma : [0, T] \times [0, T] \rightarrow \mathbb{R}$;
3. $\sup_{t \in [0, T]} \int_0^T \gamma(t, r) dr \leq 1$;
4. there exists a constant $\lambda \in (0, 1)$ such that for all $(t, r) \in [0, T]^2$ and $x, y \in \mathbb{R}$,

$$|L(t, r, x(r)) + L(t, r, y(r))| \leq \left(\frac{\lambda}{s^3}\right)^{\frac{1}{m}} \gamma(t, r) (|x(r)| + |y(r)|).$$

Then the integral equation (48) has a unique solution $x \in X$.

Proof. For $x, y \in X$, from conditions (3) and (4), for all t , we have

$$\begin{aligned} s^2 \sigma_b(fx(t), fy(t)) &= s^2 (|fx(t)| + |fy(t)|)^m \\ &= s^2 \left(\left| \int_0^T L(t, r, x(r)) dr \right| + \left| \int_0^T L(t, r, y(r)) dr \right| \right)^m \\ &\leq s^2 \left(\int_0^T |L(t, r, x(r))| dr + \int_0^T |L(t, r, y(r))| dr \right)^m \\ &\leq s^2 \left(\int_0^T \left(\frac{\lambda}{s^3}\right)^{\frac{1}{m}} \gamma(t, r) \left((|x(r)| + |y(r)|)^m\right)^{\frac{1}{m}} dr \right)^m \\ &\leq s^2 \left(\int_0^T \left(\frac{\lambda}{s^3}\right)^{\frac{1}{m}} \gamma(t, r) \sigma_b^{\frac{1}{m}}(x(r), y(r)) dr \right)^m \\ &\leq s^2 \cdot \frac{\lambda}{s^3} \sigma_b(x(r), y(r)) \left(\int_0^T \gamma(t, r) dr \right)^m \\ &= \frac{\lambda}{s} \sigma_b(x(r), y(r)) \left(\int_0^T \gamma(t, r) dr \right)^m \\ &\leq \frac{\lambda}{s} \sigma_b(x(r), y(r)) \\ &\leq \frac{\lambda}{s} N(x, y) = N(x, y) - \left(1 - \frac{\lambda}{s}\right) N(x, y) \\ &= N(x, y) - \phi(N(x, y)), \end{aligned}$$

Therefore, taking the coefficient $p = 2$, and function $\phi(x) = (1 - \lambda/s)x$, where $\lambda/s \in (0, 1)$, all of the conditions of Theorem 3.16 are satisfied, and as a result, the mapping f has a unique fixed point in X , which is a solution of the integral equation in (48). \square

5 Conclusions

Contractive conditions (13) and (29) are much wider than some previously used, and theorems related to these conditions are more general, since parameter s and the coefficient $p \geq 1$ are optional. Theorems 3.5 and 3.16 extend and generalize some existing results to a wider domain such as b -metric-like-spaces. Also, the generalized (s, p, α) -contractions and (s, p) -weak contractions unify a large class of existing contractions in the literature. Theoretical results are supported by applications.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final version of manuscript.

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