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Research Article

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On nonlinear evolution equation of second order in Banach spaces

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Abstract: Here we study the existence of a solution and also the behavior of the existing solution of the abstract nonlinear differential equation of second order that, in particular, is the nonlinear hyperbolic equation with nonlinear main parts, and in the special case, is the equation of the type of equation of traffic flow.

Keywords: Abstract nonlinear equation, Solvability, Behavior

MSC: 46T20, 47J35, 35L65, 58J45

1 Introduction

In this article we study the following nonlinear evolution equation

$$x_{tt} + A \circ F(x) = g\left(x, A^{-\frac{1}{2}}x_t\right), \quad t \in (0, T), \quad 0 < T < \infty^1 \quad (1)$$

under the initial conditions

$$x(0) = x_0, \quad x_t(0) = x_1, \quad (2)$$

where A is a linear operator in a real Hilbert space H , $F: X \rightarrow X^*$ and $g: D(g) \subseteq H \times H \rightarrow H$ are a nonlinear operators, X is a real Banach space. For example, operator A denotes $-\Delta$ with Dirichlet boundary conditions and $F(u) = |u|^\rho u$ (see, Example in Section 2), that in the one space dimension case, we can formulate in the form

$$u_{tt} - (f(u)u_x)_x = g(u), \quad (t, x) \in R_+ \times (0, l), \quad l > 0, \quad (3)$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad u(t, 0) = u(t, l) = 0, \quad (4)$$

where $u_0(x)$, $u_1(x)$ are known functions, $f(\cdot)$, $g(\cdot): R \rightarrow R$ are continuous functions and $l > 0$ is a number. The equation of type (3) describes a mathematical model of the problem from the theory of the flow in networks as is affirmed in articles [1–4] (e.g. Aw-Rascle equations, Antman–Cosserat model, etc.). As it is noted in the survey [2], such a study can find application in accelerating missiles and space crafts, components of high-speed machinery, manipulator arm, microelectronic mechanical structures, components of bridges and other structural elements. Balance laws are hyperbolic partial differential equations that are commonly used to express the fundamental dynamics of open conservative systems (e.g. [3]). As the survey [2] presents sufficiently exact explanations of the significance of equations of such type, we not discuss this theme.

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¹ see, Section 2

We would like to note only the following physical interpretation (see, [5]): "Let V be the smooth elastic body and F be the force acting on V through ∂V with the mass density is unit. Newton's law asserts the mass times the acceleration equal the net force

$$\int_V u_{tt} dx = - \int_{\partial V} F \cdot \nu dS \implies u_{tt} = - \operatorname{div} F$$

where u is the displacement in some direction of the point x at time $t \geq 0$ and F is a function of the displacement gradient ∇u ; whence

$$u_{tt} + \operatorname{div} F(\nabla u) = 0."$$

As it is well-known, F is a nonlinear function but for the study of this equation one usually uses a local linear approximation of F . Unlike the above mentioned works, one can study some variant of this equation with the nonlinear function F by use of the general result of this article.

In this article we use different approach to study proposed problem that allows us to investigate the case when the main part of the problem actually contains the nonlinear operator. As is shown in the above mentioned examples, one can investigate the nonlinear hyperbolic equations with the use of the results of this article, which haven't been studied earlier. We will note that in this approach we used the Galerkin approximation method.

This article is organized as follows. In Section 2, we study the solvability of the nonlinear equation of second order in the Banach spaces, for which we found the sufficient conditions and proved the existence theorem. In Section 3, we investigate the global behavior of solutions of the posed problem.

2 Solvability of problem (1)–(2)

Let A be a symmetric linear operator densely defined in a real Hilbert space H and positive, A has a self-adjoint extension. Moreover, there is linear operator B defined in H such that $A \equiv B^* \circ B$, here $f : R \rightarrow R$ is continuous as function, X is a real reflexive Banach space and $X \subset H$, $g : D(g) \subseteq H \times H \rightarrow H$, where $g : R^2 \rightarrow R$ is a continuous as function and $x : [0, T) \rightarrow X$ is an unknown function. Let $F(r)$ as a function be defined as $F(r) = \int_0^r f(s) ds$. Let the inequation $\|x\|_H \leq \|Bx\|_H$ be valid for any $x \in D(B)$. We denote by V , W and by Y the spaces defined as $V \equiv \{y \in H \mid By \in H\}$, $W = \{x \in H \mid Ax \in H\}$ and as $Y \equiv \{x \in X \mid Ax \in X\}$, respectively, for which inclusions $W \subset V \subset H$ are compact and $Y \subset W$.

Let H be the real separable Hilbert space, X be the reflexive Banach space and $X \subset H \subset X^*$; V is the previously defined space. It is clear that $W \subset V \subset H \subset V^* \subset W^*$ are framed spaces by H , these inclusions are compact and $X \subset V^*$. Then one can define the framed spaces $Y \subset V \subset H \subset V^* \subset Y^*$; then $X \subset V^* \subset Y^*$ are compact, with use the property of the operator A . Assume that operator $A : V_B \rightarrow V_B^*$ and $A : X^* \rightarrow Y^*$. Consequently, we get $A \circ F : X \rightarrow Y^*$ and $A \circ F \circ A : Y \rightarrow Y^*$. Moreover, we assume that $[X^*, Y]_{\frac{1}{2}} \subseteq V$.

Since operator A is invertible, here one can set the function $y(t) = A^{-1}x(t)$ for any $t \in (0, T)$, in other words one can assume the denotation $x(t) = Ay(t)$.

We will interpret the solution of the problem (1) - (2) in the following manner.

Definition 2.1. A function $x : (0, T) \rightarrow X$, $x \in C^0(0, T; X) \cap C^1(0, T; V^*) \cap C^2(0, T; Y^*)$, $x = Ay$, is called a weak solution of problem (1) - (2) if x a. e. $t \in (0, T)$ satisfies the following equation

$$\frac{d^2}{dt^2} \langle x, z \rangle + \langle A \circ F(x), z \rangle = \langle g(x, By_t), z \rangle \quad (5)$$

for any $z \in Y$ and the initial conditions (2) (here and further the expression $\langle \cdot, \cdot \rangle$ denotes the dual form for the pair: the Banach space and its dual).

Consider the following conditions

(i) Let $A : W \subset H \longrightarrow H$ be the selfadjoint and positive operator, moreover, $A : V \longrightarrow V^*$, $A : X^* \longrightarrow Y^*$, there exists a linear operator $B : V \longrightarrow H$ that satisfies the equation $Ax \equiv (B^* \circ B)x$ for any $x \in D(A)$ and $\|x\|_H \leq \|Bx\|_H = \|x\|_V$.

(ii) Let $F : X \longrightarrow X^*$ be the continuously differentiable and monotone operator with the potential Φ that is the functional defined on X (its Frechet derivative is the operator F). Moreover, for any $x \in X$ the following inequalities hold

$$\|F(x)\|_{X^*} \leq a_0 \|x\|_X^{p-1} + a_1 \|x\|_H; \quad \langle F(x), x \rangle \geq b_0 \|x\|_X^p + b_1 \|x\|_H^2,$$

where $a_0, b_0 > 0$, $a_1, b_1 \geq 0$, $p > 2$ are numbers.

(iii) Assume $g : H \times V \longrightarrow H$ is a continuous operator that satisfies the condition

$$|\langle g(x, y) - g(x_1, y_1), z \rangle| \leq g_1 |\langle x - x_1, z \rangle| + g_2 |\langle y - y_1, z \rangle|,$$

for any $(x, y), (x_1, y_1) \in H \times H$, $z \in H$ and consequently for any $(x, y) \in H \times H$ the inequation

$$\|g(x, y)\|_H \leq g_1 \|x\|_H + g_2 \|y\|_H + g_0, \quad g_0 \geq \|g(0, 0)\|_H$$

holds, where g_0 is a number.

Theorem 2.2. Let spaces H, V, W, X, Y that are defined above satisfy all above mentioned conditions and conditions (i)-(iii) are fulfilled, then problem (1) - (2) is solvable in the space $C^0(0, T; X) \cap C^1(0, T; V) \cap C^2(0, T; Y^*)$ for any $x_0 \in V \cap [X^*, Y]_{\frac{1}{2}}$ and $x_1 \in H$ in the sense of Definition 2.1.

At the beginning for the investigation of the posed problem we set the following expression in order to obtain of the a priori estimations

$$\langle x_{tt}, y_t \rangle + \langle A \circ F(x), y_t \rangle = \langle g(x, By_t), y_t \rangle,$$

where element y is defined as the solution of the equation $Ay(t) = x(t)$, i.e. $y(t) = A^{-1}x(t)$ for any $t \in (0, T)$ as was already mentioned above.

Hence follow

$$\langle By_{tt}, By_t \rangle + \langle F(x), x_t \rangle = \langle g(x, By_t), y_t \rangle,$$

or

$$\frac{1}{2} \frac{d}{dt} \|By_t\|_H^2 + \frac{d}{dt} \Phi(x) = \langle g(x, By_t), y_t \rangle, \quad (6)$$

where $\Phi(x)$ is the functional defined as $\Phi(x) = \int_0^1 \langle F(sx), x \rangle ds$ (see, [6]).

Then using condition (iii) on $g(x, By_t)$ in (6) one can obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|By_t\|_H^2 + \frac{d}{dt} \Phi(x) &\leq \|g(x, By_t)\|_H^2 + \|y_t\|_H^2 \leq \\ &2 \left(g_1^2 \|x\|_H^2 + g_2^2 \|By_t\|_H^2 + g_0^2 \right) + \|y_t\|_H^2 \leq \tilde{C} \left(\|x\|_H^2 + \frac{1}{2} \|By_t\|_H^2 + g_0^2 \right), \end{aligned}$$

where one can use the estimation $\|x\|_H^2 \leq \tilde{C}(\Phi(x) + 1)$ (if $b_1 > 0$ then $\|x\|_H^2 \leq \tilde{C}\Phi(x)$) as $2 < p$ by virtue of the condition (ii). Consequently, we get to the Cauchy problem for the inequation

$$\frac{d}{dt} \left(\frac{1}{2} \|By_t\|_H^2(t) + \Phi(x(t)) \right) \leq C_0 \left(\frac{1}{2} \|By_t\|_H^2(t) + \Phi(x(t)) \right) + C_1 \quad (7)$$

with the initial conditions

$$x(t)|_{t=0} = x_0; \quad y_t(t)|_{t=0} = A^{-1}x_t|_{t=0} = A^{-1}x_1, \quad (8)$$

where $C_j \geq 0$ are constants independent of $x(t)$. From here follows

$$\frac{1}{2} \|By_t\|_H^2(t) + \Phi(x(t)) \leq e^{tC_0} \left[\|By_1\|_H^2 + 2\Phi(x_0) \right] + \frac{C_1}{C_0} (e^{tC_0} - 1).$$

This gives the following estimations for every $T \in (0, \infty)$

$$\|By_t\|_H^2(t) \leq C(x_0, x_1) e^{C_0 T}, \quad \Phi(x(t)) \leq C(x_0, x_1) e^{C_0 T}, \quad (9)$$

for a. e. $t \in (0, T)$, i.e. $y = A^{-1}x$ is contained in the bounded subset of the space $y \in C^1(0, T; V) \cap C^0(0, T; Y)$, consequently, we obtain that if the weak solution $x(t)$ exists then it belongs to a bounded subset of the space $C^0(0, T; X) \cap C^1(0, T; V^*)$.

Hence one can see, that the following inclusion

$$y \in C^2(0, T; X^* \cap H) \cap C^1(0, T; X^* \cap V) \cap C^0(0, T; Y)$$

holds by virtue of (5) in the assumption that $x = Ay$ is a solution of the posed problem in the sense of Definition 2.1.

Proof of Theorem 2.2. In order to prove of the solvability theorem we will use the Faedo-Galerkin approach. Let the system $\{y^k\}_{k=1}^\infty \subset Y$ be total in Y such that it is complete in the spaces Y, V , and also in the spaces X, H . We will seek out of the approximative solutions $y_m(t)$, and consequently $x_m(t)$, in the form

$$x_m(t) \equiv Ay_m(t) = \sum_{k=1}^m c_k(t) Ay^k \text{ or } x_m(t) \in \text{span}\{y^1, \dots, y^m\}$$

as the solutions of the considered problem, where $c_i(t)$ are the unknown functions that will be defined as solutions of the following Cauchy problem for system of ODE

$$\frac{d^2}{dt^2} \langle x_m, y^j \rangle + \langle F(x_m), Ay^j \rangle = \langle g(x_m, By_{mt}), y^j \rangle, \quad j = 1, 2, \dots, m$$

$$x_m(0) = x_{0m}, \quad x_{tm}(0) = x_{1m},$$

where x_{0m} and x_{1m} are contained in $\text{span}\{y^1, \dots, y^m\}$, $m = 1, 2, \dots$, moreover,

$$x_{0m} \longrightarrow x_0 \quad \text{in } [X, Y]_{\frac{1}{2}} \subseteq V; \quad x_{1m} \longrightarrow x_1 \quad \text{in } X, \quad m \nearrow \infty.$$

Thus we obtain the following problem

$$\frac{d^2}{dt^2} \langle x_m, y^j \rangle + \langle F(x_m), Ay^j \rangle = \langle g(x_m, By_{mt}), y^j \rangle, \quad j = 1, 2, \dots, m \quad (10)$$

$$\langle x_m(t), y^j \rangle|_{t=0} = \langle x_{0m}, y^j \rangle, \quad \frac{d}{dt} \langle x_m(t), y^j \rangle|_{t=0} = \langle x_{1m}, y^j \rangle$$

that is solvable by virtue of estimates (9) on $(0, T)$ for any $m = 1, 2, \dots, j = 1, 2, \dots$ and $T > 0$. Hence we set

$$\frac{d^2}{dt^2} \langle x_m, z \rangle + \langle F(x_m), Az \rangle = \langle g(x_m, By_{mt}), z \rangle \quad (11)$$

for any $z \in Y$ and $m = 1, 2, \dots$

Consequently, with use of the known procedure ([7 - 9]) we obtain, $y_{mt} \in C^0(0, T; V)$, $y_m \in C^0(0, T; Y)$ and $x_m \in C^0(0, T; X)$, $x_{mt} \in C^0(0, T; V^*)$, moreover, they are contained in the bounded subset of these spaces for any $m = 1, 2, \dots$. Hence from (9) we get

$$x_{mtt} \in C^0(0, T; Y^*) \quad \text{or} \quad x_m \in C^2(0, T; Y^*), \quad (Y^* \subset Y^*).$$

Thus we obtain, that the sequence $\{x_m\}_{m=1}^\infty$ of the approximated solutions of the problem is contained in a bounded subset of the space

$$C^0(0, T; X) \cap C^1(0, T; V^*) \cap C^2(0, T; Y^*)$$

or $\{x_m\}_{m=1}^\infty$ such that for a. e. $t \in (0, T)$ the following inclusions take place $\{y_m(t)\}_{m=1}^\infty \subset Y \subset X \subset H$, $\{y_{mt}(t)\}_{m=1}^\infty \subset V$, $\{y_{mtt}(t)\}_{m=1}^\infty \subset Y^*$. So we have

$$\{y_m(t)\}_{m=1}^\infty \subset C^0(0, T; Y) \cap C^1(0, T; V) \cap C^2(0, T; Y^*).$$

Therefore, $\{y_m(t)\}_{m=1}^\infty$ possess a precompact subsequence in $C^1\left(0, T; [X^*, Y]_{\frac{1}{2}}\right)$ and in $C^1(0, T; V)$, as $[X^*, Y]_{\frac{1}{2}} \subseteq V$ by virtue of conditions on X and A (by virtue of well known results, see, e. g. [10, 11] etc.). From here follows $y_m(t) \rightarrow y(t)$ in $C^1(0, T; V)$ for $m \nearrow \infty$ (Here and hereafter in order to abate the number of index we don't change the indexes of subsequences). Then the sequence $\{F(Ay_m(t))\}_{m=1}^\infty \subset X^*$ and bounded for a. e. $t \in (0, T)$; the sequence

$$\{g(x_m(t), x_{mt}(t))\}_{m=1}^\infty \equiv \{g(Ay_m(t), By_{mt}(t))\}_{m=1}^\infty \subset H$$

and bounded for a. e. $t \in (0, T)$ also, by virtue of the condition (iii). Indeed, for any m the estimation

$$\|g(Ay_m, By_{mt})\|_H(t) \leq \|Ay_m(t)\|_H + \|By_{mt}(t)\|_H + \|g(0, 0)\|_H$$

holds and, therefore, $\{g(Ay_m(t), By_{mt}(t))\}_{m=1}^\infty$ is contained in a bounded subset of H for a. e. $t \in (0, T)$. Consequently, $\{F(Ay_m)\}_{m=1}^\infty$ and $\{g(Ay_m(t), By_{mt}(t))\}_{m=1}^\infty$ have weakly converging subsequences to $\eta(t)$ and $\theta(t)$ in X^* and H , respectively, for a. e. $t \in (0, T)$. Hence one can pass to the limit in (11) with respect to $m \nearrow \infty$. Then we obtain the following equation

$$\frac{d^2}{dt^2} \langle x, z \rangle + \langle A\eta(t), z \rangle = \langle \theta(t), z \rangle. \quad (12)$$

It remained to show the following: if the sequence $\{x_m(t)\}_{m=1}^\infty \equiv \{Ay_m(t)\}_{m=1}^\infty$ is weakly converging to $x(t) = Ay(t)$ then $\eta(t) = F(x(t))$ and $\theta(t) = g(x(t), By_t(t))$. In order to show these equations are fulfilled we will use the monotonicity of F and the condition (iii).

We start by showing $\theta(t) = g((t), By_t(t))$ as $x \in X \subset H$ and $y_t \in V$, $By_t \in H$ therefore $g(x, By_t)$ is defined for a. e. $t \in (0, T)$. Consequently, one can consider of the expression

$$\langle g(Ay_m(t), By_{mt}(t)) - g(Ay(t), By_t(t)), \widehat{y} \rangle$$

for any $\widehat{y} \in C^0(0, T; Y) \cap C^1(0, T; V)$. So we set this expression and investigate this for any $\widehat{y} \in C^0(0, T; Y) \cap C^1(0, T; V)$; then we have

$$|\langle g(Ay_m(t), By_{mt}(t)) - g(Ay(t), By_t(t)), \widehat{y} \rangle| \leq$$

$$g_1 |\langle Ay_m(t) - Ay(t), \widehat{y}(t) \rangle| + g_2 |\langle By_{mt}(t) - By_t(t), \widehat{y}(t) \rangle| \quad (13)$$

that takes place by virtue of the condition (iii). Using here the weak convergences of $Ay_m(t) \rightharpoonup Ay(t)$ and $By_{mt}(t) \rightarrow By_t(t)$ and by passing to the limit in the inequation (13) with respect to $m : m \nearrow \infty$ we get

$$|\langle \theta(t) - g(Ay(t), By_t(t)), \widehat{y} \rangle| \leq 0$$

for any $\widehat{y} \in C^0(0, T; H)$. Consequently, the equation $\theta(t) = g((t), By_t(t))$ holds, then the following equation is valid

$$\frac{d^2}{dt^2} \langle x, z \rangle + \langle A\eta(t), z \rangle = \langle g(x(t), By_t(t)), z \rangle$$

for any $z \in Y$, as $\{y^k\}_{k=1}^\infty$ is complete in Y that display fulfilling of equation

$$A\eta(t) = g(x(t), By_t(t)) - \frac{d^2 x}{dt^2} \quad (14)$$

in the sense of Y^* .

In order to show the equation $\eta(t) = F(x(t))$ one can use the monotonicity of F . So the following inequation holds

$$\begin{aligned} \langle A \circ F(Az) - A \circ F(Ay), z - y \rangle &= \langle F(Az) - F(Ay), Az - Ay \rangle = \\ &= \langle F(\widetilde{x}) - F(x), \widetilde{x} - x \rangle \geq 0 \end{aligned}$$

for any $y, z \in Y$, $Ay = x$ and $Az = \tilde{x}$ by condition (i). Then one can write

$$0 \leq \langle F(x_m) - F(\tilde{x}), x_m - \tilde{x} \rangle = \langle F(Ay_m) - F(Az), Ay_m - Az \rangle =$$

take account here the equation (9)

$$\begin{aligned} \langle F(Ay_m), Ay_m \rangle - \left\langle \frac{d^2}{dt^2} x_m - g(x_m, By_{mt}), z \right\rangle - \langle F(Az), Ay_m - Az \rangle = \\ \langle F(x_m), x_m \rangle - \left\langle \frac{d^2}{dt^2} x_m - g(x_m, By_{mt}), z \right\rangle - \langle F(\tilde{x}), x_m - \tilde{x} \rangle. \end{aligned} \quad (15)$$

Here one can use the well-known inequation

$$\limsup \langle F(x_m), x_m \rangle \leq \langle \eta, x \rangle = \langle \eta, Ay \rangle = \langle A\eta, y \rangle.$$

Then passing to the limit in (15) with respect to $m : m \nearrow \infty$ we obtain

$$\begin{aligned} 0 \leq \langle A\eta, y \rangle - \left\langle \frac{d^2}{dt^2} x - g(x, By_t), z \right\rangle - \langle F(\tilde{x}), x - \tilde{x} \rangle = \\ \langle A\eta, y \rangle - \langle A\eta, z \rangle - \langle F(\tilde{x}), Ay - Az \rangle = \langle A\eta - A \circ F(\tilde{x}), y - z \rangle \end{aligned}$$

by virtue of (14).

Consequently, we obtain that the equation $A\eta(t) = A \circ F(x)$ holds since z is arbitrary element of Y .

Now it remains to show that the obtained function $x(t) = Ay(t)$ satisfies the initial conditions. Consider the following equation

$$\begin{aligned} \langle y_{mt}, Ay_m \rangle(t) = \int_0^t \left\langle \frac{d^2}{ds^2} Ay_m, y_m \right\rangle ds + \int_0^t \left\langle \frac{d}{ds} By_m, \frac{d}{ds} By_m \right\rangle ds + \langle y_{1m}, Ay_{0m} \rangle = \\ \int_0^t \left\langle \frac{d^2}{ds^2} y_m, Ay_m \right\rangle ds + \int_0^t \left\| \frac{d}{ds} By_m \right\|_H^2 ds + \langle y_{1m}, Ay_{0m} \rangle \end{aligned}$$

for $m = 1, 2, \dots$, here $x_m(t) = Ay_m(t)$. Hence we get: the left side is bounded as far as all added items in the right side are bounded by virtue of the obtained estimations. Therefore, one can pass to limit with respect to m as here y_{mt} is continuous with respect to t for any m ; then y_{mt} strongly converges to y_t and Ay_m weakly converges to Ay in H . It must be noted the equation

$$\lim_{m \rightarrow \infty} \int_0^t \left\| \frac{d}{ds} By_m \right\|_H^2 ds = \int_0^t \left\| \frac{d}{ds} By \right\|_H^2 ds$$

holds by virtue of the above reasonings that $\{y_m(t)\}_{m=1}^\infty$ is a precompact subset in $C^1(0, T; V)$. Consequently, the left side converges to the expression of such type, i.e. to $\langle y_t, Ay \rangle(t)$. The obtained results show that the following convergences are just: $x_m(t) = Ay_m(t) \rightharpoonup Ay(t) = x(t)$ in X , $x_{mt}(t) = Ay_{mt} \rightharpoonup Ay_t = x_t(t)$ in V^* . From here follows, that the initial conditions are fulfilled in the sense of X and V^* , respectively.

Thus the existence theorem is fully proved. $\square$

Remark 2.3. This theorem shows that there exists a flow $S(t)$ defined in $V \times X$ and the solution of the problem (1) - (2) can be represented as $x(t) = S(t) \circ (x_0, x_1)$.

Example 2.4. Let $\Omega \subset R^n (n \geq 3)$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Consider on $Q = (0, T) \times \Omega$ of the following problem

$$u_{tt} - \nabla \cdot (|u|^{p-2} \nabla u) = a(u) + b \int_\Omega \frac{u_t(t, y)}{|x - y|^{n-1}} dy, \quad p > 2,$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad u|_{\partial\Omega \times (0, T)} = 0,$$

where $a(\tau)$ satisfies the Lipschitz condition, $b \in R$.

It is clear that all conditions of Theorem 2.2 are fulfilled for this problem under above conditions of this example.

We would like to note the equation with main part of such type hasn't been studied earlier.

3 Behavior of solutions of problem (1)–(2)

Here we consider a problem under the following complementary conditions:

(iv) Let $g(x, By_t) = 0$ and $\|x\|_H^p(t) \leq c_0 \Phi(x(t))$ for some $c_0 > 0$.

We set a function $E(t) = \|By\|_H^2(t)$ and consider this function on the solution of problem (1)–(2), then for $E(t) = \|By\|_H^2(t)$ we have

$$E'(t) = 2 \langle By_t, By \rangle \leq \|By_t\|_2^2(t) + \|By\|_2^2(t), \quad (16)$$

where $y = A^{-1}x$. Here we will use equation (9). For this we need the following equation

$$\frac{1}{2} \|By_s\|_H^2(s) + \Phi(x(s)) \Big|_0^t = 0$$

as $g(x, By_t) = 0$.² Hence

$$\frac{1}{2} \|By_s\|_H^2(t) + \Phi(x(t)) = \frac{1}{2} \|By_1\|_H^2(t) + \Phi(x_0)$$

and

$$\|By_t\|_H^2(t) = -2\Phi(x(t)) + \|By_1\|_H^2 + 2\Phi(x_0).$$

Granting this by (16) we get

$$E'(t) \leq E(t) - E'(t) + \|By_1\|_H^2 + 2\Phi(x_0)$$

by virtue of the condition $\Phi(x) \geq c_0 \|x\|_X^p$ and of the continuity of embedding $X \subset H$, $r = p/2$.

So denoted by $z(t) = E(t)$ we have the Cauchy problem for differential inequality

$$z'(t) \leq z(t) - cz^r(t) + C(x_0, x_1), \quad z(0) = \|By_0\|_H^2, \quad (17)$$

that we will investigate. Inequation (17) can be rewritten in the form

$$(z(t) + kC(x_0, x_1))' \leq z(t) + kC(x_0, x_1) - \delta [z(t) + kC(x_0, x_1)]^r,$$

where $k > 1$ is a number and $\delta = \delta(c, C, k, r) > 0$ is sufficiently small number. Then solving this problem we get

$$z(t) + kC(x_0, x_1) \leq \left[e^{(1-r)t} (z_0 + kC(x_0, x_1))^{1-r} + \delta (1 - e^{(1-r)t}) \right]^{\frac{1}{1-r}}$$

or

$$E(t) \leq \left[e^{(1-r)t} (\|By_0\|_H^2 + kC(x_0, x_1))^{1-r} + \delta (1 - e^{(1-r)t}) \right]^{\frac{1}{1-r}} - kC(x_0, x_1)$$

$$\|By\|_H^2(t) \leq \frac{e^t (\|By_0\|_H^2 + kC(x_0, x_1))}{\left[1 + \delta (\|By_0\|_H^2 + kC(x_0, x_1))^{r-1} (e^{(r-1)t} - 1) \right]^{\frac{1}{r-1}}} - kC(x_0, x_1). \quad (18)$$

Here the right side is greater than zero, because $\delta \leq \frac{k-1}{k^r C^r}$ and $2r = p > 2$.

Thus the result is proved.

Theorem 3.1. Under conditions (i), (ii), (iv) the function $y(t)$, defined by the solution of problem (5)–(6), for any $t > 0$ is contained in ball $B_l^{X \cap V}(0) \subset X \cap V$ depending on the initial values $(x_0, x_1) \in (X \cap V) \times H$, here $l = l(x_0, x_1, p) > 0$.

² We would like to note that this equation shows the stability of the energy of the considered system in this case.

4 Conclusion

In this article, the existence of a very weak solution for differential-operator equations of second order with nonlinear operator in the main part is proved. We would like to note that, in particular, if A is the differential operator this equation becomes a hyperbolic equation. Consequently, one can investigate previously not studied nonlinear hyperbolic equations with the use of results and the approach presented in this article. The following work will be focused on nonlinear hyperbolic equations with the nonlinearity of the same type as studied here.

Moreover, here the long-time behavior of the very weak solution of the problem is proved, and also the dependence of the behavior of the solution from initial datums is shown. In other words, here we show the behavior of the weak semi-flow (in some sense), defined by the considered problem, with respect to t when $t \rightarrow +\infty$.

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