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Stability problems and numerical integration on the Lie group $SO(3) \times \mathbf{R}^3 \times \mathbf{R}^3$

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Abstract: The paper is dealing with stability problems for a nonlinear system on the Lie group $SO(3)\times\mathbf{R}^3\times\mathbf{R}^3$. The approximate analytic solutions of the considered system via Optimal Homotopy Asymptotic Method are presented, too.

Keywords: Nonlinear ordinary differential systems, Nonlinear stability, Lie group, Optimal homotopy asymptotic method

MSC: 34H15, 65Nxx, 65P40, 70H14, 74G10, 74H10

1 Introduction

The optimal control problems on the Lie groups were studied very often in deep connection with mechanical systems. We can find a large list of such examples, like the dynamics of an underwater vehicle, with $SE(3,\mathbf{R}) = SO(3) \times \mathbf{R}^3$ as space configuration (see [1]), the ball-plate problem, with $\mathbf{R}^2 \times SO(3)$ as space configuration [2], the rolling-penny dynamics having the Lie group $SE(2,\mathbf{R}) \times SO(2)$ as space configuration [3], the control tower problem from air traffic, modeled on the Special Euclidean Group $SE(3)$, the spacecraft dynamics modeled on the special orthogonal group $SO(3)$, [4], the buoyancy's dynamics on the Lie group $SO(3) \times \mathbf{R}^3 \times \mathbf{R}^3$, (see [5] for details), and the list may go on.

Similar methods were used in [6-10].

Taking into consideration that in many cases the dynamics can be viewed as a left-invariant, drift-free control system on the considered Lie group, we became interested in the study of such systems. The problem of finding the optimal controls that minimize a quadratic cost function for the general left-invariant drift-free control system

$$\dot{X} = X\left(A_1u_1 + A_2u_2 + A_3u_3 + A_5u_5 + A_7u_7\right), \tag{1}$$

on the Lie group $G = SO(3) \times \mathbf{R}^3 \times \mathbf{R}^3$, where A_i, $i = \overline{1,9}$ is the standard basis of the Lie algebra g :

$$A_1 = \begin{bmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix}, A_2 = \begin{bmatrix} 0&0&1&0\\ 0&0&0&0\\ -1&0&0&0\\ 0&0&0&0 \end{bmatrix}, A_3 = \begin{bmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix},$$

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$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the span of the set of Lie brackets generated by A_1, A_2, A_3, A_5, A_7 coincides with g, the system (1) is controllable [11].

Considering now the cost function given by:

$$J(u_1, u_2, u_3, u_5, u_7) = \frac{1}{2} \int_0^{t_f} \left[c_1 u_1^2(t) + c_2 u_2^2(t) + c_3 u_3^2(t) + c_5 u_5^2(t) + c_7 u_7^2(t) \right] dt$$

$$c_1 > 0, c_2 > 0, c_3 > 0, c_5 > 0, c_7 > 0,$$

the controls that minimize J and steer the system (1) from $X = X_0$ at $t = 0$ to $X = X_f$ at $t = t_f$ are given by:

$$u_1 = \frac{1}{c_1} x_1, u_2 = \frac{1}{c_2} x_2, \ u_3 = \frac{1}{c_3} x_3, \ u_5 = \frac{1}{c_5} x_5, u_7 = \frac{1}{c_7} x_7,$$

where x_i's are solutions of the following nonlinear system:

$$\begin{cases} \dot{x}_1 = -x_5 x_6 \\ \dot{x}_2 = x_7 x_9 \\ \dot{x}_3 = x_4 x_5 - x_7 x_8 \\ \dot{x}_4 = -x_2 x_6 + x_3 x_5 \\ \dot{x}_5 = x_1 x_6 - x_3 x_4 \\ \dot{x}_6 = -x_1 x_5 + x_2 x_4 \\ \dot{x}_7 = -x_2 x_9 + x_3 x_8 \\ \dot{x}_8 = x_1 x_9 - x_3 x_7 \\ \dot{x}_9 = -x_1 x_8 + x_2 x_7 \end{cases} . \tag{2}$$

The main goal of our paper is to establish some stability results of the equilibrium points

$$e_6^{MNP} = (M, 0, 0, N, 0, 0, P, 0, 0), M, N, P \in \mathbf{R},$$

of the above system. Some stability results regarding the equilibrium states

$$\begin{aligned} e_1^{MNPQ} &= (0, 0, 0, M, 0, N, 0, P, Q), \ M, N, P, Q \in \mathbf{R}, \\ e_2^{MNP} &= (0, 0, M, 0, 0, N, 0, 0, P), \ M, N, P \in \mathbf{R}, \\ e_3^{MPQ} &= (0, 0, 0, 0, M, 0, 0, P, Q), \ M, P, Q \in \mathbf{R}, \\ e_4^{MNP} &= (0, M, 0, 0, N, 0, 0, P, 0), \ M, N, P \in \mathbf{R}, \\ e_5^{MNP} &= (M, N, P, 0, 0, 0, 0, 0, 0), \ M, N, P \in \mathbf{R}, \end{aligned}$$

were already obtained in [11], but the stability problem for the other equilibrium states remains unsolved.

The paper is organized as follows: in the second paragraph we find an appropriate control function in order to stabilize the equilibrium states e_6^{MNP}. The third section briefly presents the Optimal Homotopy Asymptotic Method, developed in [12-14] and used in the last part in order to obtain the approximate analytic solutions of the controlled system.

2 Stabilization of e_6^{MNP} by one linear control

Let us employ the control $u \in C^\infty(\mathbf{R}^9, \mathbf{R})$,

$$u(\bar{x}) = (0, -Mx_3 - 2Px_9, Mx_2 + 2Px_8, 0, -Mx_6, Mx_5, 0, -Mx_9, Mx_8), \tag{3}$$

for the system (2). The controlled system (2)–(3), explicitly given by

$$\begin{cases} \dot{x}_1 = -x_5 x_6 \\ \dot{x}_2 = x_7 x_9 - M x_3 - 2P x_9 \\ \dot{x}_3 = x_4 x_5 - x_7 x_8 + M x_2 + 2P x_8 \\ \dot{x}_4 = -x_2 x_6 + x_3 x_5 \\ \dot{x}_5 = x_1 x_6 - x_3 x_4 - M x_6 \\ \dot{x}_6 = -x_1 x_5 + x_2 x_4 + M x_5 \\ \dot{x}_7 = -x_2 x_9 + x_3 x_8 \\ \dot{x}_8 = x_1 x_9 - x_3 x_7 - M x_9 \\ \dot{x}_9 = -x_1 x_8 + x_2 x_7 + M x_8 \end{cases}, \tag{4}$$

has e_6^{MNP} as an equilibrium state.

Proposition 2.1. *The controlled system* (4) *has the Hamilton-Poisson realization*

$$(G, \Pi, H),$$

where $G = SO(3) \times \mathbf{R}^3 \times \mathbf{R}^3$,

$$\Pi = \begin{bmatrix} 0 & -x_3 & x_2 & 0 & -x_6 & x_5 & 0 & -x_9 & x_8 \\ x_3 & 0 & -x_1 & x_6 & 0 & -x_4 & x_9 & 0 & -x_7 \\ -x_2 & x_1 & 0 & -x_5 & x_4 & 0 & -x_8 & x_7 & 0 \\ 0 & -x_6 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_6 & 0 & -x_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_5 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_9 & x_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_9 & 0 & -x_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_8 & x_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

is the minus Lie-Poisson structure on the dual of the corresponding Lie algebra g^* *and the Hamiltonian function given by*

$$H(\bar{x}) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_7^2) - M x_1 - 2P x_7.$$

Proof. Indeed, one obtains immediately that

$$\Pi \cdot \nabla H = [\dot{x}_1\ \dot{x}_2\ \dot{x}_3\ \dot{x}_4\ \dot{x}_5\ \dot{x}_6\ \dot{x}_7\ \dot{x}_8\ \dot{x}_9]^t,$$

and Π is a minus Lie-Poisson structure, see for details [11]. □

Remark 2.2 ([11])**.** *The functions* $C_1, C_2, C_3 : \mathbf{R}^9 \to \mathbf{R}$ *given by*

$$C_1(\bar{x}) = \frac{1}{2}(x_4^2 + x_5^2 + x_6^2),$$

$$C_2(\bar{x}) = \frac{1}{2}(x_7^2 + x_8^2 + x_9^2)$$

and

$$C_3(\bar{x}) = x_4 x_7 + x_5 x_8 + x_6 x_9$$

are the Casimirs of our Poisson configuration.

The goal of this paragraph is to study the spectral and nonlinear stability of the equilibrium state e_6^{MNP} of the controlled system (4).

Let A be the matrix of linear part of our controlled system (4), that is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & -x_6 & -x_5 & 0 & 0 & 0 \\ 0 & 0 & -M & 0 & 0 & 0 & x_9 & 0 & x_7 - 2P \\ 0 & M & 0 & x_5 & x_4 & 0 & -x_8 & -x_7 + 2P & 0 \\ 0 & -x_6 & x_5 & 0 & x_3 & -x_2 & 0 & 0 & 0 \\ x_6 & 0 & -x_4 & -x_3 & 0 & x_1 - M & 0 & 0 & 0 \\ -x_5 & x_4 & 0 & x_2 & -x_1 + M & 0 & 0 & 0 & 0 \\ 0 & -x_9 & x_8 & 0 & 0 & 0 & 0 & x_3 & -x_2 \\ x_9 & 0 & -x_7 & 0 & 0 & 0 & -x_3 & 0 & x_1 - M \\ -x_8 & x_7 & 0 & 0 & 0 & 0 & x_2 & -x_1 + M & 0 \end{bmatrix}.$$

At the equilibrium of interest its characteristic polynomial has the following expression:

$$p_{A(e_2^{MNP})}(\lambda) = -\lambda^5[\lambda^4 + (M^2 + N^2 + 2P^2)\lambda^2 + P^2(N^2 + P^2)].$$

Hence we have five zero eigenvalues and four purely imaginary eigenvalues. So we can conclude:

Proposition 2.3. *The controlled system* (4) *may be spectral stabilized about the equilibrium states* e_6^{MNP} *for all* $M, N, P \in \mathbf{R}^*$.

Moreover we can prove:

Proposition 2.4. *The controlled system* (4) *may be nonlinear stabilized about the equilibrium states* e_6^{MNP} *for all* $M, N, P \in \mathbf{R}^*$.

Proof. For the proof we shall use Arnold's technique. Let us consider the following function

$$F_{\lambda,\mu,\nu} = C_1 + \lambda H + \mu C_2 + \nu C_3 =$$

$$= \frac{1}{2}(x_4^2 + x_5^2 + x_6^2) + \frac{\lambda}{2}(x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_7^2 - 2Mx_1 - 4Px_7) +$$

$$+ \frac{\mu}{2}(x_7^2 + x_8^2 + x_9^2) + \nu(x_4x_7 + x_5x_8 + x_6x_9).$$

The following conditions hold:
(i) $\nabla F_{\lambda,\mu,\nu}(e_6^{MNP}) = 0$ iff $\mu = \lambda + \frac{N^2}{P^2}$, $\nu = -\frac{N}{P}$;
(ii) Considering now

$$W = \ker[dH(e_6^{MNP})] \cap \ker[dC_2(e_6^{MNP})] \cap \ker[dC_3(e_6^{MNP})] =$$

$$= \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}\right\},$$

then, for all $v \in W$, i.e. $v = (a, b, c, 0, d, e, 0, f, g)$, $a, b, c, d, e, f, g \in \mathbf{R}$ we have

$$v \cdot \nabla^2 F_{\lambda,\lambda+\frac{N^2}{P^2},-\frac{N}{P}}(e_6^{MNP}) \cdot v^t =$$

$$\lambda a^2 + \lambda b^2 + \lambda c^2 + (1+\lambda)d^2 + e^2 + \left(\lambda + \frac{N^2}{P^2}\right)f^2 + \left(\lambda + \frac{N^2}{P^2}\right)g^2 - 2\frac{N}{P}fd - 2\frac{N}{P}eg$$

which is positive definite under the restriction $\lambda > 0$, and so

$$\nabla^2 F_{\lambda, \lambda + \frac{N^2}{P^2}, -\frac{N}{P}}(e_6^{MNP})|_{W \times W}$$

is positive definite.

Therefore, via Arnold's technique, the equilibrium states e_6^{MNP}, $M, N, P \in \mathbf{R}^*$ are nonlinear stable, as required. □

3 Basic ideas of the Optimal Homotopy Asymptotic Method

In order to compute analytical approximate solutions for the nonlinear differential system given by the equations (4) with the boundary conditions

$$x_i(0) = A_i, \ i = \overline{1, 9}, \tag{6}$$

we will use the Optimal Homotopy Asymptotic Method (OHAM) [12-14].

Let us start with a very short description of this method. The analytical approximate solutions can be obtained for equations of the general form:

$$L\Big(x(t)\Big) + N\Big(x(t)\Big) = 0, \tag{7}$$

subject to the initial conditions of the type:

$$x(0) = A, \quad A \in \mathbf{R} \text{ - given real number}, \tag{8}$$

where L is a linear operator (**which is not unique**), N is a nonlinear one and $x(t)$ is the unknown smooth function of the Eq. (7).

Following [12-14], we construct the homotopy given by:

$$\begin{aligned} \mathcal{H}\Big[L\Big(X(t,p)\Big), \ H(t, C_i), \ N\Big(X(t,p)\Big)\Big] \equiv L\Big(x_0(t)\Big) + \\ + p\Big[L\Big(x_1(t, C_i)\Big) - H(t, C_i) N\Big(x_0(t)\Big)\Big] = 0, \end{aligned} \tag{9}$$

where $p \in [0, \ 1]$ is the embedding parameter, $H(t, C_i)$, $(H \neq 0)$ is an auxiliary convergence-control function, depending on the variable t and on the parameters $C_1, C_2, ..., C_s$ and the function $X(t, p)$ has the expression:

$$X(t, p) = x_0(t) + p x_1(t, C_i). \tag{10}$$

The following properties hold:

$$\mathcal{H}\Big[L\Big(X(t,0)\Big), \ H(t, C_i), \ N\Big(X(t,0)\Big)\Big] = L\Big(x_0(t)\Big) \tag{11}$$

and

$$\mathcal{H}\Big[L\Big(X(t,1)\Big), \ H(t, C_i), \ N\Big(X(t,1)\Big)\Big] = L\Big(x_1(t, C_i)\Big) - H(t, C_i) N\Big(x_0(t)\Big). \tag{12}$$

The governing equations of $x_0(t)$ and $x_1(t, C_i)$ can be obtained by equating the coefficients of p^0 and p^1, respectively:

$$L\Big(x_0(t)\Big) = 0, \quad x(0) = A, \tag{13}$$

$$\begin{aligned} &L\Big(x_1(t, C_i)\Big) = H(t, C_i) N\Big(x_0(t)\Big), \\ &x_1(0, C_i) = 0, \ i = \overline{1, s}. \end{aligned} \tag{14}$$

The expression of $x_0(t)$ can be found by solving the linear equation (13). Also, to compute $x_1(t, C_i)$ we solve the equation (14), by taking into consideration that the nonlinear operator N presents the general form:

$$N(x_0(t)) = \sum_{i=1}^{m} h_i(t) g_i(t), \tag{15}$$

where m is a positive integer and $h_i(t)$ and $g_i(t)$ are known functions depending both on $x_0(t)$ and N.

Although the equation (14) is a nonhomogeneous linear one, in the most cases its solution can not be found.

In order to compute the function $x_1(t, C_i)$ we will use the third modified version of OHAM (see [14] for details), consisting in the following steps:

First we consider one of the following expressions for $x_1(t, C_i)$:

$$x_1(t, C_i) = \sum_{i=1}^{m} H_i(t, h_j(t), C_j) g_i(t), \quad j = \overline{1, s}, \tag{16}$$

or

$$\begin{aligned} &x_1(t, C_i) = \sum_{i=1}^{m} H_i(y, g_j(t), C_j) h_i(t), \quad j = \overline{1, s}, \\ &x_1(0, C_i) = 0. \end{aligned} \tag{17}$$

These expressions of $H_i(t, h_j(t), C_j)$ contain both linear combinations of the functions h_j and the parameters $C_j, j = \overline{1, s}$. The summation limit m is an arbitrary positive integer number.

Next, by taking into account the equation (10), for $p = 1$, the first-order analytical approximate solution of the equations (7) - (8) is:

$$\overline{x}(t, C_i) = X(t, 1) = x_0(t) + x_1(t, C_i). \tag{18}$$

Finally, the convergence-control parameters C_1, C_2, ..., C_s, which determine the first-order approximate solution (18), can be optimally computed by means of various methods, such as: the least square method, the Galerkin method, the collocation method, the Kantorowich method or the weighted residual method.

Definition 3.1. [15] *We call an ϵ-**approximate solution** of the problem (7) on the domain $(0, \infty)$ a smooth function $\overline{x}(t, C_i)$ of the form (18) which satisfies the following condition:*

$$\left| R(t, \overline{x}(t, C_i)) \right| < \epsilon,$$

together with the initial condition from Eq. (8), where the residual function $R(t, \overline{x}(t, C_i))$ is obtained by substituting the Eq. (18) into Eq. (7), i.e.

$$R(t, \overline{x}(t, C_i)) = L\Big(\overline{x}(t, C_i)\Big) + N\Big(\overline{x}(t, C_i)\Big).$$

Definition 3.2 ([15])**.** *We call a **week** ϵ-**approximate solution** of the problem (7) on the domain $(0, \infty)$ a smooth function $\overline{x}(t, C_i)$ of the form (18) which satisfies the following condition:*

$$\int_0^{\infty} R^2(t, \overline{x}(t, C_i))\, d\eta < \epsilon,$$

together with the initial condition from Eq. (8).

4 Application of Optimal Homotopy Asymptotic Method for solving the nonlinear differential system (4)

In order to solve the nonlinear differential system given by the equations (4), each equation of the system (4) can be written in the form Eq. (7), where we can choose the linear operators as:

$$\begin{aligned}
&L[x_1(t)] = \dot{x}_1(t) + K_1 x_1(t),\\
&L[x_2(t)] = \dot{x}_2(t) + K_1 x_2(t) + M x_3(t),\\
&L[x_3(t)] = \dot{x}_3(t) + K_1 x_3(t) - M x_2(t),\\
&L[x_4(t)] = \dot{x}_4(t) + K_1 x_4(t) + K x_7(t),\\
&L[x_5(t)] = \dot{x}_5(t) + K_1 x_5(t) + M x_6(t),\\
&L[x_6(t)] = \dot{x}_6(t) + K_1 x_6(t) - M x_5(t),\\
&L[x_7(t)] = \dot{x}_7(t) + K_1 x_7(t) - K x_4(t),\\
&L[x_8(t)] = \dot{x}_8(t) + K_1 x_8(t) + M x_9(t),\\
&L[x_9(t)] = \dot{x}_9(t) + K_1 x_9(t) - M x_8(t),
\end{aligned} \tag{19}$$

with $K > 0$, $K_1 > 0$ the unknown parameters at this moment.

The corresponding linear equations for initial approximations x_{i_0}, $i = \overline{1,9}$ can be obtained by means of the Eqs. (13), (19) and (6):

$$L[x_{i_0}(t)] = 0, \quad x_{i_0}(0) = A_i, \ \ i = \overline{1,9}, \tag{20}$$

whose solutions are

$$\begin{aligned}
&x_{1_0}(t) = A_1 \cdot e^{-K_1 t},\\
&x_{2_0}(t) = (A_2 \cos(Mt) - A_3 \sin(Mt)) \cdot e^{-K_1 t},\\
&x_{3_0}(t) = (A_3 \cos(Mt) + A_2 \sin(Mt)) \cdot e^{-K_1 t},\\
&x_{4_0}(t) = (A_4 \cos(Kt) - A_7 \sin(Kt)) \cdot e^{-K_1 t},\\
&x_{5_0}(t) = (A_5 \cos(Mt) - A_6 \sin(Mt)) \cdot e^{-K_1 t},\\
&x_{6_0}(t) = (A_6 \cos(Mt) + A_5 \sin(Mt)) \cdot e^{-K_1 t},\\
&x_{7_0}(t) = (A_7 \cos(Kt) + A_4 \sin(Kt)) \cdot e^{-K_1 t},\\
&x_{8_0}(t) = (A_8 \cos(Mt) - A_9 \sin(Mt)) \cdot e^{-K_1 t},\\
&x_{9_0}(t) = (A_9 \cos(Mt) + A_8 \sin(Mt)) \cdot e^{-K_1 t},
\end{aligned} \tag{21}$$

The corresponding nonlinear operators $N[x_i(t)]$, $i = \overline{1,9}$ are obtained from the equations (4):

$$\begin{aligned}
&N[x_1(t)] = -K_1 x_1(t) + x_5(t)x_6(t),\\
&N[x_2(t)] = -K_1 x_2(t) - x_7(t)x_9(t) + 2P x_9(t),\\
&N[x_3(t)] = -K_1 x_3(t) - x_4(t)x_5(t) + x_7(t)x_8(t) - 2P x_8(t),\\
&N[x_4(t)] = -K_1 x_4(t) - K x_7(t) + x_2(t)x_6(t) - x_3(t)x_5(t),\\
&N[x_5(t)] = -K_1 x_5(t) - x_1(t)x_6(t) + x_3(t)x_4(t),\\
&N[x_6(t)] = -K_1 x_6(t) + x_1(t)x_5(t) - x_2(t)x_4(t),\\
&N[x_7(t)] = -K_1 x_7(t) + x_2(t)x_9(t) - x_3(t)x_8(t),\\
&N[x_8(t)] = -K_1 x_8(t) - x_1(t)x_9(t) + x_3(t)x_7(t),\\
&N[x_9(t)] = -K_1 x_9(t) + x_1(t)x_8(t) - x_2(t)x_7(t),
\end{aligned} \tag{22}$$

such that

$$L[x_1(t)] + N[x_1(t)] = \dot{x}_1(t) + x_5(t)x_6(t), L[x_2(t)] + N[x_2(t)] = \dot{x}_2(t) - x_7(t)x_9(t) + M x_3(t) + 2P x_9(t),$$

$$L[x_3(t)] + N[x_3(t)] = \dot{x}_3(t) - x_4(t)x_5(t) + x_7(t)x_8(t) - M x_2(t) - 2P x_8(t),$$

$$L[x_4(t)] + N[x_4(t)] = \dot{x}_4(t) + x_2(t)x_6(t) - x_3(t)x_5(t)$$

$$L[x_5(t)] + N[x_5(t)] = \dot{x}_5(t) - x_1(t)x_6(t) + x_3(t)x_4(t) + M x_6(t),$$

$$L[x_6(t)] + N[x_6(t)] = \dot{x}_6(t) + x_1(t)x_5(t) - x_2(t)x_4(t) - M x_5(t),$$

$$L[x_7(t)] + N[x_7(t)] = \dot{x}_7(t) + x_2(t)x_9(t) - x_3(t)x_8(t),$$
$$L[x_8(t)] + N[x_8(t)] = \dot{x}_8(t) - x_1(t)x_9(t) + x_3(t)x_7(t) + Mx_9(t)$$
$$L[x_9(t)] + N[x_9(t)] = \dot{x}_9(t) + x_1(t)x_8(t) - x_2(t)x_7(t) - Mx_8(t),$$

and therefore, substituting Eqs. (21) into Eqs. (22), we obtain

$$\begin{aligned}
&N[x_{1_0}(t)] = -K_1x_{1_0}(t) + x_{5_0}(t)x_{6_0}(t),\\
&N[x_{2_0}(t)] = -K_1x_{2_0}(t) - x_{7_0}(t)x_{9_0}(t) + 2Px_{9_0}(t),\\
&N[x_{3_0}(t)] = -K_1x_{3_0}(t) - x_{4_0}(t)x_{5_0}(t) + x_{7_0}(t)x_{8_0}(t) - 2Px_{8_0}(t),\\
&N[x_{4_0}(t)] = -K_1x_{4_0}(t) - Kx_{7_0}(t) + x_{2_0}(t)x_{6_0}(t) - x_{3_0}(t)x_{5_0}(t),\\
&N[x_{5_0}(t)] = -K_1x_{5_0}(t) - x_{1_0}(t)x_{6_0}(t) + x_{3_0}(t)x_{4_0}(t),\\
&N[x_{6_0}(t)] = -K_1x_{6_0}(t) + x_{1_0}(t)x_{5_0}(t) - x_{2_0}(t)x_{4_0}(t),\\
&N[x_{7_0}(t)] = -K_1x_{7_0}(t) + x_{2_0}(t)x_{9_0}(t) - x_{3_0}(t)x_{8_0}(t),\\
&N[x_{8_0}(t)] = -K_1x_{8_0}(t) - x_{1_0}(t)x_{9_0}(t) + x_{3_0}(t)x_{7_0}(t),\\
&N[x_{9_0}(t)] = -K_1x_{9_0}(t) + x_{1_0}(t)x_{8_0}(t) - x_{2_0}(t)x_{7_0}(t).
\end{aligned} \tag{23}$$

Remark 4.1. *Now, we observe that the nonlinear operators $N[x_{i_0}(t)]$, $i = \overline{1,9}$ are the linear combinations between the elementary functions $e^{-K_1t}\cdot\cos(Mt)$, $e^{-K_1t}\cdot\sin(Mt)$, $e^{-2K_1t}\cdot\cos^2(Mt)$, $e^{-2K_1t}\cdot\sin^2(Mt)$, $e^{-2K_1t}\cdot\cos(Mt)\sin(Mt)$, $e^{-K_1t}\cdot\cos(Kt)$, $e^{-K_1t}\cdot\sin(Kt)$, $e^{-2K_1t}\cdot\cos(Kt)\cos(Mt)$, $e^{-2K_1t}\cdot\sin(Kt)\sin(Mt)$, $e^{-2K_1t}\cdot\cos(Kt)\sin(Mt)$, $e^{-2K_1t}\cdot\sin(Kt)\cos(Mt)$.*

On the other hand, the Eq. (14) becomes:

$$\begin{aligned}
&L\Big(x_{i_1}(t, C_j)\Big) = H(t, C_j)N\Big(x_{i_0}(t)\Big),\\
&x_{i_1}(0, C_j) = 0, \quad i = \overline{1,9}, \quad j = \overline{1,s},
\end{aligned} \tag{24}$$

where the linear operators L are given by Eq. (19) and the expressions $N\Big(x_{i_0}(t)\Big)$, $i = \overline{1,9}$ are given by Eq. (23).

The auxiliary convergence-control functions H_i are chosen such that the product between $H_i \cdot N[x_{i_0}(t)]$ has the same form of the $N[x_{i_0}(t)]$. Then, the first approximation becomes:

$$x_{i_1} = \sum_{n=1}^{7} [B_n\cos(2n+1)\omega t + C_n\sin(2n+1)\omega t]\cdot e^{-K_1t}, \quad i = \overline{1,9}. \tag{25}$$

Using now the third-alternative of OHAM and the equations (18), the first-order approximate solution can be put in the form

$$\bar{x}_i(t, C_i) = x_{i_0}(t) + x_{i_1}(t, C_i), \quad i = \overline{1,9} \tag{26}$$

where $x_{i_0}(t)$ and $x_{i_1}(t, C_i)$ are given by (21) and (25), respectively.

5 Numerical examples and discussions

In this section, the accuracy and validity of the OHAM technique is proved using a comparison of our approximate solutions with numerical results obtained via the fourth-order Runge-Kutta method in the following case: we consider the initial value problem given by (4) with initial conditions (6) $A_i = 0.0001$, $i = \overline{1,9}$, $M = 15$ and $P = 20$.

One can show that these approximate solutions are **week ϵ-approximate solutions** by computing the numerical value of the integral of square residual function (to see the Table 4), i.e. $\int_0^1 R_i^2(t)\,dt$, $i = 1, \ldots, 9$,

where

$$\begin{aligned}
R_1(t) &= \dot{\bar{x}}_1(t) + \bar{x}_5(t)\bar{x}_6(t),\\
R_2(t) &= \dot{\bar{x}}_2(t) - \bar{x}_7(t)\bar{x}_9(t) + M\bar{x}_3(t) + 2P\bar{x}_9(t),\\
R_3(t) &= \dot{\bar{x}}_3(t) - \bar{x}_4(t)\bar{x}_5(t) + \bar{x}_7(t)\bar{x}_8(t) - M\bar{x}_2(t) - 2P\bar{x}_8(t),\\
R_4(t) &= \dot{\bar{x}}_4(t) + \bar{x}_2(t)\bar{x}_6(t) - \bar{x}_3(t)\bar{x}_5(t),\\
R_5(t) &= \dot{\bar{x}}_5(t) - \bar{x}_1(t)\bar{x}_6(t) + \bar{x}_3(t)\bar{x}_4(t) + M\bar{x}_6(t),\\
R_6(t) &= \dot{\bar{x}}_6(t) + \bar{x}_1(t)\bar{x}_5(t) - \bar{x}_2(t)\bar{x}_4(t) - M\bar{x}_5(t),\\
R_7(t) &= \dot{\bar{x}}_7(t) + \bar{x}_2(t)\bar{x}_9(t) - \bar{x}_3(t)\bar{x}_8(t),\\
R_8(t) &= \dot{\bar{x}}_8(t) - \bar{x}_1(t)\bar{x}_9(t) + \bar{x}_3(t)\bar{x}_7(t) + M\bar{x}_9(t),\\
R_9(t) &= \dot{\bar{x}}_9(t) + \bar{x}_1(t)\bar{x}_8(t) - \bar{x}_2(t)\bar{x}_7(t) - M\bar{x}_8(t),
\end{aligned} \tag{27}$$

with $\bar{x}_i(t)$, $i = 1, \ldots, 9$ given by Eq. (26).

The convergence-control parameters K, K_1, ω, B_i, C_i, $i = \overline{1,9}$ are optimally determined by means of the least-square method.

– *for* $\bar{x}_1$ *:* The convergence-control parameters are respectively :

$$B_1 = 0.00243871503187,\ B_2 = -0.00359273871118,\ B_3 = 0.00052965854847,$$
$$B_4 = 0.00114795966364,\ B_5 = -0.00056092131600,\ B_6 = 6.47335130 \cdot 10^{-6},$$
$$B_7 = 0.00003434745346,\ B_8 = -3.49402157 \cdot 10^{-6},\ C_1 = -0.00350555321652,$$
$$C_2 = -0.00076136918414,\ C_3 = 0.00251066466427,\ C_4 = -0.00087065292393,$$
$$C_5 = -0.00027493879386,\ C_6 = 0.00019640146954,\ C_7 = -0.00002061370631,$$
$$C_8 = -1.94094963 \cdot 10^{-6},\ K = 2.57114627223466,$$
$$K_1 = 0.58656793719790,\ \omega = 1.01132823106464.$$

The first-order approximate solutions given by the Eq. (26) are respectively :

$$\begin{aligned}
\bar{x}_1(t) = e^{-0.586567937197 90t}\Big(&0.0001 + 0.00243871503187\cos(\omega t) -\\
&-0.00359273871118\cos(3\omega t) + 0.00052965854847\cos(5\omega t) +\\
&+0.00114795966364\cos(7\omega t) - 0.00056092131600\cos(9\omega t) +\\
&+6.47335130511943 \cdot 10^{-6}\cos(11\omega t) + 0.00003434745346\cos(13\omega t) -\\
&-3.49402157853902 \cdot 10^{-6}\cos(15\omega t)\Big) + \Big(-0.00350555321652\sin(\omega t) -\\
&-0.00076136918414\sin(3\omega t) + 0.00251066466427\sin(5\omega t) -\\
&-0.00087065292393\sin(7\omega t) - 0.00027493879386\sin(9\omega t) +\\
&+0.00019640146954\sin(11\omega t) - 0.00002061370631\sin(13\omega t) -\\
&-1.94094963 \cdot 10^{-6}\sin(15\omega t)\Big)e^{-0.58656793719790\, t}.
\end{aligned} \tag{28}$$

For all unknown functions $\bar{x}_i$, $i = \overline{1,9}$, we have $K_1 = 0.58656793719790$ and $\omega = 1.01132823106464$.

– *for* $\bar{x}_2$ *:*

$$\begin{aligned}
\bar{x}_2(t) = e^{-0.58656793719790\, t}&(-0.46417071707583\cos(\omega t) +\\
&+0.55436638784881\cos(3\omega t) + 0.23179980739892\cos(5\omega t) -\\
&-0.52457790349642\cos(7\omega t) + 0.20239755363119\cos(9\omega t) +\\
&+0.03102295410884\cos(11\omega t) - 0.04104869574268\cos(13\omega t) +\\
&+0.01021061332716\cos(15\omega t)) + e^{-0.58656793719790\, t}(0.0001\cos(15t) -\\
&-0.0001\sin(15t)) + e^{-0.586567937197\, t}(0.42041532210572\sin(\omega t) +\\
&+0.47464526230797\sin(3\omega t) - 0.66980627335479\sin(5\omega t) +\\
&+0.08449154306675\sin(7\omega t) + 0.23139718247432\sin(9\omega t) -\\
&-0.13429737097902\sin(11\omega t) + 0.02641377996654\sin(13\omega t) -\\
&-0.00262253548558\sin(15\omega t)).
\end{aligned} \tag{29}$$

– *for* $\bar{x}_3$ *:*

$$\begin{aligned}\bar{x}_3(t) = e^{-0.58656793719790\,t}&(1.62690195527876\cos(\omega t)-\\ -2.91905245722799&\cos(3\omega t) + 1.41052417394739\cos(5\omega t)+\\ +0.18894557284618&\cos(7\omega t) - 0.45976960570592\cos(9\omega t)+\\ +0.17490980799902&\cos(11\omega t) - 0.02436756038517\cos(13\omega t)+\\ +0.00190811324772&\cos(15\omega t)) + e^{-0.58656793719790\,t}(0.0001\cos(15t)+\\ +0.0001&\sin(15t)) + e^{-0.58656793719790\,t}(-3.16149837021824\sin(\omega t)+\\ +0.42959981904576&\sin(3\omega t) + 1.45120190775009\sin(5\omega t)-\\ -1.12246042334992&\sin(7\omega t) + 0.24077988316763\sin(9\omega t)+\\ +0.07360271218550&\sin(11\omega t) - 0.05047592981365\sin(13\omega t)+\\ +0.01049752842422&\sin(15\omega t)).\end{aligned} \tag{30}$$

– *for* $\bar{x}_4$ *:*

$$\begin{aligned}\bar{x}_4(t) = e^{-0.58656793719790\,t}&(1.94037601\cdot 10^{-6}\cos(\omega t)+\\ +0.00001098412604&\cos(3\omega t) - 0.00001943075801\cos(5\omega t)+\\ +7.22410959926685\cdot 10^{-6}&\cos(7\omega t) - 4.23016798075448\cdot 10^{-7}\cos(9\omega t)-\\ -3.67489997706953\cdot 10^{-7}&\cos(11\omega t) + 7.48336663534119\cdot 10^{-8}\cos(13\omega t)-\\ -2.18052201200004\cdot 10^{-9}&\cos(15\omega t)) + e^{-0.58656793719790\,t}(0.0001\cos(0.80363574138731t)-\\ -0.0001&\sin(0.80363574138731t)) + e^{-0.58656793719790\,t}(0.00024662613164\sin(\omega t)-\\ -0.00005660016305&\sin(3\omega t) + 0.00001080462985\sin(5\omega t)+\\ +3.31362820627460\cdot 10^{-6}&\sin(7\omega t) - 2.30237733420243\cdot 10^{-6}\sin(9\omega t)+\\ +3.66350337276902\cdot 10^{-7}&\sin(11\omega t) + 1.43826553061835\cdot 10^{-8}\sin(13\omega t)-\\ -5.09710156758535\cdot 10^{-9}&\sin(15\omega t)).\end{aligned} \tag{31}$$

– *for* $\bar{x}_5$ *:*

$$\begin{aligned}\bar{x}_5(t) = e^{-0.58656793719790\,t}&(0.01369917160029\cos(\omega t)-\\ -0.02516198187512&\cos(3\omega t) + 0.01340403032961\cos(5\omega t)+\\ +0.00013038596714&\cos(7\omega t) - 0.00335904322615\cos(9\omega t)+\\ +0.00158445050516&\cos(11\omega t) - 0.00035976290254\cos(13\omega t)+\\ +0.00006274960160&\cos(15\omega t)) + e^{-0.58656793719790\,t}(0.0001\cos(15t)-\\ -0.0001&Sin(15t)) + e^{-0.58656793719790\,t}(-0.02795989598259\sin(\omega t)+\\ +0.00526407948570&\sin(3\omega t) + 0.01119295776584\sin(5\omega t)-\\ -0.00969533499261&\sin(7\omega t) + 0.00279014392623\sin(9\omega t)+\\ +0.00019457655388&\sin(11\omega t) - 0.00035980785989\sin(13\omega t)+\\ +0.00010360505021&\sin(15\omega t)).\end{aligned} \tag{32}$$

– *for* $\bar{x}_6$ *:*

$$\begin{aligned}\bar{x}_6(t) = e^{-0.58656793719790\,t}&(0.01331723000622\cos(\omega t)-\\ -0.02136690535515&\cos(3\omega t) + 0.00599018344061\cos(5\omega t)+\\ +0.00533244090418&\cos(7\omega t) - 0.00400304532847\cos(9\omega t)+\\ +0.00056272420145&\cos(11\omega t) + 0.00027367239189\cos(13\omega t)-\\ -0.00010630026076&\cos(15\omega t)) + e^{-0.58656793719790\,t}(0.0001\cos(15t)+\\ +0.0001&\sin(15t)) + e^{-0.58656793719790\,t}(-0.02188514843412\sin(\omega t)-\\ -0.00146001826477&\sin(3\omega t) + 0.01365871149717\sin(5\omega t)-\\ -0.00673797188735&\sin(7\omega t) - 0.00067903495719\sin(9\omega t)+\\ +0.00146666027085&\sin(11\omega t) - 0.00045540606550\sin(13\omega t)+\\ +0.00007291228376&\sin(15\omega t)).\end{aligned} \tag{33}$$

– *for* $\bar{x}_7$ *:*

$$\begin{aligned}\bar{x}_7(t) = {} & e^{-0.58656793719790\,t}(0.00014174642928\cos(\omega t)- \\ & -0.00012283316033\cos(3\omega t) - 0.00002041841606\cos(5\omega t)+ \\ & +9.42449906\cdot 10^{-7}\cos(7\omega t) + 8.49879531\cdot 10^{-7}\cos(9\omega t)- \\ & -3.23563916\cdot 10^{-7}\cos(11\omega t) + 3.63337271\cdot 10^{-8}\cos(13\omega t)+ \\ & +4.78659301\cdot 10^{-11}\cos(15\omega t)) + (0.0001\cos(3.72653706867179t)+ \\ & +0.0001\sin(3.72653706867179t))e^{-0.58656793719790\,t}+ \\ & +(0.00013001568894\sin(\omega t) - 0.00011137613515\sin(3\omega t)- \\ & -0.00002806958703\sin(5\omega t) + 6.45157686\cdot 10^{-6}\sin(7\omega t)- \\ & 1.35713708\cdot 10^{-6}\sin(9\omega t) + 6.39039531\cdot 10^{-8}\sin(11\omega t)+ \\ & 2.93113481\cdot 10^{-8}\sin(13\omega t) - 3.32424397\cdot 10^{-9}\sin(15\omega t))e^{-0.58656793719790\,t}.\end{aligned} \tag{34}$$

– *for* $\bar{x}_8$ *:*

$$\begin{aligned}\bar{x}_8(t) = {} & e^{-0.58656793719790\,t}(0.01369917160029\cos(\omega t)- \\ & -0.02516198187512\cos(3\omega t) + 0.01340403032961\cos(5\omega t)+ \\ & +0.00013038596714\cos(7\omega t) - 0.00335904322615\cos(9\omega t)+ \\ & +0.00158445050516\cos(11\omega t) - 0.00035976290254\cos(13\omega t)+ \\ & +0.00006274960160\cos(15\omega t)) + e^{-0.58656793719790\,t}(0.0001\cos(15t)- \\ & -0.0001\sin(15t)) + e^{-0.58656793719790\,t}(-0.02795989598259\sin(\omega t)+ \\ & +0.00526407948570\sin(3\omega t) + 0.01119295776584\sin(5\omega t)- \\ & -0.00969533499261\sin(7\omega t) + 0.00279014392623\sin(9\omega t)+ \\ & +0.00019457655388\sin(11\omega t]) - 0.00035980785989\sin(13\omega t)+ \\ & +0.00010360505021\sin(15\omega t).\end{aligned} \tag{35}$$

– *for* $\bar{x}_9$ *:*

$$\begin{aligned}\bar{x}_9(t) = {} & e^{-0.58656793719790\,t}(0.01331723000622\cos[\omega t)- \\ & -0.02136690535515\cos(3\omega t) + 0.00599018344061\cos(5\omega t)+ \\ & +0.00533244090418\cos(7\omega t) - 0.00400304532847\cos(9\omega t)+ \\ & +0.00056272420145\cos(11\omega t) + 0.00027367239189\cos(13\omega t)- \\ & -0.00010630026076\cos(15\omega t)) + e^{-0.58656793719790\,t}(0.0001\cos(15t)+ \\ & +0.0001\sin(15t)) + e^{-0.58656793719790\,t}\,(-0.02188514843412\sin(\omega t)- \\ & -0.00146001826477\sin(3\omega t) + 0.01365871149717\sin(5\omega t)- \\ & -0.00673797188735\sin(7\omega t) - 0.00067903495719\sin(9\omega t)+ \\ & +0.00146666027085\sin(11\omega t) - 0.00045540606550\sin(13\omega t)+ \\ & +0.00007291228376\sin(15\omega t)).\end{aligned} \tag{36}$$

Finally, Tables 1, 2 and 3 emphasizes the accuracy of the OHAM technique by comparing the approximate analytic solutions $\bar{x}_3$, $\bar{x}_5$ and $\bar{x}_8$ respectively presented above with the corresponding numerical integration values.

The Figs. 1-9 depicted a comparison between the obtained approximate solutions given by Eqs. (28)-(36) with corresponding numerical integration.

Fig. 1. Comparison between the approximate solutions $\bar{x}_1$ given by Eq. (28) and the corresponding numerical solutions

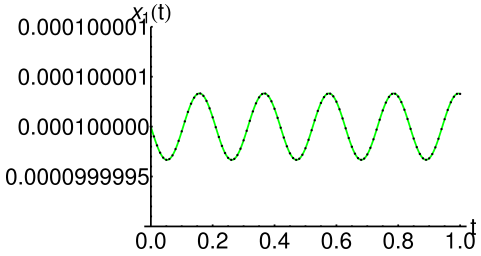


Table 1. The comparison between the approximate solutions $\bar{x}_3$ given by Eq. (30) and the corresponding numerical solutions for $M = 15$ and $P = 20$ (relative errors: $\epsilon_{x_3} = |x_{3_{numerical}} - \bar{x}_{3_{OHAM}}|$)

t	$x_{3_{numerical}}$	$\bar{x}_{3_{OHAM}}$ given by Eq. (30)	ϵ_{x_3}
0	0.0001	0.0001	0
1/10	-0.00026388686019	-0.00026388920732	$2.34712908 \cdot 10^{-9}$
1/5	-0.00098972163690	-0.00098972743135	$5.79444527 \cdot 10^{-9}$
3/10	0.00080121591193	0.00080121516755	$7.44372891 \cdot 10^{-10}$
2/5	0.00205113177145	0.00205113953903	$7.76758435 \cdot 10^{-9}$
1/2	-0.00105407568168	-0.00105407578483	$1.03151043 \cdot 10^{-10}$
3/5	-0.00322482848832	-0.00322483170127	$3.21294763 \cdot 10^{-9}$
7/10	0.00099573781977	0.00099573897878	$1.15901067 \cdot 10^{-9}$
4/5	0.00444604275839	0.00444604908979	$6.33140294 \cdot 10^{-9}$
9/10	-0.00061159885752	-0.00061159746243	$1.39509107 \cdot 10^{-9}$
1	-0.00564686848672	-0.00564687685941	$8.37269479 \cdot 10^{-9}$

Table 2. The comparison between the approximate solutions $\bar{x}_5$ given by Eq. (32) and the corresponding numerical solutions for $M = 15$ and $P = 20$ (relative errors: $\epsilon_{x_5} = |x_{5_{numerical}} - \bar{x}_{5_{OHAM}}|$)

t	$x_{5_{numerical}}$	$\bar{x}_{5_{OHAM}}$ given by Eq. (32)	ϵ_{x_5}
0	0.0001	0.0001	0
1/10	-0.00009267394569	-0.00009267384041	$1.05273724 \cdot 10^{-10}$
1/5	-0.00011309924594	-0.00011309966129	$4.15348823 \cdot 10^{-10}$
3/10	0.00007665834993	0.00007665851028	$1.60350492 \cdot 10^{-10}$
2/5	0.00012391598733	0.00012391636093	$3.73592167 \cdot 10^{-10}$
1/2	-0.00005910519311	-0.00005910537099	$1.77875786 \cdot 10^{-10}$
3/5	-0.00013222655382	-0.00013222667316	$1.19336773 \cdot 10^{-10}$
7/10	0.00004037430382	0.00004037416907	$1.34745601 \cdot 10^{-10}$
4/5	0.00013786222359	0.00013786239690	$1.73302447 \cdot 10^{-10}$
9/10	-0.00002084981273	-0.00002084984022	$2.74919318 \cdot 10^{-11}$
1	-0.00014071144636	-0.00014071122544	$2.20921854 \cdot 10^{-10}$

Fig. 2. Comparison between the approximate solutions $\bar{x}_2$ given by Eq. (29) and the corresponding numerical solutions

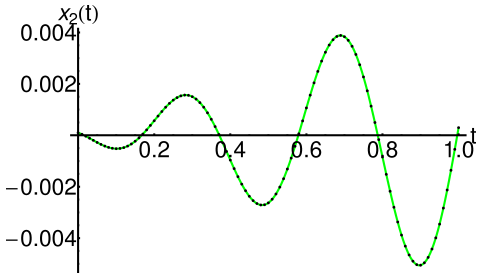


Table 3. The comparison between the approximate solutions $\bar{x}_8$ given by Eq. (35) and the corresponding numerical solutions for $M = 15$ and $P = 20$ (relative errors: $\epsilon_{x_8} = |x_{8_{numerical}} - \bar{x}_{8_{OHAM}}|$)

t	$x_{8_{numerical}}$	$\bar{x}_{8_{OHAM}}$ given by Eq. (35)	ϵ_{x_8}
0	0.0001	0.0001	0
1/10	-0.00009267394569	-0.00009267384041	$1.05273724 \cdot 10^{-10}$
1/5	-0.00011309924594	-0.00011309966129	$4.15348823 \cdot 10^{-10}$
3/10	0.00007665834993	0.00007665851028	$1.60350492 \cdot 10^{-10}$
2/5	0.00012391598733	0.00012391636093	$3.73592167 \cdot 10^{-10}$
1/2	-0.00005910519311	-0.00005910537099	$1.77875786 \cdot 10^{-10}$
3/5	-0.00013222655382	-0.00013222667316	$1.19336773 \cdot 10^{-10}$
7/10	0.00004037430382	0.00004037416907	$1.34745601 \cdot 10^{-10}$
4/5	0.00013786222359	0.00013786239690	$1.73302447 \cdot 10^{-10}$
9/10	-0.00002084981273	-0.00002084984022	$2.74919318 \cdot 10^{-11}$
1	-0.00014071144636	-0.00014071122544	$2.20921854 \cdot 10^{-10}$

Table 4. The numerical values of the integral of square residual function given by Eq. (27) corresponding to the approximate solutions given by Eqs. (28)-(36) for $M = 15$ and $P = 20$

i	$\int_0^1 R_i^2(t)\, dt$
1	$7.464737808519274 \cdot 10^{-17}$
2	$8.459118984995915 \cdot 10^{-13}$
3	$1.0225368942228159 \cdot 10^{-13}$
4	$2.6769875640036004 \cdot 10^{-23}$
5	$9.659432242298964 \cdot 10^{-17}$
6	$8.086505295586057 \cdot 10^{-17}$
7	$1.4525406989853368 \cdot 10^{-23}$
8	$9.659432355712418 \cdot 10^{-17}$
9	$8.086505295410683 \cdot 10^{-17}$

Fig. 3. Comparison between the approximate solutions $\bar{x}_3$ given by Eq. (30) and the corresponding numerical solutions

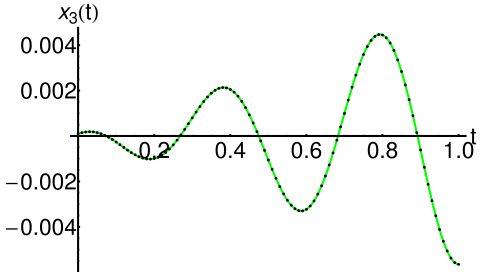


Fig. 4. Comparison between the approximate solutions $\bar{x}_4$ given by Eq. (31) and the corresponding numerical solutions

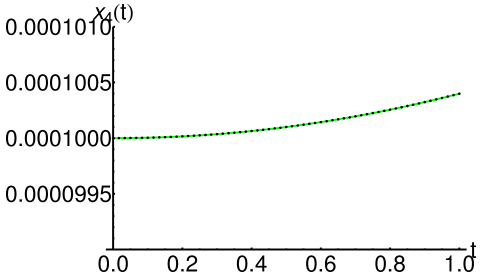


Fig. 5. Comparison between the approximate solutions $\bar{x}_5$ given by Eq. (32) and the corresponding numerical solutions

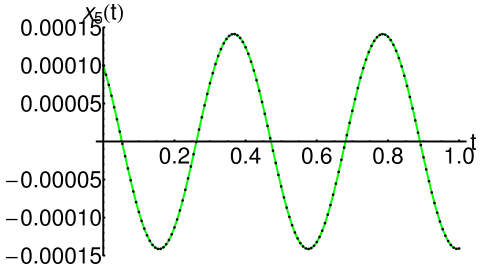


Fig. 6. Comparison between the approximate solutions $\bar{x}_6$ given by Eq. (33) and the corresponding numerical solutions

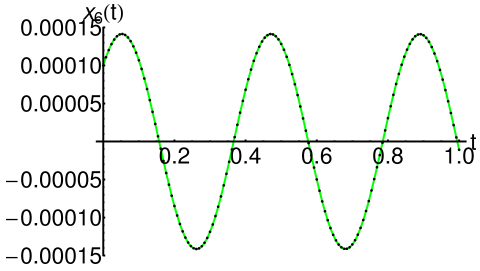


Fig. 7. Comparison between the approximate solutions $\bar{x}_7$ given by Eq. (34) and the corresponding numerical solutions

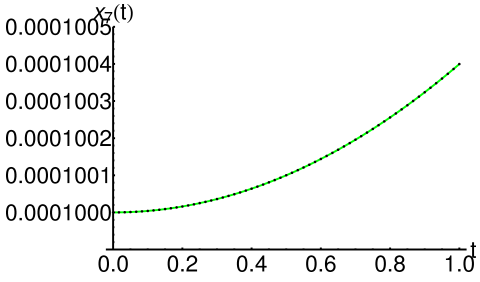


Fig. 8. Comparison between the approximate solutions $\bar{x}_8$ given by Eq. (35) and the corresponding numerical solutions

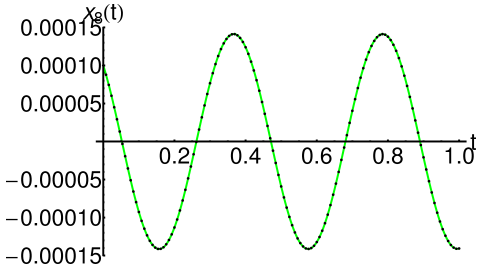
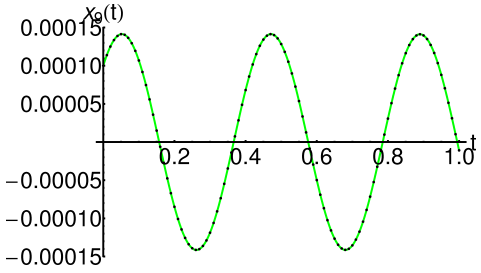


Fig. 9. Comparison between the approximate solutions $\bar{x}_9$ given by Eq. (36) and the corresponding numerical solutions



6 Conclusion

The paper presents the stabilization of a dynamical system using a linear control function. The Hamilton-Poisson formulation of the obtained system allows to use energy-methods in order to obtain stability results. In the last section the approximate analytic solutions of the considered controlled system (4) are established using the optimal homotopy asymptotic method (OHAM). Numerical simulations via Mathematica 9.0 software and the approximations deviations are presented. The accuracy of our results is pointed out by means of the approximate residual of the solutions.

The next step we intend to do is a comparison between the Lie-Trotter integrator (which is a Poisson one, see [11]) and OHAM, regarding the numerical results.

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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