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Lie n superderivations and generalized Lie n superderivations of superalgebras

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Abstract: In the paper, we study Lie n superderivations and generalized Lie n superderivations of superalgebras, using the theory of functional identities in superalgebras. We prove that if $A = A_0 \oplus A_1$ is a prime superalgebra with $\deg(A_1) \geq 2n+5$, $n \geq 2$, then any Lie n superderivation of A is the sum of a superderivation and a linear mapping, and any generalized Lie n superderivation of A is the sum of a generalized superderivation and a linear mapping.

Keywords: Lie n superderivation, Generalized Lie n superderivation, Superalgebra, Functional identity

MSC: 17A70, 16W10, 16W55

1 Introduction

Let A be an associative algebra. A linear mapping $d : A \to A$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$. A Lie derivation δ of A is a linear mapping from A into itself satisfying $\delta[x, y] = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in A$. A Lie triple derivation is a linear mapping $\psi : A \to A$ which satisfies $\psi[[x, y], z] = [[\psi(x), y], z] + [[x, \psi(y)], z] + [[x, y], \psi(z)]$ for all $x, y, z \in A$. Obviously, each derivation is a Lie derivation and each Lie derivation is a Lie triple derivation. Brešar [1] described the structure of Lie derivations and Lie triple derivations on prime rings and obtained that each Lie derivation or Lie triple derivation of a prime ring is the sum of a derivation and an additive mapping. Wang [2] studied the structure of Lie superderivations of superalgebras in 2016. Lie n-derivations were introduced by Abdullaev [3], where the form of Lie n-derivations of a certain von Neumann algebra was described. In 2012, Benkovič and Eremita [4] gave the form of Lie n-derivations on triangular rings, which has been generalized to generalized matrix algebras in [5].

The concept of a generalized derivation was introduced by Brešar [6] and generalized by Hvala [7], who has proved in [8] that each generalized Lie derivation of a prime ring is the sum of a generalized derivation and a central mapping which vanishes on all commutators.

A functional identity can be described as an identical relation involving elements in a ring together with functions. The goal when studying a functional identity is to describe the form of these functions or to determine the structure of the ring admitting the functional identity in question. The theory of functional identities in rings originated from the results of commuting mappings [9]. The name “functional identity” was introduced by Brešar in [10]. The crucial tool in the theory of functional identities in rings is the d-free set, which was developed by Beidar and Chebotar in [11, 12]. Making use of the theory of functional identities in rings, Herstein’s conjectures on Lie mappings in rings have been settled [13–15]. After this, Wang

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[16] established the theory of functional identities in superalgebras and gave the definition of d-superfree sets. As an application, Wang [17] described Lie superhomomorphisms from the set of skew elements of a superalgebra with superinvolution into a unital superalgebra. The knowledge of functional identities and d-superfree sets of superalgebras refer to [16], [18] and [19].

In the paper, our purpose is to study Lie n superderivations and generalized Lie n superderivations of superalgebras, using the theory of functional identities in superalgebras. Section 2 presents some preliminaries. In the third section we discuss the structure of Lie n superderivations. In Section 4 the results of generalized Lie n superderivations are stated and proved.

2 Preliminaries

Throughout the paper, by an algebra we shall mean an algebra over a fixed unital commutative ring Φ. We assume without further mentioning that $\frac{1}{2} \in \Phi$.

An associative algebra A over Φ is said to be an *associative superalgebra* if there exist two Φ-submodules A_0 and A_1 of A such that $A = A_0 \oplus A_1$ and $A_iA_j \subseteq A_{i+j}$, $i, j \in Z_2$. We call A_0 the *even* and A_1 the *odd* part of A. The elements of A_i are *homogeneous of degree i* and we write $|a_i| = i$ for all $a_i \in A_i$. For a superalgebra A, we define $\sigma : A \to A$ by $(a_0 + a_1)^\sigma = a_0 - a_1$, then σ is an automorphism of A such that $\sigma^2 = 1$. On the other hand, for an algebra A, if there exists an automorphism σ of A such that $\sigma^2 = 1$, then A becomes a superalgebra $A = A_0 \oplus A_1$, where $A_i = \{x \in A | x^\sigma = (-1)^i x\}$, $i = 0, 1$. A superalgebra A is called a *prime superalgebra* if and only if $aAb = 0$ implies $a = 0$ or $b = 0$, where at least one of the elements a and b is homogeneous.

On a superalgebra A, define for any $x, y \in A_0 \cup A_1$ the *Lie superproduct*

$$[x, y]_s = xy - (-1)^{|x||y|} yx.$$

Thus

$$[a, b]_s = [a_0, b_0]_s + [a_1, b_0]_s + [a_0, b_1]_s + [a_1, b_1]_s,$$

where $a = a_0 + a_1$, $b = b_0 + b_1$.

In [20] Montaner obtained that a prime superalgebra A is not necessarily a prime algebra but a semiprime algebra. Hence one can define the maximal right ring of quotients Q_{mr} of A, and the useful properties of Q_{mr} can be found in [21]. By [21, proposition 2.5.3] σ can be uniquely extended to Q_{mr}. Therefore, Q_{mr} is also a superalgebra. Moreover, we can get that Q_{mr} is a prime superalgebra.

On the other hand, we will introduce some important concepts of the theory of functional identities in superalgebras.

Let $Q = Q_0 \oplus Q_1$ be a unital superalgebra with grading automorphism σ and center $C = C_0 \oplus C_1$ satisfying $[C, Q] = 0$. Fix an element $\omega \in Q$ as follows: If either $\sigma = 1$ or σ is outer, we set $\omega = 0$. Otherwise, we denote ω as an invertible element in Q such that $\sigma(x) = \omega x \omega^{-1}$ for all $x \in Q$. It is easy to check that $\omega \in Q_0$, $\omega^2 \in C_0$, $\omega x_0 = x_0 \omega$ for all $x_0 \in Q_0$, and $\omega x_1 = -x_1 \omega$ for all $x_1 \in Q_1$. We shall call the ω the grading element of Q. If $a = b + c\omega$, $b, c \in C$, we set $\bar{a} = b - c\omega$.

Let $m \in N^*, \mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_m$ are subsets of Q such that either $\mathcal{U}_i \subseteq Q_0$ or $\mathcal{U}_i \subseteq Q_1$ for every $1 \le i \le m$. Set $\epsilon_i = \pm 1$, where either $\epsilon_i = 1$ if $\mathcal{U}_i \subseteq Q_0$ or $\epsilon_i = -1$ if $\mathcal{U}_i \subseteq Q_1$. $\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_m$ are nonempty sets, $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \cdots, m\}$ and $\delta_l : \mathcal{S}_l \to \mathcal{U}_l$, $l \in \mathcal{I} \cup \mathcal{J}$, are surjective maps. Set $\widehat{\mathcal{S}} = \prod_{k=1}^m \mathcal{S}_k$, $\widehat{\mathcal{U}} = \prod_{k=1}^m \mathcal{U}_k$ and $\Delta = \{\delta_l | l \in \mathcal{I} \cup \mathcal{J}\}$.

We shall consider *functional identities* on $\widehat{\mathcal{S}}$ of the following form

$$\sum_{i\in\mathcal{I}} E_i^i(\bar{x}_m)\delta_i(x_i) + \sum_{j\in\mathcal{J}} \delta_j(x_j)F_j^j(\bar{x}_m) = 0; \tag{1}$$

$$\sum_{i\in\mathcal{I}} E_i^i(\bar{x}_m)\delta_i(x_i) + \sum_{j\in\mathcal{J}} \delta_j(x_j)F_j^j(\bar{x}_m) \in C + C\omega, \tag{2}$$

for all $\bar{x}_m \in \widehat{\mathcal{S}}$, where $E_i : \prod_{k\neq i} \mathcal{S}_k \to Q$ and $F_j : \prod_{k\neq j} \mathcal{S}_k \to Q$.

Suppose that $\omega = 0$ or each $\mathcal{U}_i \subseteq Q_0$. There exist maps

$$p_{ij} : \prod_{k\neq i,j} \mathcal{S}_k \to Q, \quad i \in \mathcal{I}, \quad j \in \mathcal{J}, \quad i \neq j,$$

$$\lambda_l : \prod_{k \neq l} \mathcal{S}_k \to C + C\omega, \quad l \in \mathcal{I} \cup \mathcal{J},$$

such that

$$\begin{aligned} E_i^i(\bar{x}_m) &= \sum_{\substack{j \in \mathcal{J} \\ j \neq i}} \delta_j(x_j) p_{ij}^{ij}(\bar{x}_m) + \lambda_i^i(\bar{x}_m); \\ F_j^j(\bar{x}_m) &= -\sum_{\substack{i \in \mathcal{I} \\ i \neq j}} p_{ij}^{ij}(\bar{x}_m)\delta_i(x_i) - \lambda_j^j(\bar{x}_m), \end{aligned} \tag{3}$$

for all $\bar{x}_m \in \widehat{S}$, where $\lambda_l = 0$ if $l \notin \mathcal{I} \cap \mathcal{J}$.

Otherwise, there exist maps

$$\begin{aligned} p_{ij} &: \prod_{k \neq i,j} \mathcal{S}_k \to Q, \quad i \in \mathcal{I}, \quad j \in \mathcal{J}, \quad i \neq j, \\ \lambda_l, \mu_l &: \prod_{k \neq l} \mathcal{S}_k \to C \quad l \in \mathcal{I} \cup \mathcal{J}, \end{aligned}$$

such that

$$\begin{aligned} E_i^i(\bar{x}_m) &= \sum_{\substack{j \in \mathcal{J} \\ j \neq i}} \delta_j(x_j) p_{ij}^{ij}(\bar{x}_m) + \lambda_i^i(\bar{x}_m) + \mu_i^i(\bar{x}_m)\omega; \\ F_j^j(\bar{x}_m) &= -\sum_{\substack{i \in \mathcal{I} \\ i \neq j}} p_{ij}^{ij}(\bar{x}_m)\delta_i(x_i) - \lambda_j^j(\bar{x}_m) - \epsilon_j \mu_j^j(\bar{x}_m)\omega, \end{aligned} \tag{4}$$

for all $\bar{x}_m \in \widehat{S}$, where $\lambda_l = 0 = \mu_l$ if $l \notin \mathcal{I} \cap \mathcal{J}$. We shall refer to (3) and (4) as a *standard solution* of (1) and (2).

Definition 2.1 ([16, Definition 3.1]). *Let $d \in N^*$. A triple $(\widehat{\mathcal{S}}; \Delta; \widehat{\mathcal{U}})$ is called d-superfree if the following conditions are satisfied:*
(a) For all $m \in \mathcal{N}^*$ and $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \cdots, m\}$ with $\max\{|\mathcal{I}|, |\mathcal{J}|\} < d + 1$, we have that (1) implies (3) and (4).
(b) For all $m \in \mathcal{N}^*$ and $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \cdots, m\}$ with $\max\{|\mathcal{I}|, |\mathcal{J}|\} < d$, we have that (2) implies (3) and (4).

If each $\mathcal{S}_k = \mathcal{U}_k$ and each $\delta_l = \mathrm{id}_{\mathcal{U}_l}$, then the $\widehat{\mathcal{U}}$ is said to be d-superfree provided that $(\widehat{\mathcal{S}}; \Delta; \widehat{\mathcal{U}})$ is so. Let $R = R_0 \oplus R_1$ be a graded Φ-submodule of Q. For every $1 \leq i \leq m$, either $\mathcal{U}_i = R_0$ or $\mathcal{U}_i = R_1$. Then R is said to be d-superfree provided that each $\widehat{\mathcal{U}}$ is d-superfree. And, we can get the following result.

Lemma 2.2 ([16, Theorem 4.16]). *Let $A = A_0 \oplus A_1$ be a prime superalgebra. If $\deg(A_1) \geq 2d + 1$, then A is d-superfree.*

Let $\{x_1, x_2, \cdots, x_m\}$ be a finite set of variables and k be a nonnegative integer such that $k \leq m$. We denote by $\mathcal{M}_m^k$ the set of all multilinear monomials of degree k in the variables $\{x_1, x_2, \cdots, x_m\}$. It is understood that $\mathcal{M}_1^0 = \{1\}$. We write $\mathcal{M}_m = \cup_{k=0}^m \mathcal{M}_m^k$. For a given monomial $M = x_{i_1} \cdots x_{i_u} \in \mathcal{M}_m$ where $u \leq m - k$. We denote by $\mathcal{M}_m^k(M)$ the set of all multilinear monomials of degree k in the variables $\{x_1, \cdots, x_m\} \backslash \{x_{i_1}, \cdots, x_{i_u}\}$ and write $\mathcal{M}_m(M) = \cup_{k=0}^{m-u} \mathcal{M}_m^k(M)$.

For every $1 \leq t \leq m$, let $\mathcal{S}_t$ and $\mathcal{R}_t$ be two sets and let $\delta_t : \mathcal{S}_t \to \mathcal{R}_t$ be a surjective mapping. We set

$$M(\bar{s}_m) = \delta_{i_1}(s_{i_1})\delta_{i_2}(s_{i_2})\cdots\delta_{i_u}(s_{i_u}),$$

where $s_{i_t} \in \mathcal{S}_{i_t}$.

We set $\widehat{\mathcal{S}} = \prod_{i=1}^m \mathcal{S}_i$ and $\widehat{\mathcal{S}}(M) = \prod_{t=1}^{m-u} \mathcal{S}_{j_t}$. For any given $F : \widehat{\mathcal{S}}(M) \to Q$ we introduce a mapping $F^M : \widehat{\mathcal{S}} \to Q$ by the rule

$$F^M(\bar{s}_m) = F^M(s_1, \cdots, s_m) = F(s_{j_1}, \cdots, s_{j_{m-u}}),$$

for any $\bar{s}_m \in \widehat{\mathcal{S}}$.

Let $M \in \mathcal{M}_m^k$ and let $\lambda_M : \widehat{\mathcal{S}}(M) \to C + C\omega$. A mapping $\widehat{\mathcal{S}} \to Q$ defined by the rule $\bar{s}_m \to \lambda_M^M(\bar{s}_m)M(\bar{s}_m)$ for any $\bar{s}_m \in \widehat{\mathcal{S}}$ is called a *superquasi-monomial* and is denoted by $\lambda_M M$. A sum $\sum_{L\in\mathcal{M}_m} \lambda_L L$ of different superquasi-monomials will be called a *superquasi-polynomial*.

An element $x \in A_0 \cup A_1$ is said to be *algebraic* over C of degree $\leq n$ if there exist $c_0, c_1, \ldots, c_n \in C$, not all zero and such that $\sum_{i=0}^n c_i x^{n-i} = 0$. The element x is said to be algebraic over C of degree n if it is algebraic over C of degree $\leq n$ and is not algebraic over C of degree $\leq n-1$. By $\deg(x)$ we shall mean the degree of x over C (if x is algebraic over C) or ∞ (if x is not algebraic over C). Given a nonempty subset $S \subseteq A_0 \cup A_1$, we set

$$\deg(S) = \sup\{\deg(x) | x \in S\}.$$

Let A be a superalgebra. For $i \in \{0, 1\}$, a *superderivation of degree i* is actually a Φ-linear mapping $d_i : A \to A$ which satisfies $d_i(A_j) \subseteq A_{i+j}$, $j \in Z_2$, and $d_i(ab) = d_i(a)b + (-1)^{i|a|}ad_i(b)$ for all $a, b \in A_0 \cup A_1$. If $d = d_0 + d_1$, then d is called a *superderivation*.

Let A be a superalgebra. For $i \in \{0, 1\}$, a Φ-linear mapping $g_i : A \to A$ is called a *generalized superderivation of degree i* if $g_i(A_j) \subseteq A_{i+j}$, $j \in Z_2$, and $g_i(xy) = g_i(x)y + (-1)^{i|x|}xd_i(y)$ for all $x, y \in A_0 \cup A_1$, where d_i is a superderivation of degree i. If $g = g_0 + g_1$, then g is called a *generalized superderivation*.

The following identity will be used frequently,

$$[a_i b_j, c_k]_s = [a_i, b_j c_k]_s + (-1)^{ij+ik}[b_j, c_k a_i]_s \qquad a_i, b_j, c_k \in A_0 \cup A_1, \tag{5}$$

where $i, j, k \in \{0, 1\}$.

3 Lie n superderivations of superalgebras

In the section, we describe the structure of Lie n superderivations on a superalgebra.

Definition 3.1. *Let A be a superalgebra. For $m \in \{0, 1\}$, a Lie superderivation of degree m is actually a Φ-linear mapping $\alpha_m : A \to A$ which satisfies $\alpha_m(A_j) \subseteq A_{m+j}$, $j \in Z_2$, and $\alpha_m([x, y]_s) = [\alpha_m(x), y]_s + (-1)^{m|x|}[x, \alpha_m(y)]_s$ for all $x, y \in A_0 \cup A_1$. If $\alpha = \alpha_0 + \alpha_1$, then α is called a Lie superderivation on A.*

Obviously, each superderivation is a Lie superderivation on A.

Definition 3.2. *Let A be a superalgebra. For $m \in \{0, 1\}$, a Φ-linear mapping $\beta_m : A \to A$ is called a Lie triple superderivation of degree m if $\beta_m(A_j) \subseteq A_{m+j}$, $j \in Z_2$, and*

$$\beta_m([[x, y]_s, z]_s) = [[\beta_m(x), y]_s, z]_s + (-1)^{m|x|}[[x, \beta_m(y)]_s, z]_s + (-1)^{m(|x|+|y|)}[[x, y]_s, \beta_m(z)]_s,$$

for all $x, y, z \in A_0 \cup A_1$. If $\beta = \beta_0 + \beta_1$, then β is called a Lie triple superderivation on A.

Let us define the following sequence of polynomials: $p_1(x) = x$ and

$$p_n(x_1, x_2, \ldots, x_n) = [p_{n-1}(x_1, x_2, \ldots, x_{n-1}), x_n]_s \qquad n \geqslant 2.$$

Thus, $p_2(x_1, x_2) = [x_1, x_2]_s$, $p_3(x_1, x_2, x_3) = [[x_1, x_2]_s, x_3]_s$, etc.

Definition 3.3. *Let $n \geqslant 2$ be an integer. Let A be a superalgebra. For $m \in \{0, 1\}$, a Φ-linear mapping $\gamma_m : A \to A$ is called a Lie n superderivation of degree m if $\gamma_m(A_j) \subseteq A_{m+j}$, $j \in Z_2$, and*

$$\begin{aligned}\gamma_m(p_n(x_1, x_2, \ldots, x_n)) &= p_n(\gamma_m(x_1), x_2, \ldots, x_n) \\ &+ \sum_{i=2}^n (-1)^{m(|x_1|+|x_2|+\ldots+|x_{i-1}|)} p_n(x_1, x_2, \ldots, x_{i-1}, \gamma_m(x_i), x_{i+1}, \ldots, x_n),\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in A_0 \cup A_1$. If $\gamma = \gamma_0 + \gamma_1$, then γ is called a Lie n superderivation on A.

Theorem 3.4. *Let $Q = Q_0 \oplus Q_1$ be a unital superalgebra with center $C = C_0 \oplus C_1$. Let $A = A_0 \oplus A_1$ be a superalgebra and a subalgebra of Q. Suppose that $\gamma : A \to Q$ is a Lie n superderivation, $n \geq 2$. If A is an $(n+2)$-superfree subset of Q, then $\gamma = d + h$, where $d : A \to Q$ is a superderivation and $h : A \to C + C\omega$ is a linear mapping.*

Proof. By the definition of Lie n superderivations, we assume that γ_m is a Lie n superderivation of degree m, $m \in \{0, 1\}$. According to (5), we have

$$\begin{aligned}[][\dots[[a_ib_j, c_k]_s, x_3]_s, \dots, x_n]_s =& [\dots[[a_i, b_jc_k]_s, x_3]_s, \dots, x_n]_s \\ &+ (-1)^{ij+ik}[\dots[[b_j, c_ka_i]_s, x_3]_s, \dots, x_n]_s,\end{aligned} \tag{6}$$

for all $a_i, b_j, c_k, x_3, \dots, x_n \in A_0 \cup A_1$. Applying γ_m to (6), we have

$$\begin{aligned}0 =& [\dots[[\gamma_m(a_ib_j), c_k]_s, x_3]_s, \dots, x_n]_s + (-1)^{m(i+j)}[\dots[[a_ib_j, \gamma_m(c_k)]_s, x_3]_s, \dots, x_n]_s \\ &+ (-1)^{m(i+j+k)}[\dots[[a_ib_j, c_k]_s, \gamma_m(x_3)]_s, \dots, x_n]_s \\ &+ \dots + (-1)^{m(i+j+k+\dots+|x_{n-1}|)}[\dots[[a_ib_j, c_k]_s, x_3]_s, \dots, \gamma_m(x_n)]_s \\ &- [\dots[[\gamma_m(a_i), b_jc_k]_s, x_3]_s, \dots, x_n]_s - (-1)^{mi}[\dots[[a_i, \gamma_m(b_jc_k)]_s, x_3]_s, \dots, x_n]_s \\ &- (-1)^{m(i+j+k)}[\dots[[a_i, b_jc_k]_s, \gamma_m(x_3)]_s, \dots, x_n]_s \\ &- \dots - (-1)^{m(i+j+k+\dots+|x_{n-1}|)}[\dots[[a_i, b_jc_k]_s, x_3]_s, \dots, \gamma_m(x_n)]_s \\ &- (-1)^{ij+ik}([\dots[[\gamma_m(b_j), c_ka_i]_s, x_3]_s, \dots, x_n]_s + (-1)^{mj}[\dots[[b_j, \gamma_m(c_ka_i)]_s, x_3]_s, \dots, x_n]_s \\ &+ (-1)^{m(i+j+k)}[\dots[[b_j, c_ka_i]_s, \gamma_m(x_3)]_s, \dots, x_n]_s \\ &+ \dots + (-1)^{m(i+j+k+\dots+|x_{n-1}|)}[\dots[[b_j, c_ka_i]_s, x_3]_s, \dots, \gamma_m(x_n)]_s).\end{aligned}$$

It follows from (5) that

$$\begin{aligned}0 =& [\dots[[\gamma_m(a_ib_j), c_k]_s + (-1)^{m(i+j)}[a_i, b_j\gamma_m(c_k)]_s + (-1)^{mi+mj+ij+mi+ik}[b_j, \gamma_m(c_k)a_i]_s \\ &- [\gamma_m(a_i)b_j, c_k]_s + (-1)^{mj+ij+mk+ik}[b_j, c_k\gamma_m(a_i)]_s - (-1)^{mi}[a_i, \gamma_m(b_jc_k)]_s \\ &- (-1)^{ij+ik}[\gamma_m(b_j)c_k, a_i]_s + (-1)^{ij+ik+mk+jk+mi+ji}[c_k, a_i\gamma_m(b_j)]_s \\ &- (-1)^{ij+ik+mj}[b_j, \gamma_m(c_ka_i)]_s, x_3]_s, \dots, x_n]_s,\end{aligned}$$

for all $a_i, b_j, c_k, x_3, \dots, x_n \in A_0 \cup A_1$. Since $[a_i, b_j]_s = -(-1)^{ij}[b_j, a_i]_s$, it follows that

$$\begin{aligned}0 =& [\dots[[\gamma_m(a_ib_j), c_k]_s - [\gamma_m(a_i)b_j, c_k]_s - (-1)^{mi}[a_i\gamma_m(b_j), c_k]_s - (-1)^{mi}([a_i, \gamma_m(b_jc_k)]_s \\ &- [a_i, \gamma_m(b_j)c_k]_s - (-1)^{mj}[a_i, b_j\gamma_m(c_k)]_s) - (-1)^{ij+ik+mj}([b_j, \gamma_m(c_ka_i)]_s \\ &- [b_j, \gamma_m(c_k)a_i]_s - (-1)^{mk}[b_j, c_k\gamma_m(a_i)]_s), x_3]_s, \dots, x_n]_s.\end{aligned} \tag{7}$$

Define $\mathcal{B} : A \times A \to Q$ by

$$\mathcal{B}(x, y) = \gamma_m(xy) - \gamma_m(x)y - (-1)^{m|x|}x\gamma_m(y),$$

for all $x, y \in A_0 \cup A_1$. It follows from (7) that

$$[\dots[[\mathcal{B}(a_i, b_j), c_k]_s - (-1)^{mi}[a_i, \mathcal{B}(b_j, c_k)]_s - (-1)^{ij+ik+mj}[b_j, \mathcal{B}(c_k, a_i)]_s, x_3]_s, \dots, x_n]_s = 0, \tag{8}$$

for all $a_i, b_j, c_k, x_3, \dots, x_n \in A_0 \cup A_1$. Since A is an $(n+2)$-superfree subset of Q, $n \geq 2$, [16, Theorem 3.8] implies

$$\begin{aligned}\mathcal{B}(x_0, y_0) &= \lambda_1x_0y_0 + \lambda_1'y_0x_0 + \mu_1(x_0)y_0 + \mu_1'(y_0)x_0 + \nu_1(x_0, y_0); \\ \mathcal{B}(x_0, y_1) &= \lambda_2x_0y_1 + \lambda_2'y_1x_0 + \mu_2(x_0)y_1 + \mu_2'(y_1)x_0 + \nu_2(x_0, y_1); \\ \mathcal{B}(x_1, y_0) &= \lambda_3x_1y_0 + \lambda_3'y_0x_1 + \mu_3(x_1)y_0 + \mu_3'(y_0)x_1 + \nu_3(x_1, y_0); \\ \mathcal{B}(x_1, y_1) &= \lambda_4x_1y_1 + \lambda_4'y_1x_1 + \mu_4(x_1)y_1 + \mu_4'(y_1)x_1 + \nu_4(x_1, y_1),\end{aligned} \tag{9}$$

where $\lambda_k, \lambda'_k \in C_m + C_m\omega$, $\mu_k, \mu'_k : A_i \to C_{m+i} + C_{m+i}\omega$, $\nu_k : A_i \times A_j \to C + C\omega$, $k \in \{1,2,3,4\}$, $i, j \in \{0,1\}$.

We shall now compute $\gamma_m(xyz)$ in two different ways. On the one hand,

$$\begin{aligned}\gamma_m(xyz) &= \mathcal{B}(xy,z) + \gamma_m(xy)z + (-1)^{m(|x|+|y|)}xy\gamma_m(z)\\ &= \mathcal{B}(xy,z) + (\mathcal{B}(x,y) + \gamma_m(x)y + (-1)^{m|x|}x\gamma_m(y))z + (-1)^{m(|x|+|y|)}xy\gamma_m(z).\end{aligned}$$

On the other hand,

$$\begin{aligned}\gamma_m(xyz) &= \mathcal{B}(x,yz) + \gamma_m(x)yz + (-1)^{m|x|}x\gamma_m(yz)\\ &= \mathcal{B}(x,yz) + \gamma_m(x)yz + (-1)^{m|x|}x(\mathcal{B}(y,z) + \gamma_m(y)z + (-1)^{m|y|}y\gamma_m(z)).\end{aligned}$$

Comparing the above expressions, we get

$$\mathcal{B}(xy,z) - \mathcal{B}(x,yz) + \mathcal{B}(x,y)z - (-1)^{m|x|}x\mathcal{B}(y,z) = 0, \tag{10}$$

for all $x, y, z \in A_0 \cup A_1$.

When $|x| = |y| = |z| = 0$, it follows from (10) that

$$\begin{aligned}&\lambda_1 xyz + \lambda'_1 zxy + \mu_1(xy)z + \mu'_1(z)xy + \nu_1(xy,z)\\ &- \lambda_1 xyz - \lambda'_1 yzx - \mu_1(x)yz - \mu'_1(yz)x - \nu_1(x,yz)\\ &+ \lambda_1 xyz + \lambda'_1 yxz + \mu_1(x)yz + \mu'_1(y)xz + \nu_1(x,y)z\\ &- \lambda_1 xyz - \lambda'_1 xzy - \mu_1(y)xz - \mu'_1(z)xy - \nu_1(y,z)x = 0.\end{aligned}$$

An easy computation shows that:
- The coefficient of zxy is λ'_1;
- The coefficient of xz is $\mu'_1(y) - \mu_1(y)$;
- The coefficient of z is $\mu_1(xy) + \nu_1(x,y)$.

By [16, Theorem 3.7], we have

$$\lambda'_1 = 0;\ \ \mu_1(y_0) = \mu'_1(y_0);\ \ \mu_1(x_0y_0) = -\nu_1(x_0,y_0).$$

When $|x| = |z| = 0$ and $|y| = 1$, it follows from (10) that

$$\begin{aligned}&\lambda_3 xyz + \lambda'_3 zxy + \mu_3(xy)z + \mu'_3(z)xy + \nu_3(xy,z)\\ &- \lambda_2 xyz - \lambda'_2 yzx - \mu_2(x)yz - \mu'_2(yz)x - \nu_2(x,yz)\\ &+ \lambda_2 xyz + \lambda'_2 yxz + \mu_2(x)yz + \mu'_2(y)xz + \nu_2(x,y)z\\ &- \lambda_3 xyz - \lambda'_3 xzy - \mu_3(y)xz - \mu'_3(z)xy - \nu_3(y,z)x = 0.\end{aligned}$$

An easy computation shows that:
- The coefficient of zxy is λ'_3;
- The coefficient of yzx is $-\lambda'_2$;
- The coefficient of xz is $\mu'_2(y) - \mu_3(y)$;
- The coefficient of x is $-\mu'_2(yz) - \nu_3(y,z)$;
- The coefficient of z is $\mu_3(xy) + \nu_2(x,y)$.

By [16, Theorem 3.7], we have

$$\lambda'_2 = \lambda'_3 = 0;\ \ \mu_3(y_1) = \mu'_2(y_1);\ \ \mu'_2(y_1z_0) = -\nu_3(y_1,z_0);\ \ \mu_3(x_0y_1) = -\nu_2(x_0,y_1).$$

When $|x| = 0$ and $|y| = |z| = 1$, it follows from (10) that

$$\begin{aligned}&\lambda_4 xyz + \lambda'_4 zxy + \mu_4(xy)z + \mu'_4(z)xy + \nu_4(xy,z)\\ &- \lambda_1 xyz - \lambda'_1 yzx - \mu_1(x)yz - \mu'_1(yz)x - \nu_1(x,yz)\\ &+ \lambda_2 xyz + \lambda'_2 yxz + \mu_2(x)yz + \mu'_2(y)xz + \nu_2(x,y)z\\ &- \lambda_4 xyz - \lambda'_4 xzy - \mu_4(y)xz - \mu'_4(z)xy - \nu_4(y,z)x = 0.\end{aligned}$$

An easy computation shows that:

- The coefficient of xyz is $\lambda_2 - \lambda_1$;
- The coefficient of zxy is λ_4';
- The coefficient of yz is $\mu_2(x) - \mu_1(x)$;
- The coefficient of xz is $\mu_2'(y) - \mu_4(y)$;
- The coefficient of x is $-\mu_1'(yz) - \nu_4(y,z)$.

By [16, Theorem 3.7], we have

$$\lambda_1 = \lambda_2;\;\; \lambda_4' = 0;\;\; \mu_1(x_0) = \mu_2(x_0);\;\; \mu_2'(y_1) = \mu_4(y_1);\;\; \mu_1'(y_1z_1) = -\nu_4(y_1,z_1).$$

By the definition of $\mathcal{B}$, we have

$$\begin{aligned}
&\mathcal{B}([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s,\omega_1) - \mathcal{B}(\omega_1,[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s)\\
=&\gamma_m([[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s,\omega_1]_s) - \gamma_m([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s)\omega_1\\
&-[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s\gamma_m(\omega_1) + \gamma_m(\omega_1)[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s\\
&+\omega_1\gamma_m([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s)\\
=&[[\dots[\gamma_m([x_1,x_2]_s),x_3]_s,\dots,x_n]_s,\omega_1]_s + [[\dots[[x_1,x_2]_s,\gamma_m(x_3)]_s,\dots,x_n]_s,\omega_1]_s\\
&+\dots+[[\dots[[x_1,x_2]_s,x_3]_s,\dots,\gamma_m(x_n)]_s,\omega_1]_s - \mathcal{B}([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s,x_n)\omega_1\\
&+\mathcal{B}(x_n,[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s)\omega_1 + \omega_1\mathcal{B}([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s,x_n)\\
&-\omega_1\mathcal{B}(x_n,[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s) - \gamma_m([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s)x_n\omega_1\\
&-[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s\gamma_m(x_n)\omega_1 + \gamma_m(x_n)[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s\omega_1\\
&+x_n\gamma_m([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s)\omega_1 + \omega_1\gamma_m([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s)x_n\\
&+\omega_1[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s\gamma_m(x_n) - \omega_1\gamma_m(x_n)[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s\\
&-\omega_1x_n\gamma_m([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s)\\
=&[[\dots[\gamma_m([x_1,x_2]_s),x_3]_s,\dots,x_n]_s,\omega_1]_s - \mathcal{B}([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s,x_n)\omega_1\\
&+\mathcal{B}(x_n,[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s)\omega_1 + \omega_1\mathcal{B}([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s,x_n)\\
&-\omega_1\mathcal{B}(x_n,[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-1}]_s) - \mathcal{B}([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-2}]_s,x_{n-1})x_n\omega_1\\
&+\mathcal{B}(x_{n-1},[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_{n-2}]_s)x_n\omega_1 - \dots - \mathcal{B}([x_1,x_2]_s,x_3)x_4\dots x_n\omega_1\\
&+\mathcal{B}(x_3,[x_1,x_2]_s)x_4\dots x_n\omega_1 - \gamma_m([x_1,x_2]_s)x_3\dots x_n\omega_1 + \dots + \omega_1\gamma_m([x_1,x_2]_s)x_3\dots x_n,
\end{aligned}$$

for all $x_1,x_2,\dots,x_n,\omega_1 \in A_0$. Since the coefficient of $x_1x_2\dots x_n\omega_1$ is $(n-1)\lambda_1$, [16, Theorem 3.7] yields $\lambda_1 = 0$.

On the other hand, we have

$$\begin{aligned}
&\mathcal{B}([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s,\omega_1) - \mathcal{B}(\omega_1,[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s)\\
=&\gamma_m([[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s,\omega_1]_s) - \gamma_m([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s)\omega_1\\
&-[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s\gamma_m(\omega_1) + \gamma_m(\omega_1)[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s\\
&+\omega_1\gamma_m([\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s)\\
=&[[\dots[\gamma_m([x_1,x_2]_s),x_3]_s,\dots,x_n]_s,\omega_1]_s + [[\dots[[x_1,x_2]_s,\gamma_m(x_3)]_s,\dots,x_n]_s,\omega_1]_s\\
&+\dots+[[\dots[[x_1,x_2]_s,x_3]_s,\dots,\gamma_m(x_n)]_s,\omega_1]_s + [[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s,\gamma_m(\omega_1)]_s\\
&-[\dots[[\gamma_m(x_1),x_2]_s,x_3]_s,\dots,x_n]_s\omega_1 - (-1)^m[\dots[[x_1,\gamma_m(x_2)]_s,x_3]_s,\dots,x_n]_s\omega_1\\
&-[\dots[[x_1,x_2]_s,\gamma_m(x_3)]_s,\dots,x_n]_s\omega_1 - \dots - [\dots[[x_1,x_2]_s,x_3]_s,\dots,\gamma_m(x_n)]_s\omega_1\\
&-[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s\gamma_m(\omega_1) + \gamma_m(\omega_1)[\dots[[x_1,x_2]_s,x_3]_s,\dots,x_n]_s\\
&+\omega_1[\dots[[\gamma_m(x_1),x_2]_s,x_3]_s,\dots,x_n]_s + (-1)^m\omega_1[\dots[[x_1,\gamma_m(x_2)]_s,x_3]_s,\dots,x_n]_s\\
&+\omega_1[\dots[[x_1,x_2]_s,\gamma_m(x_3)]_s,\dots,x_n]_s + \dots + \omega_1[\dots[[x_1,x_2]_s,x_3]_s,\dots,\gamma_m(x_n)]_s\\
=&[[\dots[\mathcal{B}(x_1,x_2)+\mathcal{B}(x_2,x_1),x_3]_s,\dots,x_n]_s,\omega_1]_s,
\end{aligned}$$

for all $x_1,x_2 \in A_1, x_3,\dots,x_n,\omega_1 \in A_0$. Since the coefficient of $x_1x_2\dots x_n\omega_1$ is $\lambda_1-\lambda_4$, [16, Theorem 3.7] implies $\lambda_4 = \lambda_1 = 0$.

When $|y| = 0$ and $|x| = |z| = 1$, it follows from (10) that

$$\begin{aligned}&\mu_4(xy)z + \mu_4'(z)xy + \nu_4(xy,z) - \mu_4(x)yz - \mu_4'(yz)x - \nu_4(x,yz)\\&+ \lambda_3 xyz + \mu_3(x)yz + \mu_3'(y)xz + \nu_3(x,y)z - (-1)^m\overline{\mu_2(y)}xz\\&- (-1)^m\overline{\mu_2'(z)}xy - (-1)^m\overline{\nu_2(y,z)}x = 0.\end{aligned}$$

Since the coefficient of xyz is λ_3 and the coefficient of xz is $\mu_3'(y) - (-1)^m\overline{\mu_2(y)}$, [16, Theorem 3.7] yields

$$\lambda_3 = 0 \text{ and } \mu_3'(y_0) = (-1)^m\overline{\mu_2(y_0)}.$$

When $|x| = |y| = |z| = 1$, it follows from (10) that

$$\begin{aligned}&\mu_2(xy)z + \mu_2'(z)xy + \nu_2(xy,z) - \mu_3(x)yz - \mu_3'(yz)x - \nu_3(x,yz)\\&+ \mu_4(x)yz + \mu_4'(y)xz + \nu_4(x,y)z - (-1)^m\overline{\mu_4(y)}xz\\&- (-1)^m\overline{\mu_4'(z)}xy - (-1)^m\overline{\nu_4(y,z)}x = 0.\end{aligned}$$

Since the coefficient of xz is $\mu_4'(y) - (-1)^m\overline{\mu_4(y)}$, [16, Theorem 3.7] implies $\mu_4'(y_1) = (-1)^m\overline{\mu_4(y_1)}$.

According to (9), we have

$$\begin{aligned}\mathcal{B}(x_0,y_0) &= \mu_1(x_0)y_0 + \mu_1(y_0)x_0 - \mu_1(x_0y_0);\\\mathcal{B}(x_0,y_1) &= \mu_1(x_0)y_1 + \mu_4(y_1)x_0 - \mu_4(x_0y_1);\\\mathcal{B}(x_1,y_0) &= \mu_4(x_1)y_0 + (-1)^m x_1\mu_1(y_0) - \mu_4(x_1y_0);\\\mathcal{B}(x_1,y_1) &= \mu_4(x_1)y_1 + (-1)^m x_1\mu_4(y_1) - \mu_1(x_1y_1).\end{aligned} \tag{11}$$

Set

$$\mu_m(x) = \begin{cases}\mu_1(x) & x \in A_0,\\ \mu_4(x) & x \in A_1.\end{cases}$$

It follows from (11) that:

- $\gamma_m(x_0y_0) + \mu_m(x_0y_0) = \gamma_m(x_0)y_0 + \mu_m(x_0)y_0 + x_0\gamma_m(y_0) + x_0\mu_m(y_0)$;
- $\gamma_m(x_0y_1) + \mu_m(x_0y_1) = \gamma_m(x_0)y_1 + \mu_m(x_0)y_1 + x_0\gamma_m(y_1) + x_0\mu_m(y_1)$;
- $\gamma_m(x_1y_0) + \mu_m(x_1y_0) = \gamma_m(x_1)y_0 + \mu_m(x_1)y_0 + (-1)^m(x_1\gamma_m(y_0) + x_1\mu_m(y_0))$;
- $\gamma_m(x_1y_1) + \mu_m(x_1y_1) = \gamma_m(x_1)y_1 + \mu_m(x_1)y_1 + (-1)^m(x_1\gamma_m(y_1) + x_1\mu_m(y_1))$.

Let $\gamma_m + \mu_m = d_m$ and $h = -\mu_0 - \mu_1$, then $\gamma = d + h$, where $d = d_0 + d_1$ is a superderivation and $h : A \to C + C\omega$ is a linear mapping. □

By Lemma 2.2 and the above result, we have

Corollary 3.5. *Let $A = A_0 \oplus A_1$ be a prime superalgebra with maximal right ring of quotients Q and extended centroid C. Suppose that $\gamma : A \to Q$ is a Lie n superderivation, $n \geq 2$. If $\deg(A_1) \geq 2n + 5$, then $\gamma = d + h$, where $d : A \to Q$ is a superderivation and $h : A \to C + C\omega$ is a linear mapping.*

In particular, we get the following results, which will be used in the next section.

Theorem 3.6. *Let $Q = Q_0 \oplus Q_1$ be a unital superalgebra with center $C = C_0 \oplus C_1$. Let $A = A_0 \oplus A_1$ be a superalgebra and a subalgebra of Q. Suppose that $\alpha : A \to Q$ is a Lie superderivation. If A is a 4-superfree subset of Q, then $\alpha = d + h$, where $d : A \to Q$ is a superderivation and $h : A \to C + C\omega$ is a linear mapping.*

Corollary 3.7. *Let $A = A_0 \oplus A_1$ be a prime superalgebra with maximal right ring of quotients Q and extended centroid C. Suppose that $\alpha : A \to Q$ is a Lie superderivation. If $\deg(A_1) \geq 9$, then $\alpha = d + h$, where $d : A \to Q$ is a superderivation and $h : A \to C + C\omega$ is a linear mapping.*

Theorem 3.8. *Let $Q = Q_0 \oplus Q_1$ be a unital superalgebra with center $C = C_0 \oplus C_1$. Let $A = A_0 \oplus A_1$ be a superalgebra and a subalgebra of Q. Suppose that $\beta : A \to Q$ is a Lie triple superderivation. If A is a 5-superfree subset of Q, then $\beta = d + h$, where $d : A \to Q$ is a superderivation and $h : A \to C + C\omega$ is a linear mapping.*

Corollary 3.9. *Let $A = A_0 \oplus A_1$ be a prime superalgebra with maximal right ring of quotients Q and extended centroid C. Suppose that $\beta : A \to Q$ is a Lie triple superderivation. If $\deg(A_1) \geq 11$, then $\beta = d + h$, where $d : A \to Q$ is a superderivation and $h : A \to C + C\omega$ is a linear mapping.*

4 Generalized Lie *n* superderivations of superalgebras

In the section, we describe the structure of generalized Lie n superderivations on a superalgebra.

Definition 4.1. *Let A be a superalgebra. For $m \in \{0, 1\}$, a Φ-linear mapping $\eta_m : A \to A$ is called a generalized Lie superderivation of degree m if $\eta_m(A_j) \subseteq A_{m+j}$, $j \in Z_2$, and*

$$\eta_m([x,y]_s) = \eta_m(x)y - (-1)^{|x||y|}\eta_m(y)x + (-1)^{m|x|}x\alpha_m(y) - (-1)^{m|y|+|x||y|}y\alpha_m(x),$$

for all $x, y \in A_0 \cup A_1$, where α_m is a Lie superderivation of degree m on A. If $\eta = \eta_0 + \eta_1$, then η is called a generalized Lie superderivation on A.

Definition 4.2. *Let A be a superalgebra. For $m \in \{0, 1\}$, a Φ-linear mapping $\theta_m : A \to A$ is called a generalized Lie triple superderivation of degree m if $\theta_m(A_j) \subseteq A_{m+j}$, $j \in Z_2$, and*

$$\begin{aligned}\theta_m([[x,y]_s,z]_s) =& \theta_m(x)yz - (-1)^{|x||y|}\theta_m(y)xz - (-1)^{|x||z|+|y||z|}\theta_m(z)xy \\ &+ (-1)^{|x||y|+|x||z|+|y||z|}\theta_m(z)yx + (-1)^{m|x|}x\beta_m(y)z - (-1)^{m|y|+|x||y|}y\beta_m(x)z \\ &- (-1)^{m|z|+|x||z|+|y||z|}z\beta_m(x)y + (-1)^{m|z|+|x||y|+|y||z|+|x||z|}z\beta_m(y)x \\ &+ (-1)^{m|x|+m|y|}xy\beta_m(z) - (-1)^{m|x|+m|y|+|x||y|}yx\beta_m(z) \\ &- (-1)^{m|x|+m|z|+|x||z|+|y||z|}zx\beta_m(y) + (-1)^{m|y|+m|z|+|x||y|+|x||z|+|y||z|}zy\beta_m(x),\end{aligned}$$

for all $x, y, z \in A_0 \cup A_1$, where β_m is a Lie triple superderivation of degree m on A. If $\theta = \theta_0 + \theta_1$, then θ is called a generalized Lie triple superderivation on A.

According to the definition of Lie n superderivations, we give the definition of generalized Lie n superderivations.

Definition 4.3. *Let $n \geqslant 2$ be an integer. Let A be a superalgebra. For $m \in \{0, 1\}$, a Φ-linear mapping $\vartheta_m : A \to A$ is called a generalized Lie n superderivation of degree m if $\vartheta_m(A_j) \subseteq A_{m+j}$, $j \in Z_2$, and*

$$\vartheta_m(p_n(x_1, x_2, \ldots, x_n)) = \sum_{r=1}^{2^{n-1}} \sum_{l=1}^{n} A_r \tau_l \rho_r(x_1, x_2, \ldots, x_n),$$

for all $x_1, x_2, \ldots, x_n \in A_0 \cup A_1$, where

$A_1 = 1, A_{2^{i-2}+1} = -(-1)^{|x_i|(|x_1|+\ldots+|x_{i-1}|)}, i \in \{2, 3, \ldots, n\},$
$A_{2^{i-2}+j} = A_{2^{i-2}+1}A_j, 2 \leqslant j \leqslant 2^{i-2},$
$\rho_1(x_{k_1}, x_{k_2}, \ldots, x_{k_n}) = (x_{k_1}, x_{k_2}, \ldots, x_{k_n}),$
$\rho_{2^{i-2}+1}(x_{k_1}, x_{k_2}, \ldots, x_{k_n}) = (x_{k_i}, x_{k_1}, \ldots, x_{k_i-1}, x_{k_i+1}, \ldots, x_{k_n}), i \in \{2, 3, \ldots, n\},$
$\rho_{2^{i-2}+j}(x_{k_1}, x_{k_2}, \ldots, x_{k_n}) = \rho_{2^{i-2}+1}\rho_j(x_{k_1}, x_{k_2}, \ldots, x_{k_n}), 2 \leqslant j \leqslant 2^{i-2},$
$\tau_1(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = \vartheta_m(x_{t_1})x_{t_2} \ldots x_{t_n},$
$\tau_2(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = (-1)^{m|x_{t_1}|}x_{t_1}\gamma_m(x_{t_2}) \ldots x_{t_n},$
$\tau_3(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = (-1)^{m(|x_{t_1}|+|x_{t_2}|)}x_{t_1}x_{t_2}\gamma_m(x_{t_3}) \ldots x_{t_n},$
$\ldots$

$\tau_n(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = (-1)^{m(|x_{t_1}|+\ldots+|x_{t_{n-1}}|)} x_{t_1} x_{t_2} \ldots \gamma_m(x_{t_n})$,
γ_m *is a Lie n superderivation of degree m on A.*

That is, $A_r = 1$ *or* $A_r = -1$, *and* ρ_r *is a permutable. If* $\vartheta = \vartheta_0 + \vartheta_1$, *then* ϑ *is called a generalized Lie n superderivation on A.*

The expression of generalized Lie n superderivations is too complicated, so we will study the structure of generalized Lie triple superderivations firstly. In the same manner we can get the structure of generalized Lie n superderivations.

Theorem 4.4. *Let* $Q = Q_0 \oplus Q_1$ *be a unital superalgebra with center* $C = C_0 \oplus C_1$. *Let* $A = A_0 \oplus A_1$ *be a superalgebra and a subalgebra of Q. Suppose that* $\theta : A \to Q$ *is a generalized Lie triple superderivation. If A is a* 5*-superfree subset of Q, then* $\theta = g + l$, *where* $g : A \to Q$ *is a generalized superderivation and* $l : A \to C + C\omega$ *is a linear mapping.*

Proof. By the definition of generalized Lie triple superderivations, we assume that θ_m is a generalized Lie triple superderivation of degree m and β_m is a Lie triple superderivation of degree m, $m \in \{0, 1\}$. According to (5) we have

$$[[a_i b_j, c_k]_s, d_l]_s = [[a_i, b_j c_k]_s, d_l]_s + (-1)^{ij+ik}[[b_j, c_k a_i]_s, d_l]_s, \tag{12}$$

for all $a_i, b_j, c_k, d_l \in A_0 \cup A_1$. Applying θ_m to (12), we get

$$\begin{aligned}
0 =& \theta_m(a_i b_j) c_k d_l - (-1)^{ik+jk} \theta_m(c_k) a_i b_j d_l - (-1)^{il+jl+kl} \theta_m(d_l) a_i b_j c_k \\
&+ (-1)^{ik+jk+il+jl+kl} \theta_m(d_l) c_k a_i b_j + (-1)^{mi+mj} a_i b_j \beta_m(c_k) d_l - (-1)^{mk+ik+jk} c_k \beta_m(a_i b_j) d_l \\
&- (-1)^{ml+il+jl+kl} d_l \beta_m(a_i b_j) c_k + (-1)^{ml+ik+jk+il+jl+kl} d_l \beta_m(c_k) a_i b_j \\
&+ (-1)^{mi+mj+mk} a_i b_j c_k \beta_m(d_l) - (-1)^{mi+mj+mk+ik+jk} c_k a_i b_j \beta_m(d_l) \\
&- (-1)^{mi+mj+ml+il+jl+kl} d_l a_i b_j \beta_m(c_k) + (-1)^{mk+ml+ik+jk+il+jl+kl} d_l c_k \beta_m(a_i b_j) - \theta_m(a_i) b_j c_k d_l \\
&+ (-1)^{ij+ik} \theta_m(b_j c_k) a_i d_l + (-1)^{il+jl+kl} \theta_m(d_l) a_i b_j c_k - (-1)^{ij+ik+il+jl+kl} \theta_m(d_l) b_j c_k a_i \\
&- (-1)^{mi} a_i \beta_m(b_j c_k) d_l + (-1)^{mj+mk+ij+ik} b_j c_k \beta_m(a_i) d_l + (-1)^{ml+il+jl+kl} d_l \beta_m(a_i) b_j c_k \\
&- (-1)^{ml+ij+ik+il+jl+kl} d_l \beta_m(b_j c_k) a_i - (-1)^{mi+mj+mk} a_i b_j c_k \beta_m(d_l) \\
&+ (-1)^{mi+mj+mk+ij+ik} b_j c_k a_i \beta_m(d_l) + (-1)^{mi+ml+il+jl+kl} d_l a_i \beta_m(b_j c_k) \\
&- (-1)^{mj+mk+ml+ij+ik+il+jl+kl} d_l b_j c_k \beta_m(a_i) - (-1)^{ij+ik} (\theta_m(b_j) c_k a_i d_l - (-1)^{jk+ji} \theta_m(c_k a_i) b_j d_l \\
&- (-1)^{il+jl+kl} \theta_m(d_l) b_j c_k a_i + (-1)^{jk+ji+il+jl+kl} \theta_m(d_l) c_k a_i b_j - (-1)^{mk+mi+jk+ji} c_k a_i \beta_m(b_j) d_l \\
&+ (-1)^{mj} b_j \beta_m(c_k a_i) d_l - (-1)^{ml+il+jl+kl} d_l \beta_m(b_j) c_k a_i + (-1)^{ml+jk+ji+il+jl+kl} d_l \beta_m(c_k a_i) b_j \\
&+ (-1)^{mi+mj+mk} b_j c_k a_i \beta_m(d_l) - (-1)^{mi+mj+mk+jk+ji} c_k a_i b_j \beta_m(d_l) \\
&- (-1)^{mj+ml+il+jl+kl} d_l b_j \beta_m(c_k a_i) + (-1)^{mi+mk+ml+jk+ji+il+jl+kl} d_l c_k a_i \beta_m(b_j)),
\end{aligned}$$

for all $a_i, b_j, c_k, d_l \in A_0 \cup A_1$. An easy computation shows that

$$\begin{aligned}
0 =&\theta_m(a_ib_j)c_kd_l - \theta_m(a_i)b_jc_kd_l - (-1)^{ik+jk}\theta_m(c_k)a_ib_jd_l + (-1)^{ij+ik+jk+ji}\theta_m(c_ka_i)b_jd_l \\
&+ (-1)^{ij+ik}\theta_m(b_jc_k)a_id_l - (-1)^{ij+ik}\theta_m(b_j)c_ka_id_l + (-1)^{mi+mj}a_ib_j\beta_m(c_k)d_l \\
&- (-1)^{mi}a_i\beta_m(b_jc_k)d_l + (-1)^{mj+mk+ij+ik}b_jc_k\beta_m(a_i)d_l - (-1)^{mj+ij+ik}b_j\beta_m(c_ka_i)d_l \\
&- (-1)^{mk+ik+jk}c_k\beta_m(a_ib_j)d_l + (-1)^{mi+mk+ik+jk}c_ka_i\beta_m(b_j)d_l - (-1)^{ml+il+jl+kl}d_l\beta_m(a_ib_j)c_k \\
&+ (-1)^{ml+il+jl+kl}d_l\beta_m(a_i)b_jc_k - (-1)^{mi+mj+il+jl+ml+kl}d_la_ib_j\beta_m(c_k) \\
&+ (-1)^{mi+ml+il+jl+kl}d_la_i\beta_m(b_jc_k) + (-1)^{ml+ik+jk+il+jl+kl}d_l\beta_m(c_k)a_ib_j \\
&- (-1)^{ml+ik+jk+il+jl+kl}d_l\beta_m(c_ka_i)b_j + (-1)^{mk+ml+ik+jk+il+jl+kl}d_lc_k\beta_m(a_ib_j) \\
&- (-1)^{mi+mk+ml+ik+jk+kl+jl+il}d_lc_ka_i\beta_m(b_j) - (-1)^{ml+ij+ik+il+jl+kl}d_l\beta_m(b_jc_k)a_i \\
&+ (-1)^{ml+ij+ik+jl+kl+ik}d_l\beta_m(b_j)c_ka_i - (-1)^{mj+mk+ml+ij+ik+jl+kl+il}d_lb_jc_k\beta_m(a_i) \\
&+ (-1)^{mj+ml+jl+kl+il+ij+ik}d_lb_j\beta_m(c_ka_i),
\end{aligned} \tag{13}$$

for all $a_i, b_j, c_k, d_l \in A_0 \cup A_1$. By Theorem 3.8, we have $\beta_m(x) = d_m(x) + h_m(x)$, where d_m is a superderivation of degree m and $h_m : A \to C + C\omega$ is a linear mapping. Define $\mathcal{B} : A \times A \to Q$ by

$$\mathcal{B}(x, y) = \theta_m(xy) - \theta_m(x)y - (-1)^{m|x|}xd_m(y),$$

for all $x, y \in A_0 \cup A_1$. We can rewrite (13) as

$$\mathcal{B}(a_i, b_j)c_kd_l + (-1)^{ij+ik}\mathcal{B}(b_j, c_k)a_id_l + (-1)^{ik+jk}\mathcal{B}(c_k, a_i)b_jd_l + \sum \lambda_L^L L = 0,$$

where $\sum \lambda_L^L L$ is a superquasi-polynomial. By A is a 5-superfree subset of Q, [16, Theorem 3.8] implies

$$\begin{aligned}
\mathcal{B}(x_0, y_0) &= \lambda_1x_0y_0 + \lambda_1'y_0x_0 + \mu_1(x_0)y_0 + \mu_1'(y_0)x_0 + \nu_1(x_0, y_0); \\
\mathcal{B}(x_0, y_1) &= \lambda_2x_0y_1 + \lambda_2'y_1x_0 + \mu_2(x_0)y_1 + \mu_2'(y_1)x_0 + \nu_2(x_0, y_1); \\
\mathcal{B}(x_1, y_0) &= \lambda_3x_1y_0 + \lambda_3'y_0x_1 + \mu_3(x_1)y_0 + \mu_3'(y_0)x_1 + \nu_3(x_1, y_0); \\
\mathcal{B}(x_1, y_1) &= \lambda_4x_1y_1 + \lambda_4'y_1x_1 + \mu_4(x_1)y_1 + \mu_4'(y_1)x_1 + \nu_4(x_1, y_1),
\end{aligned} \tag{14}$$

where $\lambda_k, \lambda_k' \in C_m + C_m\omega$, $\mu_k, \mu_k' : A_i \to C_{m+i} + C_{m+i}\omega$, $\nu_k : A_i \times A_j \to C + C\omega$, $k \in \{1, 2, 3, 4\}$, $i, j \in \{0, 1\}$.

By computing $\theta_m(xyz)$ in two different ways, we have

$$\mathcal{B}(xy, z) + \mathcal{B}(x, y)z - \mathcal{B}(x, yz) = 0, \tag{15}$$

for all $x, y, z \in A_0 \cup A_1$.

By substituting (14) into (15), we get

$$\begin{aligned}
0 =&\lambda_1xyz + \lambda_1'zxy + \mu_1(xy)z + \mu_1'(z)xy + \nu_1(xy, z) \\
&+ \lambda_1xyz + \lambda_1'yxz + \mu_1(x)yz + \mu_1'(y)xz + \nu_1(x, y)z \\
&- \lambda_1xyz - \lambda_1'yzx - \mu_1(x)yz - \mu_1'(yz)x - \nu_1(x, yz),
\end{aligned}$$

for all $x, y, z \in A_0$;

$$\begin{aligned}
0 =&\lambda_3xyz + \lambda_3'zxy + \mu_3(xy)z + \mu_3'(z)xy + \nu_3(xy, z) \\
&+ \lambda_2xyz + \lambda_2'yxz + \mu_2(x)yz + \mu_2'(y)xz + \nu_2(x, y)z \\
&- \lambda_2xyz - \lambda_2'yzx - \mu_2(x)yz - \mu_2'(yz)x - \nu_2(x, yz),
\end{aligned}$$

for all $x, z \in A_0, y \in A_1$;

$$\begin{aligned}
0 =&\lambda_4xyz + \lambda_4'zxy + \mu_4(xy)z + \mu_4'(z)xy + \nu_4(xy, z) \\
&+ \lambda_2xyz + \lambda_2'yxz + \mu_2(x)yz + \mu_2'(y)xz + \nu_2(x, y)z \\
&- \lambda_1xyz - \lambda_1'yzx - \mu_1(x)yz - \mu_1'(yz)x - \nu_1(x, yz),
\end{aligned}$$

for all $x \in A_0, y, z \in A_1$;

$$\begin{aligned}0 =&\lambda_4 xyz + \lambda_4' zxy + \mu_4(xy)z + \mu_4'(z)xy + \nu_4(xy, z)\\ &+ \lambda_3 xyz + \lambda_3' yxz + \mu_3(x)yz + \mu_3'(y)xz + \nu_3(x, y)z\\ &- \lambda_4 xyz - \lambda_4' yzx - \mu_4(x)yz - \mu_4'(yz)x - \nu_4(x, yz),\end{aligned}$$

for all $y \in A_0, x, z \in A_1$;

$$\begin{aligned}0 =&\lambda_1 xyz + \lambda_1' zxy + \mu_1(xy)z + \mu_1'(z)xy + \nu_1(xy, z)\\ &+ \lambda_4 xyz + \lambda_4' yxz + \mu_4(x)yz + \mu_4'(y)xz + \nu_4(x, y)z\\ &- \lambda_4 xyz - \lambda_4' yzx - \mu_4(x)yz - \mu_4'(yz)x - \nu_4(x, yz),\end{aligned}$$

for all $z \in A_0, x, y \in A_1$. By [16, Theorem 3.7], we have

$$\begin{aligned}\lambda_1 = \lambda_1' = \mu_1' = 0, \mu_1(x_0y_0) = -\nu_1(x_0, y_0);\\ \lambda_3 = \lambda_3' = \lambda_2' = \mu_3' = \mu_2' = 0, \mu_3(x_0y_1) = -\nu_2(x_0, y_1);\\ \lambda_4 = -\lambda_2, \lambda_4' = \mu_4' = 0, \mu_1(x_0) = \mu_2(x_0);\\ \mu_3(x_1) = \mu_4(x_1), \mu_4(x_1y_0) = -\nu_3(x_1, y_0);\\ \mu_1(x_1y_1) = -\nu_4(x_1, y_1).\end{aligned}$$

By the definition of $\mathcal{B}$, we have

$$\begin{aligned}&\mathcal{B}([[x_0, w_0], y_0], z_1) - \mathcal{B}(z_1, [[x_0, w_0], y_0])\\ =&\theta_m([[[x_0, w_0], y_0], z_1]) - \theta_m([[x_0, w_0], y_0])z_1 - [[x_0, w_0], y_0]d_m(z_1)\\ &+ \theta_m(z_1)[[x_0, w_0], y_0] + (-1)^m z_1 d_m([[x_0, w_0], y_0])\\ =&\theta_m([x_0, w_0])y_0z_1 - \theta_m(y_0)[x_0, w_0]z_1 - \theta_m(z_1)[x_0, w_0]y_0\\ &+ \theta_m(z_1)y_0[x_0, w_0] + [x_0, w_0]\beta_m(y_0)z_1 - y_0\beta_m([x_0, w_0])z_1\\ &- (-1)^m z_1\beta_m([x_0, w_0])y_0 + (-1)^m z_1\beta_m(y_0)[x_0, w_0] + [x_0, w_0]y_0\beta_m(z_1)\\ &- y_0[x_0, w_0]\beta_m(z_1) - (-1)^m z_1[x_0, w_0]\beta_m(y_0) + (-1)^m z_1y_0\beta_m([x_0, w_0])\\ &- \mathcal{B}([x_0, w_0], y_0)z_1 + \mathcal{B}(y_0, [x_0, w_0])z_1 - \theta_m([x_0, w_0])y_0z_1\\ &- [x_0, w_0]d_m(y_0)z_1 + \theta_m(y_0)[x_0, w_0]z_1 + y_0d_m([x_0, w_0])z_1\\ &- [[x_0, w_0], y_0]d_m(z_1) + \theta_m(z_1)[[x_0, w_0], y_0] + (-1)^m z_1 d_m([[x_0, w_0], y_0]),\end{aligned}$$

for all $x_0, y_0, w_0 \in A_0, z_1 \in A_1$. Since the coefficient of $x_0w_0y_0z_1$ is λ_2, it follows from [16, Theorem 3.7] that $\lambda_2 = 0$ and $\lambda_4 = 0$. Therefore,

$$\begin{aligned}\mathcal{B}(x_0, y_0) = \mu_1(x_0)y_0 - \mu_1(x_0y_0);\\ \mathcal{B}(x_0, y_1) = \mu_1(x_0)y_1 - \mu_4(x_0y_1);\\ \mathcal{B}(x_1, y_0) = \mu_4(x_1)y_0 - \mu_4(x_1y_0);\\ \mathcal{B}(x_1, y_1) = \mu_4(x_1)y_1 - \mu_1(x_1y_1).\end{aligned} \tag{16}$$

Set

$$\mu_m(x) = \begin{cases}\mu_1(x) & x \in A_0,\\ \mu_4(x) & x \in A_1.\end{cases}$$

It follows from (16) that $\theta_m(xy) + \mu_m(xy) = \theta_m(x)y + (-1)^{m|x|}xd_m(y) + \mu_m(x)y$, for all $x, y \in A_0 \cup A_1$.

Let $\theta_m + \mu_m = g_m$ and $l = -\mu_0 - \mu_1$, then $\theta = g + l$, where $g = g_0 + g_1$ is a generalized superderivation and $l : A \to C + C\omega$ is a linear mapping. □

By Lemma 2.2 and the above result, we have

Corollary 4.5. *Let $A = A_0 \oplus A_1$ be a prime superalgebra with maximal right ring of quotients Q and extended centroid C. Suppose that $\beta : A \to Q$ is a generalized Lie triple superderivation. If $\deg(A_1) \geq 11$, then $\beta = g + l$, where $g : A \to Q$ is a generalized superderivation and $l : A \to C + C\omega$ is a linear mapping.*

According to the proof of Theorem 3.4 and Theorem 4.4, we have

Theorem 4.6. *Let $Q = Q_0 \oplus Q_1$ be a unital superalgebra with center $C = C_0 \oplus C_1$. Let $A = A_0 \oplus A_1$ be a superalgebra and a subalgebra of Q. Suppose that $\vartheta : A \to Q$ is a generalized Lie n superderivation, $n \geq 2$. If A is an $(n + 2)$-superfree subset of Q, then $\vartheta = g + l$, where $g : A \to Q$ is a generalized superderivation and $l : A \to C + C\omega$ is a linear mapping.*

By Lemma 2.2 and the above result, we have

Corollary 4.7. *Let $A = A_0 \oplus A_1$ be a prime superalgebra with maximal right ring of quotients Q and extended centroid C. Suppose that $\vartheta : A \to Q$ is a generalized Lie n superderivation, $n \geq 2$. If $\deg(A_1) \geq 2n + 5$, then $\vartheta = g + l$, where $g : A \to Q$ is a generalized superderivation and $l : A \to C + C\omega$ is a linear mapping.*

In particular, we have

Theorem 4.8. *Let $Q = Q_0 \oplus Q_1$ be a unital superalgebra with center $C = C_0 \oplus C_1$. Let $A = A_0 \oplus A_1$ be a superalgebra and a subalgebra of Q. Suppose that $\eta : A \to Q$ is a generalized Lie superderivation. If A is a 4-superfree subset of Q, then $\eta = g + l$, where $g : A \to Q$ is a generalized superderivation and $l : A \to C + C\omega$ is a linear mapping.*

Corollary 4.9. *Let $A = A_0 \oplus A_1$ be a prime superalgebra with maximal right ring of quotients Q and extended centroid C. Suppose that $\eta : A \to Q$ is a generalized Lie superderivation. If $\deg(A_1) \geq 9$, then $\eta = g + l$, where $g : A \to Q$ is a generalized superderivation and $l : A \to C + C\omega$ is a linear mapping.*

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