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## Research Article

Jian Tang, Xinyang Feng, Bijan Davvaz, and Xiang-Yun Xie\*

# A further study on ordered regular equivalence relations in ordered semihypergroups

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**Abstract:** In this paper, we study the ordered regular equivalence relations on ordered semihypergroups in detail. To begin with, we introduce the concept of weak pseudoorders on an ordered semihypergroup, and investigate several related properties. In particular, we construct an ordered regular equivalence relation on an ordered semihypergroup by a weak pseudoorder. As an application of the above result, we completely solve the open problem on ordered semihypergroups introduced in [B. Davvaz, P. Corsini and T. Changphas, Relationship between ordered semihypergroups and ordered semigroups by using pseudoorders, European J. Combinatorics 44 (2015), 208–217]. Furthermore, we establish the relationships between ordered regular equivalence relations and weak pseudoorders on an ordered semihypergroup, and give some homomorphism theorems of ordered semihypergroups, which are generalizations of similar results in ordered semigroups.

**Keywords:** Ordered semihypergroup, Weak pseudoorder, Ordered regular equivalence relation

**MSC:** 20N20, 06F05, 20M12

## 1 Introduction

The theory of algebraic hyperstructures was first introduced by Marty [1], which is a generalization of the theory of ordinary algebraic structures. Later on, people have observed that hyperstructures have many applications to several branches of both pure and applied sciences (see [2–5]). In particular, a semihypergroup is the simplest algebraic hyperstructure which possess the properties of closure and associativity. Nowadays, semihypergroups have been found useful for dealing with problems in different areas of algebraic hyperstructures. Many authors have studied different aspects of semihypergroups, for instance, see [6–13]. Especially, regular and strong regular relations on semihypergroups have been introduced and investigated in [14].

On the other hand, the theory of ordered algebraic structure was first studied by Fuchs [15] in 1960's. In particular, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have several applications in the theory of sequential machines, formal languages, computer arithmetics and error-correcting codes. There are several results which have been

**Jian Tang:** School of Mathematics and Statistics, Fuyang Normal University, Fuyang, Anhui, 236037, China, E-mail: tangjian0901@126.com

**Xinyang Feng:** School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu, 730000, China, E-mail: fxy1012@126.com

**Bijan Davvaz:** Department of Mathematics, Yazd University, Yazd, Iran, E-mail: davvaz@yazd.ac.ir

**\*Corresponding Author: Xiang-Yun Xie:** School of Mathematics and Computational Science, Wuyi University, Jiangmen, Guangdong, 529020, China, E-mail: xyxie@wyu.edu.cn

added to the theory of ordered semigroups by many researchers, for example, N. Kehayopulu, M. Tsingelis, X. Y. Xie, M. Kuil and others. For partial results, the reader is referred to [16–22].

A theory of hyperstructures on ordered semigroups has been recently developed. In [23], Heidari and Davvaz applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. Also see [2]. The work on ordered semihypergroup theory can be found in [24–27]. It is worth pointing out that Davvaz et al. [25] introduced the concept of a pseudoorder on an ordered semihypergroup, and extended some results in [16] on ordered semigroups to ordered semihypergroups. In particular, they posed an open problem about ordered semihypergroups: Is there a regular relation  $\rho$  on an ordered semihypergroup  $(S, \circ, \leq)$  for which  $S/\rho$  is an ordered semihypergroup? To answer the above open problem, Gu and Tang attempted in [28] to introduce the concept of ordered regular equivalence relations on ordered semihypergroups, and construct an ordered regular equivalence relation  $\rho_I$  on an ordered semihypergroup  $S$  by a proper hyperideal  $I$  of  $S$  such that the corresponding quotient structure is also an ordered semihypergroup. However, they only provide a partial solution to the above problem. In fact, for an ordered semihypergroup  $S$ ,  $S$  doesn't necessarily exist a proper hyperideal (see Example 3.3). As a further study, in the present paper we study the ordered regular equivalence relations on ordered semihypergroups in detail and completely solve the open problem given by Davvaz et al. in [25].

The rest of this paper is organized as follows. In Section 2 we recall some basic notions and results from the hyperstructure theory which will be used throughout this paper. In Section 3, we introduce the concept of weak pseudoorders on an ordered semihypergroup  $S$ , and illustrate this notion is a generalization of the concept of pseudoorders on  $S$  by some examples. Furthermore, the properties of weak pseudoorders on an ordered semihypergroup are investigated. In particular, we construct an ordered regular equivalence relation on an ordered semihypergroup by a weak pseudoorder. As an application of the above result, we give a complete answer to the open problem given by Davvaz et al. in [25]. In Section 4, the relationships between ordered regular equivalence relations and weak pseudoorders on an ordered semihypergroup is established, and some homomorphism theorems of ordered semihypergroups by weak pseudoorders are given. Some conclusions are given in the last Section.

## 2 Preliminaries and some notations

Recall that a *hypergroupoid*  $(S, \circ)$  is a nonempty set  $S$  together with a *hyperoperation*, that is a map  $\circ : S \times S \rightarrow P^*(S)$ , where  $P^*(S)$  denotes the set of all the nonempty subsets of  $S$ . The image of the pair  $(x, y)$  is denoted by  $x \circ y$ . If  $x \in S$  and  $A, B \in P^*(S)$ , then  $A \circ B$  is defined by  $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$ . Also  $A \circ x$  is used for  $A \circ \{x\}$  and  $x \circ A$  for  $\{x\} \circ A$ . Generally, the singleton  $\{x\}$  is identified by its element  $x$ .

We say that a hypergroupoid  $(S, \circ)$  is a *semihypergroup* if the hyperoperation “ $\circ$ ” is associative, that is,  $(x \circ y) \circ z = x \circ (y \circ z)$  for all  $x, y, z \in S$  (see [14]).

We now recall the notion of ordered semihypergroups from [23]. An algebraic hyperstructure  $(S, \circ, \leq)$  is called an *ordered semihypergroup* (also called *po-semihypergroup* in [23]) if  $(S, \circ)$  is a semihypergroup and  $(S, \leq)$  is a partially ordered set such that: for any  $x, y, a \in S$ ,  $x \leq y$  implies  $a \circ x \leq a \circ y$  and  $x \circ a \leq y \circ a$ . Here, if  $A, B \in P^*(S)$ , then we say that  $A \leq B$  if for every  $a \in A$  there exists  $b \in B$  such that  $a \leq b$ . Clearly, every ordered semigroup can be regarded as an ordered semihypergroup, see [26]. By a *subsemihypergroup* of an ordered semihypergroup  $S$  we mean a nonempty subset  $A$  of  $S$  such that  $A \circ A \subseteq A$ . A nonempty subset  $A$  of a semihypergroup  $(S, \circ)$  is called a *left* (resp. *right*) *hyperideal* of  $S$  if (1)  $S \circ A \subseteq A$  (resp.  $A \circ S \subseteq A$ ) and (2) If  $a \in A$  and  $S \ni b \leq a$ , then  $b \in A$ . If  $A$  is both a left and a right hyperideal of  $S$ , then it is called a *hyperideal* of  $S$  (see [26]).

Let  $\rho$  be a relation on a semihypergroup  $(S, \circ)$  or an ordered semihypergroup  $(S, \circ, \leq)$ . If  $A$  and  $B$  are nonempty subsets of  $S$ , then we set

$$A \xrightarrow{\rho} B \Leftrightarrow (\forall a \in A)(\exists b \in B) a \rho b,$$

$$A \overset{\leftarrow}{\rho} B \Leftrightarrow (\forall b' \in B)(\exists a' \in A) a' \rho b',$$

$$A \bar{\rho} B \Leftrightarrow A \overset{\rightarrow}{\rho} B \text{ and } A \overset{\leftarrow}{\rho} B,$$

$$A \bar{\bar{\rho}} B \Leftrightarrow (\forall a \in A, \forall b \in B) a \rho b.$$

An equivalence relation  $\rho$  on  $S$  is called *regular* [14, 25] if

$$(\forall x, y, a \in S) x \rho y \Rightarrow a \circ x \bar{\rho} a \circ y \text{ and } x \circ a \bar{\rho} y \circ a;$$

$\rho$  is called *strongly regular* [14, 25] if

$$(\forall x, y, a \in S) x \rho y \Rightarrow a \circ x \bar{\bar{\rho}} a \circ y \text{ and } x \circ a \bar{\bar{\rho}} y \circ a.$$

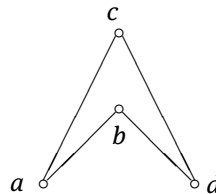
**Example 2.1.** We consider a set  $S := \{a, b, c, d\}$  with the following hyperoperation “ $\circ$ ” and the order “ $\leq$ ”:

| $\circ$ | $a$        | $b$        | $c$        | $d$        |
|---------|------------|------------|------------|------------|
| $a$     | $\{a, d\}$ | $\{a, d\}$ | $\{a, d\}$ | $\{a\}$    |
| $b$     | $\{a, d\}$ | $\{b\}$    | $\{a, d\}$ | $\{a, d\}$ |
| $c$     | $\{a, d\}$ | $\{a, d\}$ | $\{c\}$    | $\{a, d\}$ |
| $d$     | $\{a\}$    | $\{a, d\}$ | $\{a, d\}$ | $\{d\}$    |

$$\leq := \{(a, a), (a, b), (a, c), (b, b), (c, c), (d, b), (d, c), (d, d)\}.$$

We give the covering relation “ $\prec$ ” and the figure of  $S$  as follows:

$$\prec = \{(a, b), (a, c), (d, b), (d, c)\}.$$



Then  $(S, \circ, \leq)$  is an ordered semihypergroup (see [26]). Let  $\rho_1, \rho_2$  be equivalence relations on  $S$  defined as follows:

$$\rho_1 := \{(a, a), (a, d), (b, b), (c, c), (d, a), (d, d)\},$$

$$\rho_2 := \{(a, a), (a, c), (b, b), (c, a), (c, c), (d, d)\}.$$

Then

- (1)  $\rho_1$  is a strongly regular relation on  $S$ .
- (2)  $\rho_2$  is a regular relation on  $S$ , but it is not a strongly regular relation on  $S$ . In fact, since  $apc$ , while  $a \circ a \bar{\bar{\rho}} c \circ a$  doesn't hold.

**Lemma 2.2** ([14]). Let  $(S, \circ)$  be a semihypergroup and  $\rho$  an equivalence relation on  $S$ . Then

- (1) If  $\rho$  is regular, then  $(S/\rho, \otimes)$  is a semihypergroup with respect to the following hyperoperation:  $(a)_\rho \otimes (b)_\rho = \bigcup_{c \in a \circ b} (c)_\rho$ . In this case, we call  $(S/\rho, \otimes)$  a quotient semihypergroup.
- (2) If  $\rho$  is strongly regular, then  $(S/\rho, \otimes)$  is a semigroup with respect to the following operation:  $(a)_\rho \otimes (b)_\rho = (c)_\rho$  for all  $c \in a \circ b$ . In this case, we call  $(S/\rho, \otimes)$  a quotient semigroup.

A relation  $\rho$  on an ordered semihypergroup  $(S, \circ, \leq)$  is called *pseudoorder* [25] if it satisfies the following conditions: (1)  $\leq \subseteq \rho$ , (2)  $apb$  and  $bpc$  imply  $apc$ , i.e.,  $\rho \circ \rho \subseteq \rho$  and (3)  $apb$  implies  $a \circ c \bar{\bar{\rho}} b \circ c$  and  $c \circ a \bar{\bar{\rho}} c \circ b$ , for all  $c \in S$ .

**Example 2.3.** Consider the ordered semihypergroup  $(S, \circ, \leq)$  given in Example 2.1, and define a relation  $\rho$  on  $S$  as follows:

$$\rho := \{(a, a), (a, b), (a, c), (a, d), (b, b), (c, c), (d, a), (d, b), (d, c), (d, d)\}.$$

It is not difficult to verify that  $\rho$  is a pseudoorder on  $S$ .

Let  $(S, \circ, \leq)$ ,  $(T, \diamond, \preceq)$  be two ordered semihypergroups and  $f : S \rightarrow T$  a mapping from  $S$  to  $T$ .  $f$  is called *isotone* if  $x \leq y$  implies  $f(x) \preceq f(y)$ , for all  $x, y \in S$ .  $f$  is called *reverse isotone* if  $x, y \in S$ ,  $f(x) \preceq f(y)$  implies  $x \leq y$ .  $f$  is called *homomorphism* [25] if it is isotone and satisfies  $f(x) \diamond f(y) = \bigcup_{z \in x \circ y} f(z)$ , for all  $x, y \in S$ .  $f$  is called *isomorphism* if it is homomorphism, onto and reverse isotone. The ordered semihypergroups  $S$  and  $T$  are called *isomorphic*, in symbol  $S \cong T$ , if there exists an isomorphism between them.

**Remark 2.4.** Let  $S$  and  $T$  be two ordered semihypergroups. If  $f$  is a homomorphism and reverse isotone mapping from  $S$  to  $T$ , then  $S \cong \text{Im}(f)$ .

In order to investigate the structure of quotient ordered semihypergroups, Gu and Tang [28] introduced the concept of (strongly) ordered regular equivalence relations on an ordered semihypergroup. A regular (resp. strongly regular) equivalence relation  $\rho$  on an ordered semihypergroup  $S$  is called *ordered regular* (resp. *strongly ordered regular*) if there exists an order relation “ $\preceq$ ” on  $(S/\rho, \otimes)$  such that:

(1)  $(S/\rho, \otimes, \preceq)$  is an ordered semihypergroup (resp. ordered semigroup), where the hyperoperation “ $\otimes$ ” is defined as one in Lemma 2.2.

(2) The mapping  $\varphi : S \rightarrow S/\rho$ ,  $x \mapsto (x)_\rho$  is isotone, that is,  $\varphi$  is a homomorphism from  $S$  onto  $S/\rho$ .

The reader is referred to [2, 19] for notation and terminology not defined in this paper.

### 3 Weak pseudoorders on ordered semihypergroups

In [25], Davvaz et al. obtained an ordered semigroup from an ordered semihypergroup by means of pseudoorders. In the same paper they posed the following open problem about ordered semihypergroups.

**Problem 3.1.** Is there a regular equivalence relation  $\rho$  on an ordered semihypergroup  $(S, \circ, \leq)$  for which  $S/\rho$  is an ordered semihypergroup?

To answer the above open problem, Gu and Tang [28] defined an equivalence relation  $\rho_I$  on an ordered semihypergroup  $S$  as follows:

$$\rho_I := \{(x, y) \in S \setminus I \times S \setminus I \mid x = y\} \cup (I \times I),$$

where  $I$  is a proper hyperideal of  $S$ . Furthermore, they provided the following theorem:

**Theorem 3.2** ([28]). Let  $(S, \circ, \leq)$  be an ordered semihypergroup and  $I$  a hyperideal of  $S$ . Then  $\rho_I$  is an ordered regular equivalence relation on  $S$ .

As we have seen in Theorem 3.2, for an ordered semihypergroup  $S$ , there exists a regular equivalence relation  $\rho_I$  on  $S$  such that the corresponding quotient structure  $S/\rho_I$  is also an ordered semihypergroup, where  $I$  is a proper hyperideal  $I$  of  $S$ . However, in general, there need not exist proper hyperideals in  $S$ . We can illustrate it by the following example.

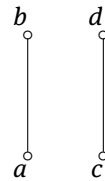
**Example 3.3.** We consider a set  $S := \{a, b, c, d\}$  with the following hyperoperation “ $\circ$ ” and the order “ $\leq$ ”:

| $\circ$ | $a$     | $b$        | $c$        | $d$        |
|---------|---------|------------|------------|------------|
| $a$     | $\{a\}$ | $\{a, b\}$ | $\{a, c\}$ | $S$        |
| $b$     | $\{b\}$ | $\{b\}$    | $\{b, d\}$ | $\{b, d\}$ |
| $c$     | $\{c\}$ | $\{c, d\}$ | $\{c\}$    | $\{c, d\}$ |
| $d$     | $\{d\}$ | $\{d\}$    | $\{d\}$    | $\{d\}$    |

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (c, d), (d, d)\}.$$

We give the covering relation “ $\prec$ ” and the figure of  $S$  as follows:

$$\prec := \{(a, b), (c, d)\}.$$



Then  $(S, \circ, \leq)$  is an ordered semihypergroup. Moreover, it is a routine matter to verify that there do not exist proper hyperideals in  $S$ .

By the above example, it can be seen that Theorem 3.2 only provides a partial solution to Problem 3.1, and not completely. In order to fully solve the open problem, we need define and study the weak pseudoorders on an ordered semihypergroup.

If  $A, B \in P^*(S)$ , then we set

$$A \tilde{\rho} B \Leftrightarrow (\forall a \in A)(\exists b \in B) apb \text{ and } bpa \ \& \ (\forall b' \in B)(\exists a' \in A) a' \rho b' \text{ and } b' \rho a'.$$

**Definition 3.4.** Let  $(S, \circ, \leq)$  be an ordered semihypergroup and  $\rho$  a relation on  $S$ .  $\rho$  is called weak pseudoorder if it satisfies the following conditions:

- (1)  $\leq \subseteq \rho$ ;
- (2)  $apb$  and  $bpc$  imply  $apc$ ;
- (3)  $apb$  implies  $a \circ c \xrightarrow{\rho} b \circ c$  and  $c \circ a \xrightarrow{\rho} c \circ b$ , for all  $c \in S$ ;
- (4)  $apb$  and  $bpa$  imply  $a \circ c \tilde{\rho} b \circ c$  and  $c \circ a \tilde{\rho} c \circ b$ , for all  $c \in S$ .

**Remark 3.5.** Note that if  $(S, \circ, \leq)$  is an ordered semigroup, then Definition 3.4 coincides with Definition 1 in [16].

**Lemma 3.6.** Let  $(S, \circ, \leq)$  be an ordered semihypergroup. Then there exists a weak pseudoorder relation on  $S$ .

*Proof.* With a small amount of effort one can verify that the order relation “ $\leq$ ” on  $S$  is a weak pseudoorder relation on  $S$ .  $\square$

One can easily observe that every pseudoorder relation on an ordered semihypergroup  $S$  is a weak pseudoorder on  $S$ . However, the converse is not true, in general, as shown in the following example.

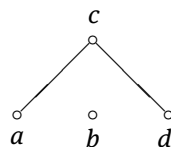
**Example 3.7.** We consider a set  $S := \{a, b, c, d\}$  with the following hyperoperation “ $\circ$ ” and the order “ $\leq$ ”:

| $\circ$ | $a$        | $b$        | $c$        | $d$        |
|---------|------------|------------|------------|------------|
| $a$     | $\{a, d\}$ | $\{a, d\}$ | $\{a, d\}$ | $\{a\}$    |
| $b$     | $\{a, d\}$ | $\{b\}$    | $\{a, d\}$ | $\{a, d\}$ |
| $c$     | $\{a, d\}$ | $\{a, d\}$ | $\{c\}$    | $\{a, d\}$ |
| $d$     | $\{a\}$    | $\{a, d\}$ | $\{a, d\}$ | $\{d\}$    |

$$\leq := \{(a, a), (a, c), (b, b), (c, c), (d, c), (d, d)\}.$$

We give the covering relation “ $\prec$ ” and the figure of  $S$  as follows:

$$\prec = \{(a, c), (d, c)\}.$$



Then  $(S, \circ, \leq)$  is an ordered semihypergroup. Let  $\rho$  be a relation on  $S$  defined as follows:

$$\rho := \{(a, a), (b, b), (c, c), (d, d), (a, c), (b, a), (b, c), (d, a), (d, c)\}.$$

We can easily verify that  $\rho$  is a weak pseudoorder on  $S$ , but it is not a pseudoorder on  $S$ . In fact, since  $apc$ , while  $a \circ b \xrightarrow{\rho} c \circ b$  doesn't hold.

In the following, we shall construct a regular equivalence relation on an ordered semihypergroup by a weak pseudoorder, from which we can answer Problem 3.1 completely.

**Theorem 3.8.** *Let  $(S, \circ, \leq)$  be an ordered semihypergroup and  $\rho$  a weak pseudoorder on  $S$ . Then, there exists a regular equivalence relation  $\rho^*$  on  $S$  such that  $S/\rho^*$  is an ordered semihypergroup.*

*Proof.* We denote by  $\rho^*$  the relation on  $S$  defined by

$$\rho^* := \{(a, b) \in S \times S \mid apb \text{ and } bpa\} (= \rho \cap \rho^{-1}).$$

First, we claim that  $\rho^*$  is a regular equivalence relation on  $S$ . In fact, for any  $a \in S$ , clearly,  $(a, a) \in \rho$ , so  $ap^*a$ . If  $(a, b) \in \rho^*$ , then  $apb$  and  $bpa$ . Thus  $(b, a) \in \rho^*$ . Let  $(a, b) \in \rho^*$  and  $(b, c) \in \rho^*$ . Then  $apb$ ,  $bpa$ ,  $bpc$  and  $cpb$ . Hence  $apc$  and  $cpa$ . It implies that  $(a, c) \in \rho^*$ . Thus  $\rho^*$  is an equivalence relation on  $S$ . Now, let  $ap^*b$  and  $c \in S$ . Then  $apb$  and  $bpa$ . Since  $\rho$  is a weak pseudoorder on  $S$ , by condition (4) of Definition 3.4, we have

$$a \circ c \tilde{\rho} b \circ c \text{ and } c \circ a \tilde{\rho} c \circ b.$$

Thus, for every  $x \in a \circ c$ , there exists  $y \in b \circ c$  such that  $xpy$  and  $ypx$  which imply that  $x\rho^*y$ , and for every  $y' \in b \circ c$ , there exists  $x' \in a \circ c$  such that  $x'\rho y'$  and  $y'\rho x'$  which mean that  $x'\rho^*y'$ . It thus follows that  $a \circ c \tilde{\rho}^* b \circ c$ . Similarly, it can be obtained that  $c \circ a \tilde{\rho}^* c \circ b$ . Hence  $\rho^*$  is indeed a regular equivalence relation on  $S$ . Thus, by Lemma 2.2,  $(S/\rho^*, \otimes)$  is a semihypergroup with respect to the following hyperoperation:

$$(\forall (a)_{\rho^*}, (b)_{\rho^*} \in S/\rho^*) (a)_{\rho^*} \otimes (b)_{\rho^*} = \bigcup_{c \in a \circ b} (c)_{\rho^*}.$$

Now, we define a relation  $\preceq_\rho$  on  $S/\rho^*$  as follows:

$$\preceq_\rho := \{((x)_{\rho^*}, (y)_{\rho^*}) \in S/\rho^* \times S/\rho^* \mid (x, y) \in \rho\}.$$

Then  $(S/\rho^*, \preceq_\rho)$  is a poset. Indeed, suppose that  $(x)_{\rho^*} \in S/\rho^*$ , where  $x \in S$ . Then  $(x, x) \in \rho$ . Hence,  $(x)_{\rho^*} \preceq_\rho (x)_{\rho^*}$ . Let  $(x)_{\rho^*} \preceq_\rho (y)_{\rho^*}$  and  $(y)_{\rho^*} \preceq_\rho (z)_{\rho^*}$ . Then  $xpy$  and  $ypz$ . Thus  $x\rho^*y$ , and we have  $(x)_{\rho^*} = (y)_{\rho^*}$ . Now, if  $(x)_{\rho^*} \preceq_\rho (y)_{\rho^*}$  and  $(y)_{\rho^*} \preceq_\rho (z)_{\rho^*}$ , then  $xpy$  and  $ypz$ . Hence, by hypothesis,  $x\rho z$ , and we conclude that  $(x)_{\rho^*} \preceq_\rho (z)_{\rho^*}$ .

Furthermore, let  $(x)_{\rho^*}, (y)_{\rho^*}, (z)_{\rho^*} \in S/\rho^*$  be such that  $(x)_{\rho^*} \preceq_\rho (y)_{\rho^*}$ . Then  $xpy$  and  $z \in S$ . Since  $\rho$  is a weak pseudoorder on  $S$ , by condition (3) of Definition 3.4, we have  $x \circ z \tilde{\rho} y \circ z$  and  $z \circ x \tilde{\rho} z \circ y$ . Thus, for any  $a \in x \circ z$  there exists  $b \in y \circ z$  such that  $apb$ . This implies that  $(a)_{\rho^*} \preceq_\rho (b)_{\rho^*}$ . Hence we have

$$(x)_{\rho^*} \otimes (z)_{\rho^*} = \bigcup_{a \in x \circ z} (a)_{\rho^*} \preceq_\rho \bigcup_{b \in y \circ z} (b)_{\rho^*} = (y)_{\rho^*} \otimes (z)_{\rho^*}.$$

In a similar way, it can be obtained that  $(z)_{\rho^*} \otimes (x)_{\rho^*} \preceq_\rho (z)_{\rho^*} \otimes (y)_{\rho^*}$ . Therefore,  $(S/\rho^*, \otimes, \preceq_\rho)$  is an ordered semihypergroup.  $\square$

Furthermore, we have the following proposition:

**Proposition 3.9.** *Let  $(S, \circ, \leq)$  be an ordered semihypergroup and  $\rho$  a weak pseudoorder on  $S$ . Then  $\rho^*$  is an ordered regular equivalence relation on  $S$ , where  $\rho^* = \rho \cap \rho^{-1}$ .*

*Proof.* By Theorem 3.8,  $(S/\rho^*, \otimes, \preceq_\rho)$  is an ordered semihypergroup, where the order relation  $\preceq_\rho$  is defined as follows:

$$\preceq_\rho := \{((x)_{\rho^*}, (y)_{\rho^*}) \in S/\rho^* \times S/\rho^* \mid (x, y) \in \rho\}.$$

Also, let  $x, y \in S$  and  $x \leq y$ . Then, since  $\rho$  is a weak pseudoorder on  $S$ ,  $(x, y) \in \rho$ , and thus  $((x)_{\rho^*}, (y)_{\rho^*}) \in \preceq_\rho$ , i.e.,  $(x)_{\rho^*} \preceq_\rho (y)_{\rho^*}$ . Therefore,  $\rho^*$  is an ordered regular equivalence relation on  $S$ .  $\square$



To begin with, we claim that  $\theta'$  is a weak pseudoorder on  $S/\rho^*$ . To prove our claim, let  $((x)_{\rho^*}, (y)_{\rho^*}) \in \preceq_{\rho}$ . Then, by Theorem 3.8,  $(x, y) \in \rho \subseteq \theta$ , which implies that  $((x)_{\rho^*}, (y)_{\rho^*}) \in \theta'$ . Thus,  $\preceq_{\rho} \subseteq \theta'$ . Now, assume that  $((x)_{\rho^*}, (y)_{\rho^*}) \in \theta'$  and  $((y)_{\rho^*}, (z)_{\rho^*}) \in \theta'$ . Then,  $(x, y) \in \theta$  and  $(y, z) \in \theta$ . It implies that  $(x, z) \in \theta$ . Hence,  $((x)_{\rho^*}, (z)_{\rho^*}) \in \theta'$ . Also, let  $((x)_{\rho^*}, (y)_{\rho^*}) \in \theta'$  and  $(z)_{\rho^*} \in S/\rho^*$ . Then,  $(x, y) \in \theta$  and  $z \in S$ . Since  $\theta$  is a weak pseudoorder on  $S$ , we have  $x \circ z \xrightarrow{\theta} y \circ z$  and  $z \circ x \xrightarrow{\theta} z \circ y$ . Thus, for every  $a \in x \circ z$  there exists  $b \in y \circ z$ , we have  $(a, b) \in \theta$ . This implies that  $((a)_{\rho^*}, (b)_{\rho^*}) \in \theta'$ , and thus we have

$$(x)_{\rho^*} \otimes (z)_{\rho^*} = \bigcup_{a \in x \circ z} (a)_{\rho^*} \xrightarrow{\theta'} \bigcup_{b \in y \circ z} (b)_{\rho^*} = (y)_{\rho^*} \otimes (z)_{\rho^*}.$$

Similarly, we obtain that  $(z)_{\rho^*} \otimes (x)_{\rho^*} \xrightarrow{\theta'} (z)_{\rho^*} \otimes (y)_{\rho^*}$ . Furthermore, let  $((x)_{\rho^*}, (y)_{\rho^*}), (z)_{\rho^*} \in S/\rho^*$  be such that  $((x)_{\rho^*}, (y)_{\rho^*}) \in \theta'$  and  $((y)_{\rho^*}, (x)_{\rho^*}) \in \theta'$ . Then  $(x, y) \in \theta$ ,  $(y, x) \in \theta$  and  $z \in S$ . By hypothesis, we have  $x \circ z \xrightarrow{\theta} y \circ z$  and  $z \circ x \xrightarrow{\theta} z \circ y$ . Then, similarly as in the above proof, it can be obtained that  $(x)_{\rho^*} \otimes (z)_{\rho^*} \xrightarrow{\theta'} (y)_{\rho^*} \otimes (z)_{\rho^*}$  and  $(z)_{\rho^*} \otimes (x)_{\rho^*} \xrightarrow{\theta'} (z)_{\rho^*} \otimes (y)_{\rho^*}$ . Therefore,  $\theta'$  is indeed a weak pseudoorder on  $S/\rho^*$ .

Now, we define the mapping  $f: \mathcal{A} \rightarrow \mathcal{B}$  by  $f(\theta) = \theta'$ ,  $\forall \theta \in \mathcal{A}$ . Then,  $f$  is a bijection from  $\mathcal{A}$  onto  $\mathcal{B}$ . In fact,

(1)  $f$  is well defined. Indeed, let  $\theta_1, \theta_2 \in \mathcal{A}$  and  $\theta_1 = \theta_2$ . Then, for any  $((x)_{\rho^*}, (y)_{\rho^*}) \in \theta'_1$ , we have  $(x, y) \in \theta_1 = \theta_2$ . It implies that  $((x)_{\rho^*}, (y)_{\rho^*}) \in \theta'_2$ . Hence  $\theta'_1 \subseteq \theta'_2$ . By symmetry, it can be obtained that  $\theta'_2 \subseteq \theta'_1$ .

(2)  $f$  is one to one. In fact, let  $\theta_1, \theta_2 \in \mathcal{A}$  and  $\theta'_1 = \theta'_2$ . Assume that  $(x, y) \in \theta_1$ . Then,  $((x)_{\rho^*}, (y)_{\rho^*}) \in \theta'_1$  and thus  $((x)_{\rho^*}, (y)_{\rho^*}) \in \theta'_2$ . This implies that  $(x, y) \in \theta_2$ . Thus,  $\theta_1 \subseteq \theta_2$ . Similarly, we obtain  $\theta_2 \subseteq \theta_1$ .

(3)  $f$  is onto. In fact, let  $\delta \in \mathcal{B}$ . We define a relation  $\theta$  on  $S$  as follows:

$$\theta := \{(x, y) \in S \times S \mid ((x)_{\rho^*}, (y)_{\rho^*}) \in \delta\}.$$

We show that  $\theta$  is a weak pseudoorder on  $S$  and  $\rho \subseteq \theta$ . Assume that  $(x, y) \in \rho$ . Then, by Theorem 3.8,  $((x)_{\rho^*}, (y)_{\rho^*}) \in \preceq_{\rho} \subseteq \delta$ , and thus  $(x, y) \in \theta$ . This implies that  $\rho \subseteq \theta$ . If  $(x, y) \in \theta$ , then  $((x)_{\rho^*}, (y)_{\rho^*}) \in \delta$ . Hence,  $\preceq_{\rho} \subseteq \delta$ . Let now  $(x, y) \in \theta$  and  $(y, z) \in \theta$ . Then  $((x)_{\rho^*}, (y)_{\rho^*}) \in \delta$  and  $((y)_{\rho^*}, (z)_{\rho^*}) \in \delta$ . Hence  $((x)_{\rho^*}, (z)_{\rho^*}) \in \delta$ , which implies that  $(x, z) \in \theta$ . Also, let  $x, y, z \in S$  be such that  $(x, y) \in \theta$ . Then  $((x)_{\rho^*}, (y)_{\rho^*}) \in \delta$  and  $(z)_{\rho^*} \in S/\rho^*$ . Since  $\delta$  is a weak pseudoorder on  $S/\rho^*$ , we have  $(x)_{\rho^*} \otimes (z)_{\rho^*} \xrightarrow{\delta} (y)_{\rho^*} \otimes (z)_{\rho^*}$  and  $(z)_{\rho^*} \otimes (x)_{\rho^*} \xrightarrow{\delta} (z)_{\rho^*} \otimes (y)_{\rho^*}$ , i.e.,  $\bigcup_{a \in x \circ z} (a)_{\rho^*} \xrightarrow{\delta} \bigcup_{b \in y \circ z} (b)_{\rho^*}$  and  $\bigcup_{a' \in z \circ x} (a')_{\rho^*} \xrightarrow{\delta} \bigcup_{b' \in z \circ y} (b')_{\rho^*}$ . Thus, for every  $a \in x \circ z$  there exists  $b \in y \circ z$  such that  $((a)_{\rho^*}, (b)_{\rho^*}) \in \delta$ , and for every  $a' \in z \circ x$  there exists  $b' \in z \circ y$  such that  $((a')_{\rho^*}, (b')_{\rho^*}) \in \delta$ . It implies that  $(a, b) \in \theta$  and  $(a', b') \in \theta$ . Hence we conclude that  $x \circ z \xrightarrow{\theta} y \circ z$  and  $z \circ x \xrightarrow{\theta} z \circ y$ . Furthermore, let  $x, y, z \in S$  be such that  $(x, y) \in \theta$  and  $(y, x) \in \theta$ . Then, similarly as in the previous proof, we can deduce that  $x \circ z \xrightarrow{\theta} y \circ z$  and  $z \circ x \xrightarrow{\theta} z \circ y$ . Thus  $\theta$  is a weak pseudoorder on  $S$  and  $\rho \subseteq \theta$ . In other words,  $\theta \in \mathcal{A}$ . Moreover, clearly,  $\theta' = \delta$ . It thus follows that  $f(\theta) = \delta$ .

Therefore,  $f$  is a bijection from  $\mathcal{A}$  onto  $\mathcal{B}$ , and  $\text{card}(\mathcal{A}) = \text{card}(\mathcal{B})$ .  $\square$

By the proof of Theorem 3.11, we immediately obtain the following corollary:

**Corollary 3.12.** *Let  $(S, \circ, \preceq)$  be an ordered semihypergroup and  $\rho, \theta$  be weak pseudoorders on  $S$  such that  $\rho \subseteq \theta$ . We define a relation  $\theta/\rho$  on  $S/\rho^*$  as follows:*

$$\theta/\rho := \{((x)_{\rho^*}, (y)_{\rho^*}) \in S/\rho^* \times S/\rho^* \mid (x, y) \in \theta\}.$$

*Then  $\theta/\rho$  is a weak pseudoorder on  $S/\rho^*$ .*

## 4 Homomorphism theorems of ordered semihypergroups

In the current section, we shall establish the relationships between ordered regular equivalence relations and weak pseudoorders on an ordered semihypergroup, and discuss homomorphism theorems of ordered semihypergroups by weak pseudoorders.

In order to establish the relationships between ordered regular equivalence relations and weak pseudoorders on an ordered semihypergroup, the following lemma is essential.

**Lemma 4.1.** *Let  $(S, \circ, \leq)$  be an ordered semihypergroup and  $\sigma$  a relation on  $S$ . Then the following statements are equivalent:*

- (1)  $\sigma$  is a weak pseudoorder on  $S$ .
- (2) There exist an ordered semihypergroup  $(T, \diamond, \preceq)$  and a homomorphism  $\varphi : S \rightarrow T$  such that

$$\overrightarrow{\ker \varphi} := \{(a, b) \in S \times S \mid \varphi(a) \preceq \varphi(b)\} = \sigma.$$

*Proof.* (1)  $\Rightarrow$  (2). Let  $\sigma$  be a weak pseudoorder on  $S$ . We denote by  $\sigma^*$  the regular equivalence relation on  $S$  defined by

$$\sigma^* := \{(a, b) \in S \times S \mid (a, b) \in \sigma, (b, a) \in \sigma\} (= \sigma \cap \sigma^{-1}).$$

Then, by Theorem 3.8, the set  $S/\sigma^* := \{(a)_{\sigma^*} \mid a \in S\}$  with the hyperoperation  $(a)_{\sigma^*} \otimes (b)_{\sigma^*} = \bigcup_{c \in a \circ b} (c)_{\sigma^*}$ , for all  $a, b \in S$  and the order

$$\preceq_{\sigma} := \{((x)_{\sigma^*}, (y)_{\sigma^*}) \in S/\sigma^* \times S/\sigma^* \mid (x, y) \in \sigma\}$$

is an ordered semihypergroup. Let  $T = (S/\sigma^*, \otimes, \preceq_{\sigma})$  and  $\varphi$  be the mapping of  $S$  onto  $S/\sigma^*$  defined by  $\varphi : S \rightarrow S/\sigma^* \mid a \mapsto (a)_{\sigma^*}$ . Then, by Proposition 3.9,  $\varphi$  is a homomorphism from  $S$  onto  $S/\sigma^*$  and clearly,  $\overrightarrow{\ker \varphi} = \sigma$ .

(2)  $\Rightarrow$  (1). Let  $(S, \circ, \leq)$  be an ordered semihypergroup. If there exist an ordered semihypergroup  $(T, \diamond, \preceq)$  and a homomorphism  $\varphi : S \rightarrow T$  such that  $\overrightarrow{\ker \varphi} = \sigma$ , then  $\sigma$  is a weak pseudoorder on  $S$ . Indeed, let  $(a, b) \in \sigma$ . Then, by hypothesis,  $\varphi(a) \preceq \varphi(b)$ . Thus  $(a, b) \in \overrightarrow{\ker \varphi} = \sigma$ , and we have  $\sigma \subseteq \overrightarrow{\ker \varphi}$ . Moreover, let  $(a, b) \in \sigma$  and  $(b, c) \in \sigma$ . Then  $\varphi(a) \preceq \varphi(b) \preceq \varphi(c)$ . Hence  $\varphi(a) \preceq \varphi(c)$ , i.e.,  $(a, c) \in \overrightarrow{\ker \varphi} = \sigma$ . Also, if  $(a, b) \in \sigma$ , then  $\varphi(a) \preceq \varphi(b)$ . Since  $(T, \diamond, \preceq)$  is an ordered semihypergroup, for any  $c \in S$  we have  $\varphi(a) \diamond \varphi(c) \preceq \varphi(b) \diamond \varphi(c)$ . Since  $\varphi$  is a homomorphism from  $S$  to  $T$ , we have

$$\bigcup_{x \in a \circ c} \varphi(x) = \varphi(a) \diamond \varphi(c) \preceq \varphi(b) \diamond \varphi(c) = \bigcup_{y \in b \circ c} \varphi(y).$$

Thus, for every  $x \in a \circ c$  there exists  $y \in b \circ c$  such that  $\varphi(x) \preceq \varphi(y)$ , and we have  $(x, y) \in \overrightarrow{\ker \varphi} = \sigma$ , which implies that  $a \circ c \xrightarrow{\sigma} b \circ c$ . In the same way, it can be shown that  $c \circ a \xrightarrow{\sigma} c \circ b$ . Furthermore, let  $(a, b) \in \sigma$ ,  $(b, a) \in \sigma$  and  $c \in S$ . Then  $\varphi(a) \preceq \varphi(b)$  and  $\varphi(b) \preceq \varphi(a)$ . Thus  $\varphi(a) = \varphi(b)$ , which implies that  $\varphi(a) \diamond \varphi(c) = \varphi(b) \diamond \varphi(c)$ , i.e.,  $\bigcup_{x \in a \circ c} \varphi(x) = \bigcup_{y \in b \circ c} \varphi(y)$ . Hence, for any  $x \in a \circ c$ , there exists  $y \in b \circ c$  such that  $\varphi(x) = \varphi(y)$ , and we have  $\varphi(x) \preceq \varphi(y)$  and  $\varphi(y) \preceq \varphi(x)$ . It thus follows that  $x \sigma y$  and  $y \sigma x$ . On the other hand, for any  $y' \in b \circ c$ , there exists  $x' \in a \circ c$  such that  $\varphi(x') = \varphi(y')$ . Hence we have  $\varphi(x') \preceq \varphi(y')$  and  $\varphi(y') \preceq \varphi(x')$ , which imply that  $x' \sigma y'$  and  $y' \sigma x'$ . Thus,  $a \circ c \tilde{\sigma} b \circ c$ , exactly as required. Similarly, it can be obtained that  $c \circ a \tilde{\sigma} c \circ b$ .  $\square$

In the following, we give a characterization of ordered regular equivalence relations in terms of weak pseudoorders.

**Theorem 4.2.** *Let  $(S, \circ, \leq)$  be an ordered semihypergroup and  $\rho$  an equivalence relation on  $S$ . Then the following statements are equivalent:*

- (1)  $\rho$  is an ordered regular equivalence relation on  $S$ .
- (2) There exists a weak pseudoorder  $\sigma$  on  $S$  such that  $\rho = \sigma \cap \sigma^{-1}$ .
- (3) There exist an ordered semihypergroup  $T$  and a homomorphism  $\varphi : S \rightarrow T$  such that  $\rho = \ker(\varphi)$ , where  $\ker \varphi = \{(a, b) \in S \times S \mid \varphi(a) = \varphi(b)\}$  is the kernel of  $\varphi$ .

*Proof.* (1)  $\Rightarrow$  (2). Let  $\rho$  be an ordered regular equivalence relation on  $S$ . Then there exists an order relation " $\preceq$ " on the quotient semihypergroup  $(S/\rho, \otimes)$  such that  $(S/\rho, \otimes, \preceq)$  is an ordered semihypergroup, and  $\varphi : S \rightarrow S/\rho$  is a homomorphism. Let  $\sigma = \overrightarrow{\ker \varphi}$ . By Lemma 4.1,  $\sigma$  is a weak pseudoorder on  $S$  and it is easy to check that  $\rho = \sigma \cap \sigma^{-1}$ .

(2)  $\Rightarrow$  (3). For a weak pseudoorder  $\sigma$  on  $S$ , by Lemma 4.1, there exist an ordered semihypergroup  $T$  and a homomorphism  $\varphi : S \rightarrow T$  such that  $\sigma = \overrightarrow{\ker \varphi}$ . Then we have

$$\ker \varphi = \overrightarrow{\ker \varphi} \cap (\overrightarrow{\ker \varphi})^{-1} = \sigma \cap \sigma^{-1} = \rho.$$

(3)  $\Rightarrow$  (1). By hypothesis and Lemma 4.1,  $\overrightarrow{\ker \varphi}$  is a weak pseudoorder on  $S$ . Then, by Theorem 3.8,  $\rho = \overrightarrow{\ker \varphi} \cap (\overrightarrow{\ker \varphi})^{-1}$  is a regular equivalence relation on  $S$ . Thus, by the proof of Lemma 4.1,  $\rho$  is an ordered regular equivalence relation on  $S$ .  $\square$

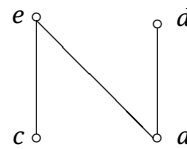
**Example 4.3.** We consider a set  $S := \{a, b, c, d, e\}$  with the following hyperoperation “ $\circ$ ” and the order “ $\leq_S$ ”:

| $\circ$ | $a$     | $b$        | $c$     | $d$     | $e$     |
|---------|---------|------------|---------|---------|---------|
| $a$     | $\{a\}$ | $\{e\}$    | $\{c\}$ | $\{d\}$ | $\{e\}$ |
| $b$     | $\{a\}$ | $\{b, e\}$ | $\{c\}$ | $\{d\}$ | $\{e\}$ |
| $c$     | $\{a\}$ | $\{e\}$    | $\{c\}$ | $\{d\}$ | $\{e\}$ |
| $d$     | $\{a\}$ | $\{e\}$    | $\{c\}$ | $\{d\}$ | $\{e\}$ |
| $e$     | $\{a\}$ | $\{e\}$    | $\{c\}$ | $\{d\}$ | $\{e\}$ |

$$\leq_S := \{(a, a), (a, d), (a, e), (b, b), (c, c), (c, e), (d, d), (e, e)\}.$$

We give the covering relation “ $\prec_S$ ” and the figure of  $S$  as follows:

$$\prec_S = \{(a, d), (a, e), (c, e)\}.$$



Then  $(S, \circ, \leq_S)$  is an ordered semihypergroup (see [28]). Let  $\rho$  be an equivalence relation on  $S$  defined as follows:

$$\rho = \{(a, a), (b, b), (c, c), (c, d), (d, c), (d, d), (e, e)\}.$$

Then  $S/\rho = \{\{a\}, \{b\}, \{c, d\}, \{e\}\}$ , and  $\rho$  is regular. Moreover, we have

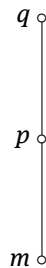
(1)  $\rho$  is an ordered regular equivalence relation on  $S$ . In fact, let  $S/\rho = \{m, n, p, q\}$ , where  $m = \{a\}$ ,  $n = \{b\}$ ,  $p = \{c, d\}$ ,  $q = \{e\}$ . The hyperoperation “ $\otimes$ ” and the order “ $\preceq$ ” on  $S/\rho$  are the follows:

| $\otimes$ | $m$     | $n$        | $p$     | $q$     |
|-----------|---------|------------|---------|---------|
| $m$       | $\{m\}$ | $\{q\}$    | $\{p\}$ | $\{q\}$ |
| $n$       | $\{m\}$ | $\{n, q\}$ | $\{p\}$ | $\{q\}$ |
| $p$       | $\{m\}$ | $\{q\}$    | $\{p\}$ | $\{q\}$ |
| $q$       | $\{m\}$ | $\{q\}$    | $\{p\}$ | $\{q\}$ |

$$\preceq := \{(m, m), (n, n), (p, p), (q, q), (m, p), (m, q), (p, q)\}.$$

We give the covering relation “ $\prec$ ” and the figure of  $S/\rho_2$  as follows:

$$\prec = \{(m, p), (p, q)\}.$$



Then  $(S/\rho, \otimes, \preceq)$  is an ordered semihypergroup and the mapping  $\psi : S \rightarrow S/\rho, x \mapsto (x)_\rho$  is isotone. Hence  $\rho$  is an ordered regular equivalence relation on  $S$ .

(2) Let  $\sigma$  be a relation on  $S$  defined as follows:

$$\sigma = \{(a, a), (a, c), (a, d), (a, e), (b, b), (c, c), (c, d), (c, e), (d, c), (d, d), (e, e)\}.$$

With a small amount of effort one can verify that  $\sigma$  is a weak pseudoorder on  $S$ , and  $\rho = \sigma \cap \sigma^{-1}$ .

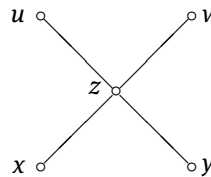
(3) Let  $T := \{x, y, z, u, v\}$  with the operation “ $\bullet$ ” and the order relation “ $\leq_T$ ” below:

| $\circ$ | $x$     | $y$     | $z$     | $u$     | $v$        |
|---------|---------|---------|---------|---------|------------|
| $x$     | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{u\}$ | $\{u\}$    |
| $y$     | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{u\}$ | $\{u\}$    |
| $z$     | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{u\}$ | $\{u\}$    |
| $u$     | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{u\}$ | $\{u\}$    |
| $v$     | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{u\}$ | $\{u, v\}$ |

$$\leq_T := \{(x, x), (x, z), (x, u), (x, v), (y, y), (y, z), (y, u), (y, v), (z, z), (z, u), (z, v), (u, u), (v, v)\}.$$

We give the covering relation “ $\prec_T$ ” and the figure of  $S$  as follows:

$$\prec_T = \{(x, z), (y, z), (z, u), (z, v)\}.$$



It is easy to check that  $(T, \bullet, \leq_T)$  is also an ordered semihypergroup. We define a mapping  $\varphi$  from  $S$  to  $T$  such that  $\varphi(a) = x, \varphi(b) = v, \varphi(c) = \varphi(d) = z, \varphi(e) = u$ . It is a routine matter to verify that  $\varphi : S \rightarrow T$  is a homomorphism and  $\rho = \ker(\varphi)$ .

**Remark 4.4.** (1) For an ordered regular equivalence relation  $\rho$  on  $S$ , since the order “ $\preceq$ ” such that  $(S/\rho, \otimes, \preceq)$  is an ordered semihypergroup is not unique in general, we have the weak pseudoorder  $\sigma$  containing  $\rho$  such that  $\rho = \sigma \cap \sigma^{-1}$  is not unique.

(2) If  $\sigma$  is a weak pseudoorder on an ordered semihypergroup  $S$ , then  $\rho = \sigma \cap \sigma^{-1}$  is the greatest ordered regular equivalence relation on  $S$  contained in  $\sigma$ . In fact, if  $\rho_1$  is an ordered regular equivalence relation on  $S$  contained in  $\sigma$ , then  $\rho_1 = \rho_1 \cap \rho_1^{-1} \subseteq \sigma \cap \sigma^{-1} = \rho$ .

Let  $\sigma$  be a weak pseudoorder on an ordered semihypergroup  $(S, \circ, \leq)$ . Then, by Theorem 4.2,  $\rho = \sigma \cap \sigma^{-1}$  is an ordered regular equivalence relation on  $S$ . We denote by  $\rho^\#$  the mapping from  $S$  onto  $S/\rho$ , i.e.,  $\rho^\# : S \rightarrow S/\rho \mid x \mapsto (x)_\rho$ , which is a homomorphism. In the following, we give out a homomorphism theorem of ordered semihypergroups by weak pseudoorders, which is a generalization of Theorem 1 in [17].

**Theorem 4.5.** Let  $(S, \circ, \leq)$  and  $(T, \diamond, \preceq)$  be two ordered semihypergroups,  $\varphi : S \rightarrow T$  a homomorphism. Then, if  $\sigma$  is a weak pseudoorder on  $S$  such that  $\sigma \subseteq \overrightarrow{\ker \varphi}$ , then there exists the unique homomorphism  $f : S/\rho \rightarrow T \mid (a)_\rho \mapsto \varphi(a)$  such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ \rho^\# \downarrow & \nearrow f & \\ S/\rho & & \end{array}$$

commutes, where  $\rho = \sigma \cap \sigma^{-1}$ . Moreover,  $\text{Im}(\varphi) = \text{Im}(f)$ . Conversely, if  $\sigma$  is a weak pseudoorder on  $S$  for which there exists a homomorphism  $f : (S/\rho, \otimes, \preceq_\sigma) \rightarrow (T, \diamond, \preceq)$  ( $\rho = \sigma \cap \sigma^{-1}$ ) such that the above diagram commutes, then  $\sigma \subseteq \overrightarrow{\ker \varphi}$ .

*Proof.* Let  $\sigma$  be a weak pseudoorder on  $S$  such that  $\sigma \subseteq \overrightarrow{\ker\varphi}$ ,  $f : S/\rho \rightarrow T \mid (a)_\rho \mapsto \varphi(a)$ . Then

(1)  $f$  is well defined. Indeed, if  $(a)_\rho = (b)_\rho$ , then  $(a, b) \in \rho \subseteq \sigma$ . Since  $\sigma \subseteq \overrightarrow{\ker\varphi}$ , we have  $(\varphi(a), \varphi(b)) \in \preceq$ . Furthermore, since  $(b, a) \in \sigma \subseteq \overrightarrow{\ker\varphi}$ , we have  $(\varphi(b), \varphi(a)) \in \preceq$ . Therefore,  $\varphi(a) = \varphi(b)$ .

(2)  $f$  is a homomorphism and  $\varphi = f \circ \rho^\sharp$ . In fact, by Lemma 4.1, there exists an order relation “ $\preceq_\sigma$ ” on the quotient semihypergroup  $(S/\rho, \otimes)$  such that  $(S/\rho, \otimes, \preceq_\sigma)$  is an ordered semihypergroup and the mapping  $\rho^\sharp$  is a homomorphism. Moreover, we have

$$\begin{aligned} (a)_\rho \preceq_\sigma (b)_\rho &\Rightarrow (a, b) \in \sigma \subseteq \overrightarrow{\ker\varphi} \\ &\Rightarrow \varphi(a) \preceq \varphi(b) \\ &\Rightarrow f((a)_\rho) \preceq f((b)_\rho). \end{aligned}$$

Also, let  $(a)_\rho, (b)_\rho \in S/\rho$ . Since  $\varphi$  is a homomorphism from  $S$  to  $T$ , we have

$$f((a)_\rho) \diamond f((b)_\rho) = \varphi(a) \diamond \varphi(b) = \bigcup_{c \in a \circ b} \varphi(c) = \bigcup_{(c)_\rho \in (a)_\rho \otimes (b)_\rho} f((c)_\rho).$$

Furthermore, for any  $a \in S$ ,  $(f \circ \rho^\sharp)(a) = f((a)_\rho) = \varphi(a)$ , and thus  $\varphi = f \circ \rho^\sharp$ .

We claim that  $f$  is the unique homomorphism from  $S/\rho$  to  $T$ . To prove our claim, let  $g$  is a homomorphism from  $S/\rho$  to  $T$  such that  $\varphi = g \circ \rho^\sharp$ . Then

$$f((a)_\rho) = \varphi(a) = (g \circ \rho^\sharp)(a) = g((a)_\rho).$$

Moreover,  $\text{Im}(f) = \{f((a)_\rho) \mid a \in S\} = \{\varphi(a) \mid a \in S\} = \text{Im}(\varphi)$ .

Conversely, let  $\sigma$  be a weak pseudoorder on  $S$ ,  $f : S/\rho \rightarrow T$  is a homomorphism and  $\varphi = f \circ \rho^\sharp$ . Then  $\sigma \subseteq \overrightarrow{\ker\varphi}$ . Indeed, by hypothesis, we have

$$\begin{aligned} (a, b) \in \sigma &\Leftrightarrow (a)_\rho \preceq_\sigma (b)_\rho \Rightarrow f((a)_\rho) \preceq f((b)_\rho) \\ &\Rightarrow (f \circ \rho^\sharp)(a) \preceq (f \circ \rho^\sharp)(b) \\ &\Rightarrow \varphi(a) \preceq \varphi(b) \Rightarrow (a, b) \in \overrightarrow{\ker\varphi}, \end{aligned}$$

where the order  $\preceq_\sigma$  on  $S/\rho$  is defined as in the proof of Lemma 4.1, that is

$$\preceq_\sigma := \{((x)_\rho, (y)_\rho) \in S/\rho \times S/\rho \mid (x, y) \in \sigma\}.$$

□

**Corollary 4.6.** Let  $(S, \circ, \leq)$  and  $(T, \diamond, \preceq)$  be two ordered semihypergroups and  $\varphi : S \rightarrow T$  a homomorphism. Then  $S/\ker\varphi \cong \text{Im}(\varphi)$ , where  $\ker\varphi$  is the kernel of  $\varphi$ .

*Proof.* Let  $\sigma = \overrightarrow{\ker\varphi}$  and  $\rho = \overrightarrow{\ker\varphi} \cap (\overrightarrow{\ker\varphi})^{-1}$ . Then, by Theorems 4.2 and 4.5,  $\rho$  is an ordered regular equivalence relation on  $S$  and  $f : S/\rho \rightarrow T \mid (a)_\rho \mapsto \varphi(a)$  is a homomorphism. Moreover,  $f$  is inverse isotone. In fact, let  $(a)_\rho, (b)_\rho$  be two elements of  $S/\rho$  such that  $f((a)_\rho) \preceq f((b)_\rho)$ . Then  $\varphi(a) \preceq \varphi(b)$ , and we have  $(a, b) \in \overrightarrow{\ker\varphi}$ . Thus, by Lemma 4.1,  $((a)_\rho, (b)_\rho) \in \preceq_\sigma$ , i.e.,  $(a)_\rho \preceq_\sigma (b)_\rho$ . Clearly,  $\rho = \ker\varphi$ . By Remark 2.4,  $S/\ker\varphi \cong \text{Im}(f)$ . Also, by Theorem 4.5,  $\text{Im}(f) = \text{Im}(\varphi)$ . Therefore,  $S/\ker\varphi \cong \text{Im}(\varphi)$ . □

**Remark 4.7.** Note that if  $S$  and  $T$  are both ordered semigroups, then Corollary 4.6 coincides with Corollary in [17].

Let  $(S, \circ, \leq)$  be an ordered semihypergroup,  $\rho, \theta$  be weak pseudoorders on  $S$  such that  $\rho \subseteq \theta$ . We define a relation  $\theta/\rho$  on  $(S/\rho^*, \otimes, \preceq_\rho)$  as follows:

$$\theta/\rho := \{((a)_\rho, (b)_\rho) \in S/\rho^* \times S/\rho^* \mid (a, b) \in \theta\},$$

where  $\preceq_\rho := \{((a)_\rho, (b)_\rho) \mid (a, b) \in \rho\}$ ,  $\rho^* = \rho \cap \rho^{-1}$ . By Corollary 3.12,  $\theta/\rho$  is a weak pseudoorder on  $S/\rho^*$ . Moreover, we have the following theorem.

**Theorem 4.8.** Let  $(S, \circ, \leq)$  be an ordered semihypergroup,  $\rho, \theta$  be weak pseudoorders on  $S$  such that  $\rho \subseteq \theta$ . Then  $(S/\rho^*)/(\theta/\rho)^* \cong S/\theta^*$ .

*Proof.* We claim that the mapping  $\varphi : S/\rho^* \rightarrow S/\theta^* \mid (a)_{\rho^*} \mapsto (a)_{\theta^*}$  is a homomorphism. In fact:

(1)  $\varphi$  is well-defined. Indeed, let  $(a)_{\rho^*} = (b)_{\rho^*}$ . Then  $(a, b) \in \rho^*$ . Thus, by the definition of  $\rho^*$ ,  $(a, b) \in \rho \subseteq \theta$  and  $(b, a) \in \rho \subseteq \theta$ . This implies that  $(a, b) \in \theta^*$ , and thus  $(a)_{\theta^*} = (b)_{\theta^*}$ .

(2)  $\varphi$  is a homomorphism. In fact, let  $(a)_{\rho^*}, (b)_{\rho^*} \in S/\rho^*$ . Then, since  $\rho^*, \theta^*$  are both ordered regular equivalence relations on  $S$ , we have

$$(a)_{\rho^*} \otimes_{\rho} (b)_{\rho^*} = \bigcup_{x \in a \circ b} (x)_{\rho^*}, (a)_{\theta^*} \otimes_{\theta} (b)_{\theta^*} = \bigcup_{x \in a \circ b} (x)_{\theta^*},$$

where “ $\otimes_{\rho}$ ” and “ $\otimes_{\theta}$ ” are the hyperactions on  $S/\rho^*$  and  $S/\theta^*$ , respectively. Thus

$$\varphi((a)_{\rho^*} \otimes_{\rho} (b)_{\rho^*}) = (a)_{\theta^*} \otimes_{\theta} (b)_{\theta^*} = \bigcup_{x \in a \circ b} (x)_{\theta^*} = \bigcup_{(x)_{\rho^*} \in (a)_{\rho^*} \otimes_{\rho} (b)_{\rho^*}} \varphi((x)_{\rho^*}).$$

Also, if  $(a)_{\rho^*} \preceq_{\rho} (b)_{\rho^*}$ , then  $(a, b) \in \rho \subseteq \theta$ . It implies that  $(a)_{\theta^*} \preceq_{\theta} (b)_{\theta^*}$ , and thus  $\varphi$  is isotone.

On the other hand, it is easy to see that  $\varphi$  is onto, since

$$\text{Im}(\varphi) = \{\varphi((a)_{\rho^*}) \mid a \in S\} = \{(a)_{\theta^*} \mid a \in S\} = S/\theta^*.$$

It thus follows from Corollary 4.6 that  $(S/\rho^*)/\text{Ker}\varphi \cong \text{Im}(\varphi) = S/\theta^*$ .

Furthermore, let  $\overrightarrow{\text{ker}}\varphi := \{((a)_{\rho^*}, (b)_{\rho^*}) \in S/\rho^* \times S/\rho^* \mid \varphi((a)_{\rho^*}) \preceq_{\theta} \varphi((b)_{\rho^*})\}$ . Then

$$((a)_{\rho^*}, (b)_{\rho^*}) \in \overrightarrow{\text{ker}}\varphi \iff (a)_{\theta^*} \preceq_{\theta} (b)_{\theta^*} \iff (a, b) \in \theta \iff ((a)_{\rho^*}, (b)_{\rho^*}) \in \theta/\rho.$$

Therefore,  $\text{Ker}\varphi = \overrightarrow{\text{ker}}\varphi \cap (\overrightarrow{\text{ker}}\varphi)^{-1} = (\theta/\rho) \cap (\theta/\rho)^{-1} = (\theta/\rho)^*$ . We have thus shown that  $(S/\rho^*)/(\theta/\rho)^* \cong S/\theta^*$ .  $\square$

**Definition 4.9.** Let  $(S, \circ, \leq)$  and  $(T, \diamond, \preceq)$  be two ordered semihypergroups,  $\rho, \theta$  be two weak pseudoorders on  $S$  and  $T$ , respectively, and the mapping  $f : S \rightarrow T$  a homomorphism. Then,  $f$  is called a  $(\rho, \theta)$ -homomorphism if  $(x, y) \in \rho$  implies  $(f(x), f(y)) \in \theta$ , for all  $x, y \in S$ .

**Example 4.10.** Consider the ordered semihypergroups  $(S, \circ, \leq_S)$  and  $(T, \bullet, \leq_T)$  given in Example 4.3, and define the relations  $\rho, \theta$  on  $S$  and  $T$ , respectively, as follows:

$$\begin{aligned} \rho &:= \{(a, a), (a, c), (a, d), (a, e), (b, b), (c, c), (c, d), (c, e), (d, c), (d, d), (e, e)\}, \\ \theta &:= \{(x, x), (x, z), (x, u), (x, v), (y, y), (y, z), (y, u), (y, v), (z, z), (z, u), (z, v), (u, u), (v, v)\}. \end{aligned}$$

It is not difficult to verify that  $\rho, \theta$  is pseudoorders on  $S$  and  $T$ , respectively. Furthermore, we define a mapping  $f$  from  $S$  to  $T$  such that  $f(a) = x, f(b) = v, f(c) = f(d) = z, f(e) = u$ . By Example 4.3,  $f : S \rightarrow T$  is a homomorphism. Hence  $f$  is a  $(\rho, \theta)$ -homomorphism.

**Lemma 4.11.** Let  $(S, \circ, \leq)$  and  $(T, \diamond, \preceq)$  be two ordered semihypergroups,  $\rho, \theta$  be two weak pseudoorders on  $S$  and  $T$ , respectively, and the mapping  $f : S \rightarrow T$  a  $(\rho, \theta)$ -homomorphism. Then, the mapping  $\bar{f} : (S/\rho^*, \otimes_{\rho}, \preceq_{\rho}) \rightarrow (T/\theta^*, \otimes_{\theta}, \preceq_{\theta})$  defined by

$$(\forall x \in S) \bar{f}((x)_{\rho^*}) := (f(x))_{\theta^*}$$

is a homomorphism of ordered semihypergroups, where the orders  $\preceq_{\rho}, \preceq_{\theta}$  on  $S/\rho^*$  and  $T/\theta^*$ , respectively, are both defined as in the proof of Lemma 4.1.

*Proof.* Let  $f : S \rightarrow T$  be a  $(\rho, \theta)$ -homomorphism and  $\bar{f} : S/\rho^* \rightarrow T/\theta^* \mid (x)_{\rho^*} \mapsto (f(x))_{\theta^*}$ . Then

(1)  $\bar{f}$  is well defined. In fact, let  $(x)_{\rho^*}, (y)_{\rho^*} \in S/\rho^*$  be such that  $(x)_{\rho^*} = (y)_{\rho^*}$ . Then  $(x, y) \in \rho^* \subseteq \rho$ . Since  $f$  is a  $(\rho, \theta)$ -homomorphism, we have  $(f(x), f(y)) \in \theta$ . It implies that  $((f(x))_{\theta^*}, (f(y))_{\theta^*}) \in \preceq_{\theta}$ . Similarly, since  $(y, x) \in \rho$ , we have  $((f(y))_{\theta^*}, (f(x))_{\theta^*}) \in \preceq_{\theta}$ . Therefore,  $((f(x))_{\theta^*} = (f(y))_{\theta^*}$ , i.e.,  $\bar{f}((x)_{\rho^*}) = \bar{f}((y)_{\rho^*})$ .

(2)  $\bar{f}$  is a homomorphism. Indeed, let  $(x)_{\rho^*}, (y)_{\rho^*} \in S/\rho^*$ . By Theorem 4.2,  $\rho^*, \theta^*$  are ordered regular equivalence relations on  $S$  and  $T$ , respectively. Since  $f$  is a homomorphism, by Lemma 2.2 we have

$$\bar{f}((x)_{\rho^*}) \otimes_{\theta} \bar{f}((y)_{\rho^*}) = (f(x))_{\theta^*} \otimes_{\theta} (f(y))_{\theta^*} = \bigcup_{a \in x \circ y} (f(a))_{\theta^*} = \bigcup_{(a)_{\rho^*} \in (x)_{\rho^*} \otimes_{\rho} (y)_{\rho^*}} \bar{f}((a)_{\rho^*}).$$

Also, since  $f$  is a  $(\rho, \theta)$ -homomorphism, we have

$$\begin{aligned} (x)_{\rho^*} \preceq_{\rho} (y)_{\rho^*} &\Rightarrow (x, y) \in \rho \Rightarrow (f(x), f(y)) \in \theta \\ &\Rightarrow (f(x))_{\theta^*} \preceq_{\theta} (f(y))_{\theta^*} \Rightarrow \bar{f}((x)_{\rho^*}) \preceq_{\theta} \bar{f}((y)_{\rho^*}). \end{aligned}$$

Hence  $\bar{f}$  is isotone. We have thus shown that  $\bar{f}$  is a homomorphism, as required.  $\square$

**Lemma 4.12.** Let  $(S, \circ, \preceq)$  and  $(T, \diamond, \preceq)$  be two ordered semihypergroups,  $\rho, \theta$  be two weak pseudoorders on  $S$  and  $T$ , respectively, and the mapping  $f : S \rightarrow T$  a  $(\rho, \theta)$ -homomorphism. Then, We define a relation on  $S/\rho^*$  denoted by  $\rho_f$  as follows:

$$\rho_f := \{((x)_{\rho^*}, (y)_{\rho^*}) \in S/\rho^* \times S/\rho^* \mid (f(x))_{\theta^*} \preceq_{\theta} (f(y))_{\theta^*}\}.$$

Then  $\rho_f$  is a weak pseudoorder on  $S/\rho^*$ .

*Proof.* Assume that  $((x)_{\rho^*}, (y)_{\rho^*}) \in \preceq_{\rho}$ . By Lemma 4.11,  $\bar{f}$  is a homomorphism. Then  $\bar{f}((x)_{\rho^*}) \preceq_{\theta} \bar{f}((y)_{\rho^*})$ , i.e.,  $(f(x))_{\theta^*} \preceq_{\theta} (f(y))_{\theta^*}$ . It implies that  $((x)_{\rho^*}, (y)_{\rho^*}) \in \rho_f$ , and thus  $\preceq_{\rho} \subseteq \rho_f$ . Now, let  $((x)_{\rho^*}, (y)_{\rho^*}) \in \rho_f$  and  $((y)_{\rho^*}, (z)_{\rho^*}) \in \rho_f$ . Then  $(f(x))_{\theta^*} \preceq_{\theta} (f(y))_{\theta^*}$  and  $(f(y))_{\theta^*} \preceq_{\theta} (f(z))_{\theta^*}$ . Thus, by the transitivity of  $\preceq_{\theta}$ ,  $(f(x))_{\theta^*} \preceq_{\theta} (f(z))_{\theta^*}$ . This implies that  $((x)_{\rho^*}, (z)_{\rho^*}) \in \rho_f$ . Also, let  $((x)_{\rho^*}, (y)_{\rho^*}) \in \rho_f$  and  $(z)_{\rho^*} \in S/\rho^*$ . Then  $(f(x))_{\theta^*} \preceq_{\theta} (f(y))_{\theta^*}$ . Since  $(T/\theta^*, \otimes_{\theta}, \preceq_{\theta})$  is an ordered semihypergroup, it can be obtained that  $(f(x))_{\theta^*} \otimes_{\theta} (f(z))_{\theta^*} \preceq_{\theta} (f(y))_{\theta^*} \otimes_{\theta} (f(z))_{\theta^*}$ , that is,  $\bar{f}((x)_{\rho^*}) \otimes_{\theta} \bar{f}((z)_{\rho^*}) \preceq_{\theta} \bar{f}((y)_{\rho^*}) \otimes_{\theta} \bar{f}((z)_{\rho^*})$ . Then, since  $\bar{f}$  is a homomorphism, we have

$$\bigcup_{(a)_{\rho^*} \in (x)_{\rho^*} \otimes_{\rho} (z)_{\rho^*}} \bar{f}((a)_{\rho^*}) \preceq_{\theta} \bigcup_{(b)_{\rho^*} \in (y)_{\rho^*} \otimes_{\rho} (z)_{\rho^*}} \bar{f}((b)_{\rho^*}),$$

which means that  $\bigcup_{a \in x \circ z} (f(a))_{\theta^*} \preceq_{\theta} \bigcup_{b \in y \circ z} (f(b))_{\theta^*}$ . Thus, for any  $a \in x \circ z$ , there exists  $b \in y \circ z$  such that

$(f(a))_{\theta^*} \preceq_{\theta} (f(b))_{\theta^*}$ . Hence  $((a)_{\rho^*}, (b)_{\rho^*}) \in \rho_f$ . It thus implies that  $(x)_{\rho^*} \otimes_{\rho} (z)_{\rho^*} \xrightarrow{\rho_f} (y)_{\rho^*} \otimes_{\rho} (z)_{\rho^*}$ . Similarly, it can be shown that  $(z)_{\rho^*} \otimes_{\rho} (x)_{\rho^*} \xrightarrow{\rho_f} (z)_{\rho^*} \otimes_{\rho} (y)_{\rho^*}$ . Furthermore, let  $(x)_{\rho^*}, (y)_{\rho^*}, (z)_{\rho^*} \in S/\rho^*$  be such that  $((x)_{\rho^*}, (y)_{\rho^*}) \in \rho_f$  and  $((y)_{\rho^*}, (x)_{\rho^*}) \in \rho_f$ . Then, similarly as in the above proof, it can be verified that  $(x)_{\rho^*} \otimes_{\rho} (z)_{\rho^*} \xrightarrow{\rho_f} (y)_{\rho^*} \otimes_{\rho} (z)_{\rho^*}$  and  $(z)_{\rho^*} \otimes_{\rho} (x)_{\rho^*} \xrightarrow{\rho_f} (z)_{\rho^*} \otimes_{\rho} (y)_{\rho^*}$ . Therefore,  $\rho_f$  is a weak pseudoorder on  $S/\rho^*$ .  $\square$

By Lemmas 4.11 and 4.12, we immediately obtain the following corollary:

**Corollary 4.13.** Let  $(S, \circ, \preceq)$  and  $(T, \diamond, \preceq)$  be two ordered semihypergroups,  $\rho, \theta$  be two weak pseudoorders on  $S$  and  $T$ , respectively, and the mapping  $f : S \rightarrow T$  be a  $(\rho, \theta)$ -homomorphism. Then, the following diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ (\rho^*)^{\#} \downarrow & & \downarrow (\theta^*)^{\#} \\ S/\rho^* & \xrightarrow{\bar{f}} & T/\theta^* \end{array}$$

commutes.

**Theorem 4.14.** Let  $(S, \circ, \preceq)$  and  $(T, \diamond, \preceq)$  be two ordered semihypergroups,  $\rho, \theta$  be two weak pseudoorders on  $S$  and  $T$ , respectively, and the mapping  $f : S \rightarrow T$  a  $(\rho, \theta)$ -homomorphism. If  $\sigma$  is a weak pseudoorder on  $S/\rho^*$

such that  $\sigma \subseteq \rho_f$ , then there exists the unique homomorphism  $\varphi : (S/\rho^*)/\sigma^* \rightarrow T/\theta^* \mid ((a)_{\rho^*})_{\sigma^*} \mapsto f((a)_{\rho^*})$  such that the diagram

$$\begin{array}{ccc} S/\rho^* & \xrightarrow{\bar{f}} & T/\theta^* \\ (\sigma^*)^\# \downarrow & \nearrow \varphi & \\ (S/\rho^*)/\sigma^* & & \end{array}$$

commutes.

Conversely, if  $\sigma$  is a weak pseudoorder on  $S/\rho^*$  for which there exists a homomorphism  $\varphi : (S/\rho^*)/\sigma^* \rightarrow T/\theta^*$  such that the above diagram commutes, then  $\sigma \subseteq \rho_f$ .

*Proof.* The proof is straightforward by Lemma 4.11, Lemma 4.12 and Theorem 4.5, and we omit the details.  $\square$

Let  $(S, \circ, \leq_S)$  and  $(T, \diamond, \leq_T)$  be two ordered semihypergroups. In [25], it is shown that  $(S \times T, *, \leq_{S \times T})$  is also an ordered semihypergroup with the hyperoperation “ $*$ ” and the order relation “ $\leq_{S \times T}$ ” below: for any  $(s_1, t_1), (s_2, t_2) \in S \times T$ ,

$$(s_1, t_1) * (s_2, t_2) = (s_1 \circ s_2) \times (t_1 \diamond t_2);$$

$$(s_1, t_1) \leq_{S \times T} (s_2, t_2) \Leftrightarrow s_1 \leq_S s_2 \text{ and } t_1 \leq_T t_2.$$

**Lemma 4.15.** Let  $(S, \circ, \leq_S)$  and  $(T, \diamond, \leq_T)$  be ordered semihypergroups,  $\rho_1, \rho_2$  be two weak pseudoorders on  $S$  and  $T$ , respectively. We define a relation  $\rho$  on  $S \times T$  as follows:

$$(\forall (s_1, t_1), (s_2, t_2) \in S \times T) (s_1, t_1) \rho (s_2, t_2) \Leftrightarrow s_1 \rho_1 s_2 \text{ and } t_1 \rho_2 t_2.$$

Then  $\rho$  is a weak pseudoorder on  $S \times T$ .

*Proof.* The verifications of the conditions (1), (2) and (3) of Definition 3.4 are straightforward. We only need show that the condition (4) of Definition 3.4 is satisfied.

Let  $(s_1, t_1), (s_2, t_2), (s, t) \in S \times T$  be such that  $(s_1, t_1) \rho (s_2, t_2)$  and  $(s_2, t_2) \rho (s, t)$ . Then  $s_1 \rho_1 s_2, t_1 \rho_2 t_2, s_2 \rho_1 s_1, t_2 \rho_2 t_1$  and  $s \in S, t \in T$ . Since  $\rho_1, \rho_2$  are weak pseudoorders on  $S$  and  $T$ , respectively, we have

$$s_1 \circ s \tilde{\rho}_1 s_2 \circ s \text{ and } t_1 \circ t \tilde{\rho}_2 t_2 \circ t.$$

Thus

$$\begin{aligned} &(\forall x \in s_1 \circ s) (\exists u \in s_2 \circ s) x \rho_1 u \text{ and } u \rho_1 x \ \& \ (\forall u' \in s_2 \circ s) (\exists x' \in s_1 \circ s) x' \rho_1 u' \text{ and } u' \rho_1 x', \\ &(\forall y \in t_1 \circ t) (\exists v \in t_2 \circ t) y \rho_2 v \text{ and } v \rho_2 y \ \& \ (\forall v' \in t_2 \circ t) (\exists y' \in t_1 \circ t) y' \rho_2 v' \text{ and } v' \rho_2 y'. \end{aligned}$$

Hence we have

$$\begin{aligned} &(\forall (x, y) \in (s_1 \circ s) \times (t_1 \circ t)) (\exists (u, v) \in (s_2 \circ s) \times (t_2 \circ t)) (x, y) \rho (u, v) \text{ and } (u, v) \rho (x, y), \\ &(\forall (u', v') \in (s_2 \circ s) \times (t_2 \circ t)) (\exists (x', y') \in (s_1 \circ s) \times (t_1 \circ t)) (x', y') \rho (u', v') \text{ and } (u', v') \rho (x', y'). \end{aligned}$$

It thus follows that  $(s_1 \circ s) \times (t_1 \circ t) \tilde{\rho} (s_2 \circ s) \times (t_2 \circ t)$ , i.e.,  $(s_1, t_1) * (s, t) \tilde{\rho} (s_2, t_2) * (s, t)$ . Similarly, it can be verified that  $(s, t) * (s_1, t_1) \tilde{\rho} (s, t) * (s_2, t_2)$ .  $\square$

By Theorem 3.8 and Lemma 4.15,  $((S \times T)/\rho^*, \otimes_\rho, \preceq_\rho)$  and  $(S/\rho_1^* \times T/\rho_2^*, *, \preceq_{S \times T})$  are both ordered semihypergroups, where the hyperoperation “ $*$ ” and the order relation  $\preceq_{S \times T}$  on  $S/\rho_1^* \times T/\rho_2^*$  are similar to the hyperoperation “ $*$ ” and the order relation  $\leq_{S \times T}$  on  $S \times T$ , respectively. Furthermore, we have the following theorem:

**Theorem 4.16.** Let  $(S, \circ, \leq_S)$  and  $(T, \diamond, \leq_T)$  be ordered semihypergroups,  $\rho_1, \rho_2$  be two weak pseudoorders on  $S$  and  $T$ , respectively. Then  $(S \times T)/\rho^* \cong S/\rho_1^* \times T/\rho_2^*$ .

*Proof.* We claim that the mapping  $\varphi : (S \times T)/\rho^* \rightarrow S/\rho_1^* \times T/\rho_2^* \mid ((s, t))_{\rho^*} \mapsto ((s)_{\rho_1^*}, (t)_{\rho_2^*})$  is an isomorphism. In fact:

(1)  $\varphi$  is well-defined. Indeed, let  $((s_1, t_1))_{\rho^*} = ((s_2, t_2))_{\rho^*}$ . Then  $(s_1, t_1)\rho^*(s_2, t_2)$ . Hence, by the definition of  $\rho^*$ ,  $(s_1, t_1)\rho(s_2, t_2)$  and  $(s_2, t_2)\rho(s_1, t_1)$ . It thus follows that  $s_1\rho_1 s_2$ ,  $t_1\rho_2 t_2$ ,  $s_2\rho_1 s_1$  and  $t_2\rho_2 t_1$ , which imply  $s_1\rho_1^* s_2$  and  $t_1\rho_2^* t_2$ . Thus,  $((s_1)_{\rho_1^*}, (t_1)_{\rho_2^*}) = ((s_2)_{\rho_1^*}, (t_2)_{\rho_2^*})$ .

(2)  $\varphi$  is an isomorphism. In fact, for any  $((s_1, t_1))_{\rho^*}, ((s_2, t_2))_{\rho^*} \in (S \times T)/\rho^*$ , we have

$$\begin{aligned} \varphi(((s_1, t_1))_{\rho^*}) \star' \varphi(((s_2, t_2))_{\rho^*}) &= ((s_1)_{\rho_1^*}, (t_1)_{\rho_2^*}) \star' ((s_2)_{\rho_1^*}, (t_2)_{\rho_2^*}) \\ &= ((s_1)_{\rho_1^*} \otimes_{\rho_1} (s_2)_{\rho_1^*}, (t_1)_{\rho_2^*} \otimes_{\rho_2} (t_2)_{\rho_2^*}) \\ &= \left( \bigcup_{s \in s_1 \circ s_2} (s)_{\rho_1^*}, \bigcup_{t \in t_1 \diamond t_2} (t)_{\rho_2^*} \right) \\ &= \bigcup_{\substack{s \in s_1 \circ s_2 \\ t \in t_1 \diamond t_2}} ((s)_{\rho_1^*}, (t)_{\rho_2^*}) \\ &= \bigcup_{(s, t) \in (s_1 \circ s_2) \times (t_1 \diamond t_2)} \varphi(((s, t))_{\rho^*}) \\ &= \bigcup_{(s, t) \in (s_1, t_1) \star (s_2, t_2)} \varphi(((s, t))_{\rho^*}) \\ &= \bigcup_{((s, t))_{\rho^*} \in ((s_1, t_1))_{\rho^*} \otimes_{\rho} ((s_2, t_2))_{\rho^*}} \varphi(((s, t))_{\rho^*}). \end{aligned}$$

Moreover, it can be easily seen that  $\varphi$  is onto. Also,  $\varphi$  is isotone and reverse isotone. Indeed, let  $((s_1, t_1))_{\rho^*}, ((s_2, t_2))_{\rho^*} \in (S \times T)/\rho^*$ . Then we have

$$\begin{aligned} ((s_1, t_1))_{\rho^*} \preceq_{\rho} ((s_2, t_2))_{\rho^*} &\iff (s_1, t_1)\rho(s_2, t_2) \iff s_1\rho_1 s_2 \text{ and } t_1\rho_2 t_2 \\ &\iff (s_1)_{\rho_1^*} \preceq_{\rho_1} (s_2)_{\rho_1^*} \text{ and } (t_1)_{\rho_2^*} \preceq_{\rho_2} (t_2)_{\rho_2^*} \\ &\iff ((s_1)_{\rho_1^*}, (t_1)_{\rho_2^*}) \preceq_{S \times T} ((s_2)_{\rho_1^*}, (t_2)_{\rho_2^*}) \\ &\iff \varphi(((s_1, t_1))_{\rho^*}) \preceq_{S \times T} \varphi(((s_2, t_2))_{\rho^*}). \end{aligned}$$

Therefore,  $\varphi$  is indeed an isomorphism, which means that  $(S \times T)/\rho^* \cong S/\rho_1^* \times T/\rho_2^*$ .  $\square$

## 5 Conclusions

As we know, (strongly) regular equivalence relations on ordered semihypergroups play an important role in studying the structure of ordered semihypergroups. In this paper, we further studied the ordered regular equivalence relations on ordered semihypergroups. We introduced the concept of weak pseudoorders on an ordered semihypergroup, and established the relationships between ordered regular equivalence relations and weak pseudoorders on an ordered semihypergroup. Especially, we constructed an ordered regular equivalence relation on an ordered semihypergroup by a weak pseudoorder, and gave a complete answer to the open problem given by Davvaz et al. in [25].

In [25], for a pseudoorder  $\rho$  on an ordered semihypergroup  $S$ , Davvaz et al. obtained an ordered semigroup  $S/\rho^*$ , where  $\rho^*(= \rho \cap \rho^{-1})$  is a strongly regular equivalence relation on  $S$ . Moreover, they provided some homomorphism theorems of ordered semigroups by pseudoorders, for example see Theorems 4.5, 4.11 and 4.15 in [25]. However, they have not established the homomorphisms of ordered semihypergroups. As a further study, in the present paper we also discussed the homomorphic theory of ordered semihypergroups by weak pseudoorders, and generalized some similar results in ordered semigroups.

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