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Generalized state maps and states on pseudo equality algebras

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Abstract: In this paper, we attempt to cope with states in a universal algebraic setting, that is, introduce a notion of generalized state map from a pseudo equality algebra X to an arbitrary pseudo equality algebra Y. We give two types of special generalized state maps, namely, generalized states and generalized internal states. Also, we study two types of states, namely, Bosbach states and Riečan states. Finally, we discuss the relations among generalized state maps, states and internal states (or state operators) on pseudo equality algebras. We verify the results that generalized internal states are the generalization of internal states, and generalized states are the generalization of state-morphisms on pseudo equality algebras. Furthermore, we obtain that generalized states are the generalization of Bosbach states and Riečan states on linearly ordered and involutive pseudo equality algebras, respectively. Hence we can come to the conclusion that, in a sense, generalized state maps can be viewed as a possible united framework of the states and the internal states, the state-morphisms and the internal state-morphisms on pseudo equality algebras.

Keywords: Pseudo equality algebra, Generalized state map, Bosbach state, Riečan state, Internal state

MSC: 03G25, 06F99

1 Introduction

Logical algebras are the corresponding algebraic semantics with all sorts of propositional calculus, which are the algebraic foundations of reasoning mechanism of many fields such as computer sciences, information sciences, cybernetics, artificial intelligence and so on. EQ-algebra is a new class of logical algebra which was proposed by Nov*á*k in [1], which generalizes the residuated lattice. One of the motivations is to introduce a special algebra as the correspondence of truth values for high-order fuzzy type theory (FTT). Another motivation is from the equational style of proof in logic. It has three connectives: meet $\wedge$, product $\otimes$ and fuzzy equality $\sim$. The product in EQ-algebras is quite loose which can be replaced by any other smaller binary operation, but still obtains an EQ-algebra. Based on the above reasons, Jenei [2] introduced equality algebras in 2012 similar to EQ-algebras but without a product, and the author proved the term equivalence of equivalential equality algebras to BCK-meet-semilattice. Then in 2014, Jenei introduced pseudo equality algebras in [3] in order to find a connection with pseudo BCK-algebras. About BCK/pseudo-BCK algebras and their application, one can see [4-7]. Recently, Dvurečenskij found the fact that every pseudo equality algebra in the Jenei's version is an equality algebra and so presents the new revision of pseudo equality algebras in

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[8]. It generalizes equality algebras and seems to be more reasonable as a candidate for a possible algebraic semantics of fuzzy type theory than the Jenei's version.

The notion of states on MV-algebras was introduced by Mundici [9] in 1995 with the intent of capturing the notion of average degree of truth of a proposition in Łukasiewicz logic, and so the states have been used as a semantical interpretation of the probability of fuzzy events a. That is, if s is a state and a is a fuzzy event, then $s(a)$ is presented as the average of the appearance of the event a. Different approaches to the generalization mainly gave rise to two different notions, namely, Bosbach states and Riečan states. Hence it is meaningful to extend the notion of states to other algebraic structures and their noncommutative cases [4,10-14]. For example, Liu Lianzhen studied the existence of Bosbach states and Riečan states on finite monoidal t-norm based algebras (MTL-algebra for short) in [11]. Some examples show that there exist MTL-algebras having no Bosbach states and Riečan states. It is well known that in many cases the evaluation of truth degree of sentences is made in an abstract structure, and not in the standard algebra $[0, 1]$ (see [15]). For this reason it is interesting to define a probability with values in an abstract algebra. In this case, Flaminio and Montagna [16] were the first to present a unified approach to states and probabilistic many-valued logic in a logical and algebraic setting. They added a unary operation, called internal state (or state operator) to the language of MV-algebras which preserves the usual properties of states. Correspondingly, the pair (M, σ) is called a state MV-algebra. From the viewpoint of probability, if a is a fuzzy event, then the internal state $Pr(a)$ is presented as truth value of appearing a. A more powerful type of logic can be given by algebraic structures with internal states, and they also constitute the varieties of universal algebras. Consequently, the internal states have been extended and intensively studied in other algebraic structures [17-19], etc. Recently, the notions of internal states have been applied to algebraic structures of higher order fuzzy logic, for example, equality algebras [20] and pseudo equality algebra [21] where one of the main results is about the relevance with the corresponding state BCK/pseudo-BCK meet-semilatices. Also we observe that there exist some interesting fields of states on pseudo equality algebras, which can be investigated including state-morphisms and Riečan states, etc. Based on the above research results, indeed, it is meaningful using internal states to extend the concepts of states of algebraic structures, instead of the real unit interval $[0, 1]$, to a more universal algebraic setting. This is our motivation to introduce and study generalized state maps and revelent states on pseudo equality algebras in this paper.

This paper is organized as follows: In Section 2, we recalls some basic notions and results which will be used later in the paper. In Section 3, we introduce the notion of generalized state maps (or simply, GS-map) including two special classes, namely, G-states and GI-states on pseudo equality algebras. Moreover, we give some examples and investigate basic properties of them. In Section 4, we mainly study the Bosbach states, Riečan states and state-morphisms on pseudo equality algebras and discuss relations between them. In Section 5, we emphasis on the relevances between generalized state maps, states and internal states on pseudo equality algebras and get some important results.

2 Preliminaries

In this section, we recollect some definitions and results which will be used in the following.

Definition 2.1 ([2]). *An equality algebra is an algebra* $(E; \sim, \wedge, 1)$ *of type* $(2, 2, 0)$ *such that for all* $x, y, z \in X$*:*
(E1) $(E, \wedge, 1)$ *is a meet-semilattice with top element* 1*;*
(E2) $x \sim y = y \sim x$*;*
(E3) $x \sim x = 1$*;*
(E4) $x \sim 1 = x$*;*
(E5) $x \leq y \leq z$ *implies* $x \sim z \leq y \sim z$ *and* $x \sim z \leq x \sim y$*;*
(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$*;*
(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

In any equality algebra $(E; \sim, \wedge, 1)$, defines the operation $\to$ by $x \to y := (x \wedge y) \sim x$ for all $x, y \in E$.

Definition 2.2 ([20]). *Let $(X; \sim, \wedge, 1)$ be an equality algebra. A subset D containing 1 of X is called a deductive system of X if for all $x, y \in X$:*
(1) $x \in D$ and $x \leq y$ imply $y \in D$;
(2) $x \in D$ and $y \sim x \in D$ imply $y \in D$.

Definition 2.3 ([8]). *A pseudo equality algebra is an algebra $(X; \sim, \backsim, \wedge, 1)$ of type $(2, 2, 2, 0)$ such that for all $x, y, z, t \in X$:*
(X1) $(X; \wedge, 1)$ is a meet-semilattice with top element 1;
(X2) $x \sim x = 1 = x \backsim x$;
(X3) $x \sim 1 = x = 1 \backsim x$;
(X4) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$, $x \sim z \leq x \sim y$, $z \backsim x \leq z \backsim y$ and $z \backsim x \leq y \backsim x$;
(X5) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$ and $x \backsim y \leq (x \wedge z) \backsim (y \wedge z)$;
(X6) $x \sim y \leq (z \sim x) \backsim (z \sim y)$ and $x \backsim y \leq (x \backsim z) \sim (y \backsim z)$;
(X7) $x \sim y \leq (x \sim z) \sim (y \sim z)$ and $x \backsim y \leq (z \backsim x) \backsim (z \backsim y)$.

In any pseudo equality algebra $(X; \sim, \backsim, \wedge, 1)$, define two derived binary operations $\to$ and $\rightsquigarrow$ by $x \to y := (x \wedge y) \sim x$ and $x \rightsquigarrow y := x \backsim (x \wedge y)$ for all $x, y \in X$, respectively. Note that when $\sim = \backsim$ a pseudo equality algebra is an equality algebra.

Proposition 2.4 ([8]). *In any pseudo equality algebra $(X; \sim, \backsim, \wedge, 1)$, the following properties hold for all $x, y, z \in X$:*
(1) $z \leq x$ implies $x \to y \leq z \to y$ and $x \rightsquigarrow y \leq z \rightsquigarrow y$;
(2) $x \leq ((y \sim x) \backsim y) \wedge (y \sim (x \backsim y))$;
(3) $x \leq y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$;
(4) $x \rightsquigarrow 1 = x \rightsquigarrow x = x \to x = x \to 1 = 1$, $1 \rightsquigarrow x = x$ and $1 \to x = x$;
(5) $x \leq (y \to x) \wedge (y \rightsquigarrow x)$;
(6) $x \leq ((x \to y) \rightsquigarrow y) \wedge ((x \rightsquigarrow y) \to y)$;
(7) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$ and $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$;
(8) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$;
(9) $x \to y = x \to (x \wedge y)$ and $x \rightsquigarrow y = x \rightsquigarrow (x \wedge y)$;
(10) $x \leq y$ implies $y \to x = x \sim y$ and $y \rightsquigarrow x = y \backsim x$.

Lemma 2.5 ([21]). *Let $(X; \sim, \backsim, \wedge, 1)$ be a pseudo equality algebra. Then the following hold for all $x, y, z \in X$:*
(1) $y \sim (((x \wedge y) \sim x) \backsim y = x \wedge y \sim x$;
(2) $(y \sim (x \backsim x \wedge y)) \backsim y = x \backsim x \wedge y$.

A pseudo equality algebra $(X; \sim, \backsim, \wedge, 1)$ is called bounded if it has bottom element 0. In this case, we define two negations $^-$ and $^\sim$ by $x^- := x \to 0$ and $x^\sim := x \rightsquigarrow 0$ for all $x \in X$. Clearly, $x^- = 0 \sim x$ and $x^\sim = x \backsim 0$.

Proposition 2.6 ([21]). *In any bounded pseudo equality algebra $(X; \sim, \backsim, \wedge, 0, 1)$, the following properties hold for all $x, y, z \in X$:*
(1) $1^- = 0 = 1^\sim$ and $0^- = 1 = 0^\sim$;
(2) $1^{-\sim} = 1 = 1^{\sim-}$ and $0^{-\sim} = 1 = 0^{\sim-}$;
(3) $x \leq x^{-\sim}$ and $x \leq x^{\sim-}$;
(4) $x^{-\sim-} = x^-$ and $x^{\sim-\sim} = x^\sim$;
(5) $x \leq y$ implies $y^- \leq x^-$ and $y^\sim \leq x^\sim$;
(6) $x \to y^\sim = y \rightsquigarrow x^-$ and $x \rightsquigarrow y^- = y \to x^\sim$;
(7) $x^\sim \to y^{-\sim} = y^- \rightsquigarrow x^{\sim-}$;
(8) $x \to y^{-\sim} = y^- \rightsquigarrow x^-$ and $x \rightsquigarrow y^{\sim-} = y^\sim \to x^\sim$;
(9) $x \to y^\sim = y^{\sim-} \rightsquigarrow x^-$ and $x \rightsquigarrow y^- = y^{-\sim} \to x^\sim$.

Definition 2.7 ([8]). *Let $(X;\rightsquigarrow,\leftsquigarrow,\wedge,1)$ be a pseudo equality algebra. A subset D containing 1 of X is called a $(\rightarrow,\rightsquigarrow)$ deductive system of X if for all $x,y\in X$, $x\in D$ and $x\rightarrow y\in D$ imply $y\in D$.*

Note that in any pseudo equality algebra $(X;\rightsquigarrow,\leftsquigarrow,\wedge,1)$, $(\rightsquigarrow,\leftsquigarrow)$ deductive systems and $(\rightarrow,\rightsquigarrow)$ deductive systems are equivalent, so we call them deductive systems.

Remark 2.8. *Assume that $([0,1],\oplus,^{-},0)$ is a standard MV-algebra. Then $([0,1],\rightarrow_R,0,1)$ is a bounded commutative BCK-algebra, where $\rightarrow_R$ is the Łukasiewicz implication defined by $x\rightarrow_R y=x^{-}\oplus y=min\{1,1-x+y\}$. Furthermore, $([0,1],\rightarrow_R,\wedge_R,0,1)$ is a BCK-meet-semilattice. According to Theorem 2.3 of [2], $([0,1];\sim_R,\wedge_R,1)$ is an equality algebra, where $x\sim_R y=(x\rightarrow_R y)\wedge_R(y\rightarrow_R x)=1-\mid x-y\mid$.*

The following are some notions and results about pseudo-hoops.

Definition 2.9 ([22]). *A pseudo-hoop is an algebra $(H;\odot,\rightarrow,\rightsquigarrow,1)$ of type $(2,2,2,0)$ such that for all $x,y,z\in H$:*
(H1) $x\odot 1=x=1\odot x$;
(H2) $x\rightarrow x=1=x\rightsquigarrow x$;
(H3) $(x\odot y)\rightarrow z=x\rightarrow(y\rightarrow z)$;
(H4) $(x\odot y)\rightsquigarrow z=x\rightsquigarrow(y\rightsquigarrow z)$;
(H5) $(x\rightarrow y)\odot x=(y\rightarrow x)\odot y=x\odot(x\rightsquigarrow y)=y\odot(y\rightsquigarrow x)$.

Lemma 2.10 ([22]). *Let $(H;\odot,\rightarrow,\rightsquigarrow,1)$ be a pseudo-hoop. Then for all $x,y,z\in H$:*
(1) $(X;\leq)$ is a meet-semilattice with $x\wedge y=(x\rightarrow y)\odot x=x\odot(x\rightsquigarrow y)$;
(2) $x\leq y\rightarrow x$ and $x\leq y\rightsquigarrow x$;
(3) $x\rightarrow y\wedge z=(x\rightarrow y)\wedge(x\rightarrow z)$ and $x\rightsquigarrow y\wedge z=(x\rightsquigarrow y)\wedge(x\rightsquigarrow z)$;
(4) $x\leq y$ implies $(x\rightarrow y)\rightsquigarrow y=y$ and $(x\rightsquigarrow y)\rightarrow y=y$.

Lemma 2.11. *Let $(H;\odot,\rightarrow,\rightsquigarrow,1)$ be a pseudo-hoop. Then for all $x,y\in H$, $x\rightarrow y\wedge x=x\rightarrow y$ and $x\rightsquigarrow y\wedge x=x\rightsquigarrow y$.*

Proof. By replacing z by x in Lemma 2.10 (3). □

Definition 2.12 ([22]). *A state pseudo-hoop is a structure $(H,\sigma)=(H;\sim,\backsim,\wedge,\sigma,0,1)$, where $(H;\sim,\backsim,\wedge,0,1)$ is a bounded pseudo-hoop and $\sigma:H\rightarrow H$ is a unary operator on H, called state operator (or internal state), satisfying the following conditions for all $x,y\in H$:*
(SH0) $\sigma(0)=0$;
(SH1) $\sigma(x\rightarrow y)=\sigma(x)\rightarrow\sigma(x\wedge y)$ and $\sigma(x\rightsquigarrow y)=\sigma(x)\rightsquigarrow\sigma(x\wedge y)$;
(SH2) $\sigma(x\odot y)=\sigma(x)\odot\sigma(x\rightsquigarrow x\odot y)=\sigma(y\rightarrow x\odot y)\odot\sigma(y)$;
(SH3) $\sigma(\sigma(x)\odot\sigma(y))=\sigma(x)\odot\sigma(y)$;
(SH4) $\sigma(\mu(x)\rightarrow\sigma(y))=\sigma(x)\rightarrow\sigma(y)$ and $\sigma(\sigma(x)\rightsquigarrow\sigma(y))=\sigma(x)\rightsquigarrow\sigma(y)$.

Note that it follows that $x\leq y$ implies $\sigma(x)\leq\sigma(y)$ for all $x,y\in H$ in any state pseudo-hoop (H,σ).

3 Generalized state maps on pseudo equality algebras

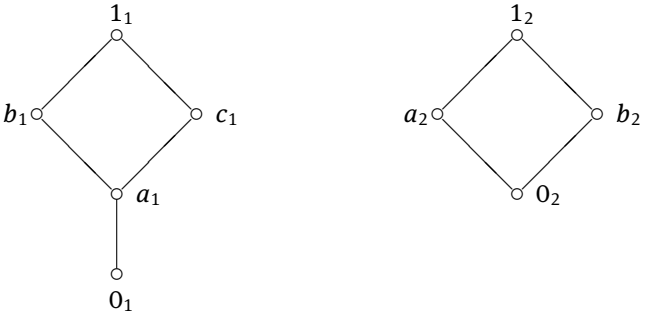
In this section, we introduce a new notion of generalized state map by extending the domain X of a state operator to a more universal setting Y. Moreover, according to the structure of Y, we give two special types of generalized state maps, that is, generalized states and generalized internal states.

Definition 3.1. *Let $(X;\sim_1,\backsim_1,\wedge_1,1_1)$ and $(Y;\sim_2,\backsim_2,\wedge_2,1_2)$ be two pseudo equality algebras. A map $\mu:X\rightarrow Y$ is called a generalized state map from X to Y (or briefly, GS-map) if it satisfies the following conditions for all*

$x, y \in X$:
(GSX1) $\mu(x) \leq_2 \mu(y)$*, whenever* $x \leq_1 y$*;*
(GSX2) $\mu((x \wedge_1 y) \rightsquigarrow_1 x) = \mu(y) \rightsquigarrow_2 \mu(((x \wedge_1 y) \rightsquigarrow_1 x) \rightsquigarrow_1 y)$ *and* $\mu(x \rightsquigarrow_1 (x \wedge_1 y)) = \mu(y \rightsquigarrow_1 (x \rightsquigarrow_1 (x \wedge_1 y))) \rightsquigarrow_2 \mu(y)$*;*
(GSX3) $\mu(x) \rightsquigarrow_2 \mu(y) \in \mu(X)$ *and* $\mu(x) \rightsquigarrow_2 \mu(y) \in \mu(X)$*;*
(GSX4) $\mu(x) \wedge_2 \mu(y) \in \mu(X)$.
Moreover, we give two special types of generalized state maps from X to Y.
(1) If $Y = ([0,1]; \rightsquigarrow_R, \rightsquigarrow_R, \wedge_R, 1)$*, then* μ *is called a generalized state (or briefly, G-state) from X to* $[0,1]$*;*
(2) If $Y = X$*, then* μ *is called a generalized internal state (or briefly, GI-state) from X to X.*

Example 3.2. *Let* $(X; \rightsquigarrow_1, \rightsquigarrow_1, \wedge_1, 1_1)$ *and* $(Y; \rightsquigarrow_2, \rightsquigarrow_2, \wedge_2, 1_2)$ *be two pseudo equality algebras. Define a map* $\mu : X \to Y$ *by* $\mu(x) = 1_2$ *for all* $x \in X$*, then* μ *is a GS-map from X to Y, in this case,* μ *is called trivial.*

Example 3.3. *Let* $X = \{0_1, a_1, b_1, c_1, 1_1\}$ *and* $Y = \{0_2, a_2, b_2, 1_2\}$ *in which the order of elements in X and Y are as the following Hasse diagrams, respectively:*



And the operations $\rightsquigarrow_1, \rightsquigarrow_1$ *on X and* $\rightsquigarrow_2, \rightsquigarrow_2$ *on Y may be given as follows, respectively.*

$\rightsquigarrow_1$	0_1	a_1	b_1	c_1	1_1
0_1	1_1	b_1	0_1	b_1	0_1
a_1	1_1	1_1	c_1	b_1	a_1
b_1	1_1	1_1	1_1	b_1	b_1
c_1	1_1	1_1	c_1	1_1	c_1
1_1	1_1	1_1	1_1	1_1	1_1

$\rightsquigarrow_1$	0_1	a_1	b_1	c_1	1_1
0_1	1_1	1_1	1_1	1_1	1_1
a_1	c_1	1_1	1_1	1_1	1_1
b_1	c_1	c_1	1_1	c_1	1_1
c_1	0_1	b_1	b_1	1_1	1_1
1_1	0_1	a_1	b_1	c_1	1_1

$\rightsquigarrow_2$	0_2	a_2	b_2	1_2
0_2	1_2	b_2	a_2	0_2
a_2	b_2	1_2	b_2	a_2
b_2	a_2	b_2	1_2	b_2
1_2	0_2	a_2	b_2	1_2

$\rightsquigarrow_2$	0_2	a_2	b_2	1_2
0_2	1_2	b_2	1_2	b_2
a_2	b_2	1_2	b_2	a_2
b_2	b_2	b_2	1_2	b_2
1_2	0_2	a_2	b_2	1_2

Then $(X; \rightsquigarrow_1, \rightsquigarrow_1, \wedge_1, 1_1)$ *and* $(Y; \rightsquigarrow_2, \rightsquigarrow_2, \wedge_2, 1_2)$ *are two pseudo equality algebras. Let the map* $\mu : X \to Y$ *be a GS-map from X to Y. Taking* $x = a, y = 0$ *in (GSX2), we get that* $\mu(b_1) = \mu(0_1) \rightsquigarrow_2 \mu(b_1)$ *and* $\mu(b_1) = 1_2$*. This shows that there doesn't exists any nontrivial GS-map from X to Y.*

Example 3.4. *Let* $X = \{0_1, a_1, b_1, 1_1\}$ *with* $0_1 < a_1 < b_1 < 1_1$ *and* $Y = \{0_2, a_2, b_2, 1_2\}$ *be given by Example 3.3. Define the operations* $\rightsquigarrow_1, \rightsquigarrow_1$ *on X as follows:*

$\rightsquigarrow_1$	0_1	a_1	b_1	1_1
0_1	1_1	a_1	a_1	0_1
a_1	1_1	1_1	a_1	a_1
b_1	1_1	1_1	1_1	b_1
1_1	1_1	1_1	1_1	1_1

$\rightsquigarrow_1$	0_1	a_1	b_1	1_1
0_1	1_1	1_1	1_1	1_1
a_1	b_1	1_1	1_1	1_1
b_1	0_1	a_1	1_1	1_1
1_1	0_1	a_1	b_1	1_1

Then $(X; \rightsquigarrow_1, \leftsquigarrow_1, \wedge_1, 1_1)$ *is a pseudo equality algebra. We define a map* $\mu : X \to Y$ *by* $\mu(0_1) = \mu(a_1) = a_2, \mu(b_1) = \mu(1_1) = 1_2$. *One can check that* μ *is a GS-map from X to Y.*

Example 3.5. *Let* $X = \{0_1, a_1, b_1, c_1, 1_1\}$ *with* $0_1 < a_1 < b_1 < c_1 < 1_1$ *and* $Y = \{0_2, a_2, b_2, 1_2\}$ *be given by Example 3.3. Define the operations* $\rightsquigarrow_1, \leftsquigarrow_1$ *as follows:*

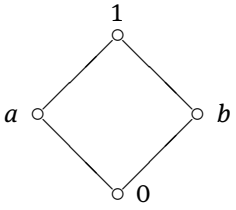
$\rightsquigarrow_1$	0_1	a_1	b_1	c_1	1_1
0_1	1_1	0_1	0_1	0_1	0_1
a_1	1_1	1_1	b_1	b_1	a_1
b_1	1_1	1_1	1_1	b_1	b_1
c_1	1_1	1_1	1_1	1_1	c_1
1_1	1_1	1_1	1_1	1_1	1_1

$\leftsquigarrow_1$	0_1	a_1	b_1	c_1	1_1
0_1	1_1	1_1	1_1	1_1	1_1
a_1	0_1	1_1	1_1	1_1	1_1
b_1	0_1	c_1	1_1	1_1	1_1
c_1	0_1	a_1	b_1	1_1	1_1
1_1	0_1	a_1	b_1	c_1	1_1

Then $(X; \rightsquigarrow_1, \leftsquigarrow_1, \wedge_1, 1_1)$ *is a pseudo equality algebra. One can check that the map* $\mu : X \to Y$ *defined by* $\mu(0_1) = 0_2, \mu(a_1) = \mu(b_1) = \mu(c_1) = \mu(1_1) = 1_2$ *is a GS-map from X to Y.*

Example 3.6. *Let* $(X; \rightsquigarrow_1, \leftsquigarrow_1, \wedge_1, 1_1)$ *be a pseudo equality given in Example 3.4. Define a map* $\mu : X \to [0, 1]$ *by* $\mu(0_1) = 0, \mu(a_1) = \mu(b_1) = 0.5, \mu(1_1) = 1$. *Then one can check that* μ *is a G-state from* $(X; \rightsquigarrow_1, \leftsquigarrow_1, \wedge_1, 1_1)$ *to* $([0, 1]; \rightsquigarrow_R, \wedge_R, 1)$.

Example 3.7. *Let* $X = \{0, a, b, 1\}$ *in which the Hasse diagram and the operations* $\rightsquigarrow, \leftsquigarrow$ *on X are as follows:*



$\rightsquigarrow$	0	a	b	1
0	1	b	a	0
a	1	1	a	a
b	1	b	1	b
1	1	1	1	1

$\leftsquigarrow$	0	a	b	1
0	1	1	1	1
a	b	1	a	1
b	a	a	1	1
1	0	a	b	1

Then $(X; \rightsquigarrow, \leftsquigarrow, \wedge, 1)$ *is a pseudo equality algebra. Define a map* $\mu : X \to X$ *by* $\mu(0) = \mu(a) = b, \mu(b) = \mu(1) = 1$. *One can check that* μ *is a GI-state from X to X.*

Proposition 3.8. *Let* $(X; \rightsquigarrow_1, \leftsquigarrow_1, \wedge_1, 1_1)$ *and* $(Y; \rightsquigarrow_2, \leftsquigarrow_2, \wedge_2, 1_2)$ *be two pseudo equality algebras, and* μ *be a GS-map from X to Y. Then for all* $x, y \in X$, *the following axioms hold:*
(G1) $\mu(1_1) = 1_2$;
(G2) $\mu(x) \to_2 \mu(y) \in \mu(X)$ *and* $\mu(x) \rightsquigarrow_2 \mu(y) \in \mu(X)$;
(G3) $\mu(X)$ *is a subalgebra of Y;*

(G4) $\mu(x \wedge_1 y) \leq_2 \mu(x) \wedge_2 \mu(y)$*;*
(G5) $y \leq_1 x$ *implies* $\mu(y) \rightsquigarrow_2 \mu(x) = \mu(x) \rightarrow_2 \mu(y)$, $\mu(x) \leftsquigarrow_2 \mu(y) = \mu(x) \rightsquigarrow_2 \mu(y)$*;*
(G6) $y \leq_1 x$ *implies* $\mu(y \leftsquigarrow_1 x) \leq_2 \mu(y) \rightsquigarrow_2 \mu(x)$ *and* $\mu(x \leftsquigarrow_1 y) \leq_2 \mu(x) \leftsquigarrow_2 \mu(y)$*;*
(G7) x, y *comparable implies* $\mu(x \rightarrow_1 y) \leq_2 \mu(x) \rightarrow_2 \mu(y)$ *and* $\mu(x \rightsquigarrow_1 y) \leq_2 \mu(x) \rightsquigarrow_2 \mu(y)$*;*
(G8) $Ker(\mu) := \{x \in X : \mu(x) = 1_2\}$ *is a deductive system of* X*.*

Proof. (G1) By taking $x = y = 1_1$ in (GSX2).
(G2)(G3) Evident by (GSX3) and (GSX4).
(G4) By (GSX1) and $x \wedge_1 y \leq_1 x, y$.
(G5) Let $y \leq_1 x$. Then $\mu(y) \leq_2 \mu(x)$. Hence by Proposition 2.4, we have $\mu(y) \rightsquigarrow_2 \mu(x) = \mu(x) \rightarrow_2 \mu(y)$ and $\mu(x) \leftsquigarrow_2 \mu(y) = \mu(x) \rightsquigarrow_2 \mu(y)$.

(G6) Let $y \leq x$. Then by (GSX2), we have $\mu(y \leftsquigarrow_1 x) = \mu(y) \rightsquigarrow_2 \mu((y \leftsquigarrow_1 x) \leftsquigarrow_1 y)$ and $\mu(x \leftsquigarrow_1 y) = \mu(y \leftsquigarrow_1 (x \leftsquigarrow_1 y)) \leftsquigarrow_2 \mu(y)$. On the other hand, according to 2.4 (2), we get $y \leq_1 x \leq_1 (y \leftsquigarrow_1 x) \leftsquigarrow_1 y$ and $y \leq_1 x \leq_1 y \leftsquigarrow_1 (x \leftsquigarrow_1 y)$. Hence $\mu(y) \leq_2 \mu(x) \leq_2 \mu((y \leftsquigarrow_1 x) \leftsquigarrow_1 y)$ and $\mu(y) \leq_2 \mu(x) \leq_2 \mu(y \leftsquigarrow_(x \leftsquigarrow_1 y))$. It follows from (X4) that $\mu(y) \rightsquigarrow_2 \mu((y \leftsquigarrow_1 x) \leftsquigarrow_1 y) \leq_2 \mu(y) \rightsquigarrow_2 \mu(x)$ and $\mu(y \leftsquigarrow_1 (x \leftsquigarrow_1 y)) \leftsquigarrow_2 \mu(y) \leq_2 \mu(x) \leftsquigarrow_2 \mu(y)$. Therefore $\mu(y \leftsquigarrow_1 x) \leq_2 \mu(y) \rightsquigarrow_2 \mu(x)$ and $\mu(x \leftsquigarrow_1 y) \leq_2 \mu(x) \leftsquigarrow_2 \mu(y)$.

(G7) Let $y \leq x$. Then by Proposition 2.4 and (G5), (G6), we get that $\mu(x \rightarrow_1 y) = \mu(y \leftsquigarrow_1 x) \leq_2 \mu(y) \rightsquigarrow_2 \mu(x) = \mu(x) \rightarrow_2 \mu(y)$ and $\mu(x \rightsquigarrow_1 y) = \mu(x \leftsquigarrow_1 y) \leq_2 \mu(x) \leftsquigarrow_2 \mu(y) \leq_2 \mu(x) \rightsquigarrow_2 \mu(y)$.

(G8) Clearly, $1_1 \in Ker(\mu)$ by (G1). Let $x, y \in X$ such that $x, x \rightarrow_1 y \in Ker(\mu)$. Then $\mu(x) = \mu(x \rightarrow_1 y) = 1_2$. Since $y \leq_1 x \rightarrow_1 y$ and $x \leq_1 (x \rightarrow_1 y) \rightsquigarrow_1 y$, it follows from (G7) that $1 = \mu(x) \leq_2 \mu((x \rightarrow_1 y) \rightsquigarrow_1 y) \leq_2 \mu(x \rightarrow_1 y) \rightsquigarrow_1 \mu(y) = 1 \rightsquigarrow_2 \mu(y) = \mu(y)$. Hence $\mu(y) = 1_2$ and so $y \in Ker(\mu)$. Therefore $Ker(\mu)$ is a deductive system of X. □

4 States on pseudo equality algebras

In this section, we introduce the notions of Riečan states and state-morphisms on pseudo equality algebras. We mainly study some of their properties and investigate the relations between Riečan states, state-morphisms and Bosbach states.

Definition 4.1 ([21])**.** *Let* $(X; \sim, \backsim, \wedge, 0, 1)$ *be a bounded pseudo equality algebra. A function* $s : X \rightarrow [0, 1]$ *is said to be a Bosbach state on* X*, if the following hold:*
(BS1) $s(0) = 0$ *and* $s(1) = 1$*;*
(BS2) $s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x)$*;*
(BS3) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$*,*
for all $x, y \in X$*.*

Example 4.2. *Let* $(Y; \sim_2, \backsim_2, \wedge_2, 0_2, 1_2)$ *be a bounded pseudo equality algebra given by Example 3.3. Define a function* $s : Y \rightarrow [0, 1]$ *by* $s(0_2) = 0, s(a_2) = s(b_2) = 0.5, s(1_2) = 1$*. Then one can check that* s *is a unique Bosbach state on* Y*.*

Example 4.3. *Let* $(X; \sim_1, \backsim_1, \wedge_1, 0_1, 1_1)$ *be a pseudo equality algebra defined by Example 3.5. Then one can check that the function* $s : X \rightarrow [0, 1]$ *defined by* $s(0_1) = 0, s(a_1) = s(b_1) = s(c_1) = s(1_1) = 1$ *is a unique Bosbach state on* X*.*

The following example shows that not every pseudo equality algebra has a Bosbach state.

Example 4.4. *Let* $(X; \sim_1, \backsim_1, \wedge_1, 0_1, 1_1)$ *be a bounded pseudo equality algebra given by Example 3.3, and the function* $s : X \rightarrow [0, 1]$ *defined by* $s(0_1) = 0, s(a_1) = \alpha, s(b_1) = \beta, s(c_1) = \gamma, s(1_1) = 1$*, be a Bosbach state on* X*. In (BS2), (BS3), taking* $x = 0_1, y = b_1$*, we obtain* $\beta = 1$ *and* $\beta + \gamma = 1$*, respectively. Hence* $\gamma = 0$*. On the other*

hand, taking $x = 0_1, y = a_1$, *we get* $\alpha + \beta = 1$ *and* $\alpha + \gamma = 1$, *respectively. According to* $\beta = 1$, *it implies* $\alpha = 0$ *and so* $\gamma = 1$, *which is a contraction. This shows that X admits no Bosbach state.*

Proposition 4.5 ([21]). *Let* $(X; \sim, \backsim, \wedge, 0, 1)$ *be a bounded pseudo equality algebra and s be a Bosbach state on X. Then for all* $x, y \in X$, *the following hold:*
(1) If $x \leq y$, *then* $s(x) \leq s(y)$;
(2) If $x \leq y$, *then* $s(y \to x) = 1 - s(y) + s(x) = s(y \rightsquigarrow x)$;
(3) $s(x \to y) = 1 - s(x) + s(x \wedge y) = s(x \rightsquigarrow y)$;
(4) $s(x^-) = 1 - s(x) = s(x^\sim)$;
(5) $s(x^{-\sim}) = s(x) = s(x^{\sim-})$.

Proposition 4.6. *Let* $(X; \sim, \backsim, \wedge, 0, 1)$ *be a bounded pseudo equality algebra and* $s : X \to [0, 1]$ *be a function such that* $s(0) = 0$. *Then the following conditions are equivalent:*
(1) s is a Bosbach state;
(2) If $x \leq y$, *then* $s(y \to x) = 1 + s(x) - s(y) = s(y \rightsquigarrow x)$;
(3) $s(x \to y) = 1 - s(x) + s(x \wedge y) = s(x \rightsquigarrow y)$.

Proof. $(1) \Rightarrow (2)$ By Proposition 4.5 (2).
$(2) \Rightarrow (3)$ By this proof of Proposition 4.5 (3).
$(3) \Rightarrow (1)$ Assume (3) holds and $x, y \in X$. Then $s(1) = s(x \to 1) = 1 - s(x) + s(x \wedge 1) = 1 - s(x) + s(x) = 1$. Also, $s(x) + s(x \to y) = s(x) + 1 - s(x) + s(x \wedge y) = 1 + s(x \wedge y) = s(y) + 1 - s(y) + s(x \wedge y) = s(y) + s(y \to x)$. Similarly, we can prove (BS3). Thus s is a Bosbach state on X.

Let $(X; \sim, \wedge, 0, 1)$ be a bounded pseudo equality algebra and $s : X \to [0, 1]$ be a Bosbach state on X, we define the kernel of s by $Ker(s) := \{x \in X : s(x) = 1\}$. □

Proposition 4.7. *Let* $(X; \sim, \backsim, \wedge, 0, 1)$ *be a bounded pseudo equality algebra and s be a Bosbach state on X. Then* $Ker(s)$ *is a deductive system of X.*

Proof. Assume that s is a Bosbach state on X. Then for any $x, y \in X$, it follows from $s(1) = 1$ that $1 \in Ker(s)$. Let $x, x \to y \in Ker(s)$. Then $s(x) = s(x \to y) = 1$. Since $x \leq y \to x$, then $1 = s(x) \leq s(y \to x)$. This implies that $s(y \to x) = 1$. Again applying (BS2), we obtain $s(y) = 1$ and thus $y \in Ker(s)$. Therefore, $Ker(s)$ is a deductive system of X.

Let $(X; \sim, \wedge, 0, 1)$ be a bounded equality algebra and s be a Bosbach state on X. Then by Proposition 4.7 and [20] Proposition 3.9, the relation $\theta : x\theta y$ iff $x \sim y \in Ker(s)$ is a congruence relation on X. In this case, we denote the quotient algebra X/θ by $X/Ker(s)$ and the congruence class of $x \in X$ by $x/Ker(s)$, where $x/Ker(s) \wedge y/Ker(s) = (x \wedge y)/Ker(s)$, $x/Ker(s) \sim y/Ker(s) = (x \sim y)/Ker(s)$, $x/Ker(s) \leq y/Ker(s)$ iff $x/Ker(s) \wedge y/Ker(s) = x/Ker(s)$. □

Definition 4.8. *A pseudo equality algebra* $(X; \sim, \backsim, \wedge, 1)$ *is said to be*
• *good provided that* $x^{-\sim} = x^{\sim-}$ *for all* $x \in X$;
• *involutive provided that* $x^{-\sim} = x = x^{\sim-}$ *for all* $x \in X$.

Example 4.9. *(1) In Example 3.5 it is evident that* $(X; \sim_1, \backsim_1, \wedge_1, 1_1)$ *is an involutive pseudo equality algebra.*
(2) Let $X = \{0, a, b, c, 1\}$ *with* $0 < a < b < c < 1$ *and the operations* $\sim, \backsim$ *be given as follows:*

$\sim$	0	a	b	c	1
0	1	b	b	0	0
a	1	1	c	a	a
b	1	1	1	b	b
c	1	1	1	1	c
1	1	1	1	1	1

$\backsim$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

One can check that $(X; \sim, \backsim, \wedge, 1)$ *is a good pseudo equality algebra, but it is not involutive, since* $a^{-\sim} = a^{\sim -} = b \neq a$.

Theorem 4.10. *Let* $(X; \sim, \wedge, 0, 1)$ *be a bounded equality algebra and* s *be a Bosbach state on* X. *Then* $(X/Ker(s); \sim, \wedge, 0/Ker(s), 1/Ker(s))$ *is an involutive equality algebra.*

Proof. Assume s is a Bosbach state on X. First, it follows that $(X/Ker(s); \sim, \wedge, 0/Ker(s), 1/Ker(s))$ is a bounded equality algebra. In the following, we prove that $X/Ker(s)$ is involutive. By (S2), we have $s(x)+s(x \to x^{--}) = s(x^{--}) + s(x^{--} \to x)$. Since $s(x) = s(x^{--})$ by Proposition 4.5 (5), then $s(x \to x^{--}) = s(x^{--} \to x)$. Again, $x \leq x^{--}$ by Proposition 2.6 (3), we get $s(x \to x^{--}) = s(1) = 1$. This implies $s(x^{--} \to x) = 1$ and so $x^{--} \to x \in Ker(s)$. Notice that $x \leq x^{--}$, we obtain $x^{--} \sim x = x^{--} \to x \in Ker(s)$. Hence $x^{--} \theta x$. This shows $x^{--}/Ker(s) = x/Ker(s)$. Therefore, $(x/Ker(s))^{--} = x^{--}/Ker(s) = x/Ker(s)$ and this proof is complete. □

Definition 4.11. *Let* $(X; \sim, \backsim, \wedge, 0, 1)$ *be a bounded pseudo equality algebra. A state-morphism on* X *is a function* $m : X \to [0, 1]$ *such that*
(M1) $m(0) = 0$;
(M2) $m(x \to y) = m(x) \to_R m(y) = m(x \rightsquigarrow y)$ *for all* $x, y \in X$.

Proposition 4.12. *A state-morphism* m *is a Bosbach state on a bounded pseudo equality algebra* X.

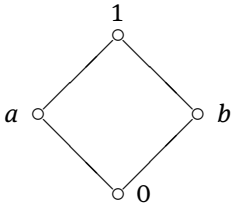
Proof. Let m be a state-morphism on X. For any $x, y \in X$, $m(1) = m(x \to x) = min\{1, 1 - m(x) + m(x)\} = 1$, and $m(x) + m(x \to y) = m(x) + min\{1, 1 - m(x) + m(y)\} = min\{1 + m(x), 1 + m(y)\} = m(y) + min\{1, 1 - m(y) + m(x)\} = m(y) + m(y \to x)$. Similarly, we can prove (BS3). This shows m is a Bosbach state on X. □

Proposition 4.13. *Let* $(X; \sim, \backsim, \wedge, 0, 1)$ *be a bounded pseudo equality algebra and* s *be a Bosbach state on* X. *Then* s *is a state-morphism on* X *if and only if* $s(x \wedge y) = min\{s(x), s(y)\}$ *for all* $x, y \in X$.

Proof. Let s be a state-morphism on X. Then by Proposition 4.6, $s(x \wedge y) = s(x) + s(x \to y) - 1 = s(x) + min\{1, 1 - s(x) + s(y)\} - 1 = min\{s(x), s(y)\}$ for all $x, y \in X$. Conversely, let $s(x \wedge y) = min\{s(x), s(y)\}$ for all $x, y \in X$. Taking $x = y = 0$, then $s(0) = 0$. Again by Proposition 4.6, we obtain $s(x \to y) = s(x \rightsquigarrow y) = 1 - s(x) + s(x \wedge y) = 1 - m(x) + min\{s(x), s(y)\} = min\{1, 1 - m(x) + m(y)\} = m(x) \to_R m(y)$. Thus s is a state-morphism on X. □

Example 4.14. *Let* $(Y; \sim_2, \backsim_2, \wedge_2, 0_2, 1_2)$ *be a bounded pseudo equality algebra given by Example 3.3. Define a function* $s : Y \to [0, 1]$ *by* $s(0_2) = 0, s(a_2) = s(b_2) = 0.5, s(1_2) = 1$. *Then one can check that* s *is a Bosbach state on* Y, *but it is not a state-morphism on* Y *since* $s(a_2 \wedge b_2) = s(0_2) = 0 \neq 0.5 = min\{s(a_2), s(b_2)\}$.

Example 4.15. *Let* $X = \{0, a, b, 1\}$ *in which the Hasse diagram and the operation* $\sim$ *on* X *is below:*



$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then $(X; \sim, \wedge, 1)$ *is an equality algebra [20], where the derived operation* $\rightarrow$ *as the above. The function* $s : X \rightarrow [0, 1]$ *is given by* $s(0) = s(a) = 0, s(b) = s(1) = 1$. *Then* s *is a Bosbach state on* X. *Furthermore,* s *is a state-morphism on* X, *since* $s(a \wedge b) = s(0) = 0 = min\{s(a), s(b)\}$.

Corollary 4.16. *In any linearly ordered bounded pseudo equality algebra* $(X; \sim, \backsim, \wedge, 0, 1)$, *the Bosbach states coincide with the state-morphisms.*

Proposition 4.17. *Let* $(X; \sim, \backsim, \wedge, 0, 1)$ *be a bounded involutive pseudo equality algebra and* s *be a Bosbach state on* X. *Then the following are equivalent:*
(1) s *is a state-morphism on* X;
(2) $s(x^{\sim} \rightarrow y^{-\sim}) = min\{1, s(x) + s(y)\}$;
(3) $s(y^{-} \rightsquigarrow x^{\sim-}) = min\{1, s(x) + s(y)\}$,
for all $x, y \in X$.

Proof. $(1) \Rightarrow (2)$ Let s be a state-morphism on X. Then by (M2) and Proposition 4.5, we get $s(x^{\sim} \rightarrow y^{-\sim}) = min\{1, 1 - s(x^{\sim}) + s(y^{-\sim})\} = min\{1, 1 - 1 + s(x) + s(y)\} = min\{1, s(x) + s(y)\}$, for all $x, y \in X$.

$(2) \Rightarrow (3)$ Assume that (2) holds. By Proposition 2.6 (7), $x^{\sim} \rightarrow y^{-\sim} = y^{-} \rightsquigarrow x^{\sim-}$. Hence $s(y^{-} \rightsquigarrow x^{\sim-}) = s(x^{\sim} \rightarrow y^{-\sim}) = min\{1, s(x) + s(y)\}$.

$(3) \Rightarrow (1)$ Assume that (3) holds. Since X is involutive, then $s(x \rightsquigarrow y) = s(x^{\sim-} \rightsquigarrow y^{\sim-}) = min\{1, s(y) + s(x^{\sim})\} = min\{1, 1 - s(x) + s(y)\}$. Again since s is a Bosbach state, we have $s(0) = 0$. Hence s is a state-morphism on X. □

Definition 4.18. *Let* $(X; \sim, \backsim, \wedge, 0, 1)$ *be a bounded pseudo equality algebra. Two elements* $x, y \in X$ *are said to be orthogonal, if* $x^{-\sim} \leq y^{\sim}$, *we write by* $x \perp y$. *If* $x, y \in X$ *are orthogonal, we define a binary operation* $+$ *on* X *by* $x + y := y^{\sim} \rightarrow x^{-\sim}$.

Proposition 4.19. *In any bounded pseudo equality algebra* $(X; \sim, \backsim, \wedge, 0, 1)$, *the following properties hold for all* $x, y \in X$:
(1) $x \perp y$ *iff* $y^{\sim-} \leq x^{-}$;
(2) $x \perp y$ *iff* $x \leq y^{\sim}$ *and* $x \perp y$ *iff* $y \leq x^{-}$;
(3) $x \perp y$ *implies* $x + y = x^{-} \rightsquigarrow y^{\sim-}$;
(4) $x \perp x^{-}$ *and* $x + x^{-} = 1$;
(5) $x^{\sim} \perp x$ *and* $x^{\sim} + x = 1$;
(6) $0 \perp x$ *and* $0 + x = x^{\sim-}$;
(7) $x \perp 0$ *and* $x + 0 = x^{-\sim}$;
(8) $x \leq y$ *implies* $x \perp y^{-}$ *and* $x + y^{-} = y^{-\sim} \rightarrow x^{-\sim}$;
(9) $x \leq y$ *implies* $y^{\sim} \perp x$ *and* $y^{\sim} + x = y \rightsquigarrow x^{\sim-}$.

Proof. (1) Let $x \perp y$, $x, y \in X$. Then $x^{-\sim} \leq y^{\sim}$. By Proposition 2.6 (3) and (4), $y^{\sim-} \leq x^{-\sim-} = x^{-}$. Conversely, let $y^{\sim-} \leq x^{-}$. Using Proposition 2.6 (3) and (4) again, we get $x^{-\sim} \leq y^{\sim-\sim} = y^{\sim}$.
(2) Let $x \perp y$. Then by (2) and Proposition 2.6 (3), $x \leq x^{-\sim} \leq y^{\sim}$ and $y \leq y^{\sim-} \leq x^{-}$.
(3) Let $x \perp y$. Since $x^{-} \rightsquigarrow y^{\sim-} = \sim \rightarrow x^{-\sim}$ by Proposition 2.6 (7), we have $x + y = x^{-} \rightsquigarrow y^{\sim-}$.
(4) Since $x^{-\sim} \leq x^{-\sim}$, then $x \perp x^{-}$ and $x + x^{-} = x^{-\sim} \rightarrow x^{-\sim} = 1$.
(5) Since $x^{\sim-} \leq x^{\sim-}$, then $x^{\sim} \perp x$ and $x^{\sim} + x = x^{\sim-} \rightarrow x^{\sim-} = 1$.
(6) By Proposition 2.6 (2), $0^{-\sim} = 0 \leq x^{\sim}$. Hence $0 \perp x$, and $0 + x = x^{\sim} \rightarrow 0^{-\sim} = x^{\sim} \rightarrow 0 = x^{\sim-}$.
(7) By Proposition 2.6 (1), $x^{-\sim} \leq 1 = 0^{\sim}$. Hence $x \perp 0$. Again by Proposition 2.4 (10), we get that $x + 0 = 0^{\sim} \rightarrow$

$x^{-\sim} = 1 \to x^{-\sim} = x^{-\sim}$.
(8) Let $x \le y$. Then by Proposition 2.6 (5), $x^{-\sim} \le y^{-\sim}$. Hence $x \perp y^{-}$ and $x + y^{-} = y^{-\sim} \to x^{-\sim}$.
(9) Let $x \le y$. Then by Proposition 2.6 (4) and (5), we have $y^{\sim} \le x^{\sim}$ and so $y^{\sim-\sim} = y^{\sim} \le x^{\sim}$. Hence $y^{\sim} \perp x$ and $y^{\sim} + x = x^{\sim} \to y^{\sim-\sim} = x^{\sim} \to y^{\sim} = y \rightsquigarrow x^{\sim-}$ by Proposition 2.6 (8). □

Definition 4.20. *Let $(X; \sim, \backsim, \wedge, 0, 1)$ be a good bounded pseudo equality algebra. A Riečan state on X is a function $s : X \to [0, 1]$ such that*
(RS1) $s(1) = 1$;
(RS2) $s(x + y) = s(x) + s(y)$ whenever $x \perp y$ for all $x, y \in X$.

Example 4.21. *Consider the good bounded pseudo equality algebra $(X; \sim, \backsim, \wedge, 0, 1)$ given by Example 3.7. Define the function $s : X \to [0, 1]$ by $s(0) = 0, s(a) = s(b) = 0.5, s(1) = 1$, then one can check that s is a Riečan state on X.*

Proposition 4.22. *Let s be a Riečan state on a good bounded pseudo equality algebra $(X; \sim, \backsim, \wedge, 0, 1)$. Then for all $x, y \in X$, the following hold:*
(1) $s(x^{-}) = 1 - s(x) = s(x^{\sim})$;
(2) $s(0) = 0$;
(3) $s(x^{-\sim}) = s(x) = s(x^{\sim-})$;
(4) $x \le y$ implies $s(x) \le s(y)$ and $s(y^{-\sim} \to x^{-\sim}) = 1 + s(x) - s(y) = s(y \rightsquigarrow x^{\sim-})$;
(5) $s(x^{-\sim} \to (x \wedge y)^{-\sim}) = 1 - s(x) + s(x \wedge y) = s(x \rightsquigarrow (x \wedge y)^{\sim-})$.

Proof. (1) By Proposition 4.19 (4), we have $s(x + x^{-}) = s(x) + s(x^{-}) = s(1) = 1$. Hence $s(x^{-}) = 1 - s(x)$. Similarly, by Proposition 4.19 (5), $s(x^{\sim}) = 1 - s(x)$.

(2) By (1) and Proposition 2.6 (1), $s(0) = s(1^{-}) = 1 - s(1) = 1 - 1 = 0$.

(3) By (2) and Proposition 4.19 (6), $s(x^{\sim-}) = s(0 + x) = s(0) + s(x) = 0 + s(x) = s(x)$. Similarly, by Proposition 4.19 (7), $s(x^{-\sim}) = s(x)$.

(4) Let $x \le y$. Then by Proposition 4.19 (9), $y^{\sim} \perp x$ and $y^{\sim} + x = y \rightsquigarrow x^{\sim-}$. Hence $s(y \rightsquigarrow x^{\sim-}) = s(y^{\sim} + x) = 1 + s(x) - s(y)$, and so $s(x) - s(y) = s(y^{\sim} + x) - 1 \le 0$. Hence $s(x) \le s(y)$. It is similar that $s(y^{-\sim} \to x^{-\sim}) = 1 + s(x) - s(y)$ by Proposition 4.19 (8).

(5) It follows from $x \wedge y \le x$ and (4). □

Theorem 4.23. *In any good bounded pseudo equality algebra $(X; \sim, \backsim, \wedge, 0, 1)$, each Bosbach state on X is a Riečan state.*

Proof. Assume that s is a Bosbach state on X. Then $s(1) = 1$. Let $x \perp y$ for $x, y \in X$. Then $x^{-\sim} \le y^{\sim}$. By (2),(4) and (5) of Proposition 4.5 , we have $s(x+y) = s(y^{\sim} \to x^{-\sim}) = 1 - s(y^{\sim}) + s(x^{-\sim}) = 1 - (1 - s(y)) + s(x) = s(x) + s(y)$. Therefore s is a Riečan state on X. □

Note that the converse of the above theorem is not true in general. Let us see the following example.

Example 4.24. *Let $(X; \sim, \backsim, \wedge, 0, 1)$ be the good pseudo equality algebra given by 4.9 (2). Define a map $s : X \to [0, 1]$ by $s(0) = 0, s(a) = s(b) = s(c) = 0.5, s(1) = 1$. Then s is a Riečan state on X, but s is not a Bosbach state on X. Taking $x = a, y = b$ in (BS2), we can obtain that $0.5 + 1 = 0.5 + s(c)$ and so $s(c) = 1$, which is a contradiction.*

Theorem 4.25. *In any bounded involutive pseudo equality algebra $(X; \sim, \backsim, \wedge, 0, 1)$, the Bosbach states and the Riečan states coincide on X.*

Proof. Assume that s is a Riečan state on X. Then $s(1) = 1$ and $s(0) = 0$ by Proposition 4.22 (2). Let $x \le y$, then by Proposition 4.22 (4), $s(y^{-\sim} \to x^{-\sim}) = 1 - s(x) + s(y) = s(y \rightsquigarrow x^{\sim-})$. Since X is involutive, we obtain

$s(y \to x) = s(y^{-\sim} \to x^{-\sim}) = 1 - s(x) + s(y) = s(y \rightsquigarrow x^{\sim-}) = s(y \rightsquigarrow x)$. Hence by Proposition 4.6, it follows that s is a Bosbach state on X. □

5 The relations between generalized state maps, states and internal states on pseudo equality algebras

In this section, we focus on discussing the relations between generalized state maps, states and internal states on pseudo equality algebras. First, we recall some related notions and results of internal states on pseudo equality algebras based on [20,21].

Definition 5.1 ([21]). *A state pseudo equality algebra is a structure* $(X, \mu) = (X; \sim, \backsim, \wedge, \mu, 1)$, *where* $(X; \sim, \backsim, \wedge, 1)$ *be a pseudo equality algebra and* $\mu : X \to X$ *is a unary operator on X, called an internal state (or state operator), satisfying the following conditions for all* $x, y \in X$*:*
(SX1) $\mu(x) \leq \mu(y)$, *whenever* $x \leq y$*;*
(SX2) $\mu((x \wedge y) \sim x) = \mu(y) \sim \mu(((x \wedge y) \sim x) \backsim y), \mu(x \backsim (x \wedge y)) = \mu(y \sim (x \backsim (x \wedge y))) \backsim \mu(y)$*;*
(SX3) $\mu(\mu(x) \sim \mu(y)) = \mu(x) \sim \mu(y), \mu(\mu(x) \backsim \mu(y)) = \mu(x) \backsim \mu(y)$*;*
(SX4) $\mu(\mu(x) \wedge \mu(y)) = \mu(x) \wedge \mu(y)$.

It is clear that a state equality algebra (see [20]) is a state pseudo equality algebra, a pseudo equality algebra can be seen as a state pseudo equality algebra.

Proposition 5.2. *Let* (X, μ) *be a state pseudo equality algebra. Then for all* $x, y \in X$*, we have:*
(1) $\mu(\mu(x)) = \mu(x)$*;*
(2) $\mu(\mu(x) \to \mu(y)) = \mu(x) \to \mu(y)$ *and* $\mu(\mu(x) \rightsquigarrow \mu(y)) = \mu(x) \rightsquigarrow \mu(y)$.

Proof. (2) is evident and (1) is similar to the proof of Proposition 5.6 in [20]. □

Definition 5.3. *Let* $(X; \sim, \backsim, \wedge, 1)$ *be a pseudo equality algebra. A strong internal state* μ *on X is an internal state on X satisfying:*
(SX5) $\mu(x \to y) = \mu(x) \to \mu(x \wedge y), \mu(x \rightsquigarrow y) = \mu(x) \rightsquigarrow \mu(x \wedge y)$ *for all* $x, y \in X$.
Accordingly, the pair (X, μ) *is said to a strong state pseudo equality algebra.*

Example 5.4. *Let* $(X; \sim, \backsim, \wedge, 1)$ *be a pseudo equality algebra given in Example 4.9. Define a map* $\mu : X \to X$ *by* $\mu(0) = 0, \mu(a) = \mu(b) = b, \mu(c) = \mu(1) = 1$. *Then we can calculate that* (X, μ) *is a strong state pseudo equality algebra.*

Proposition 5.5. *Let* (H, σ) *be a state pseudo-hoop. Then* (H, σ) *is a strong state pseudo equality algebra, where* $x \wedge y = x \odot (x \rightsquigarrow y), x \sim y = y \to x$ *and* $x \backsim y = x \rightsquigarrow y$.

Proof. Let (H, σ) be a state pseudo-hoop. Then according to Example 2.6 of [3], $(H; \sim, \backsim, \wedge, 0, 1)$ is a bounded pseudo equality algebra. In the following, we will show that σ is a strong internal state on H. Clearly, (SX0),(SX1) and (SX5) hold. (SX3) and (SX4) follow from (SH3) and (SH4). Next we prove (SX2). By Lemma 2.11 and (SH2), we have $\sigma(x \wedge y \sim x) = \sigma(x \to x \wedge y) = \sigma(x \to y)$, and $\sigma(y) \sim \sigma((x \wedge y \sim x) \backsim y) = \sigma((x \to x \wedge y) \rightsquigarrow y) \to \sigma(y) = \sigma((x \to y) \rightsquigarrow y) \to \sigma(y) = \sigma(x \to y) \rightsquigarrow ((x \to y) \wedge y)) \to \sigma(y) = (\sigma(x \to y) \rightsquigarrow \sigma(y)) \to \sigma(y)$. Since $y \leq x \to y$ by Lemma 2.10 (2), then $\sigma(y) \leq \sigma(x \to y)$. Hence by Lemma 2.10 (4), we obtain that $\sigma(x \to y) = (\sigma(x \to y) \rightsquigarrow \sigma(y)) \to \sigma(y)$. This shows that $\sigma(x \wedge y \sim x) = \sigma(y) \sim \sigma((x \wedge y \sim x) \backsim y)$. In a similar way, we can prove $\sigma(x \backsim x \wedge y) = \sigma(y \sim (x \backsim x \wedge y)) \backsim \sigma(y)$. □

Remark 5.6. *(1) In any pseudo equality algebra, a strong internal state is an internal state, but the converse is not true in general. For example, let* $X = \{0, a, b, c, 1\}$ *with* $0 < a < b < c < 1$ *and the operations* $\sim, \backsim$ *be given as follows:*

$\sim$	0	a	b	c	1
0	1	b	b	b	0
a	1	1	b	b	a
b	1	1	1	b	b
c	1	1	1	1	c
1	1	1	1	1	1

$\backsim$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then one can check that μ *is an internal state on* X*, but* μ *is not a strong internal state, because* $\mu(b \to a) = \mu(b \wedge a \sim b) = \mu(b) = 0 \neq 1 = \mu(b) \to \mu(a)$.
(2) In any bounded pseudo equality algebra, a strong internal state μ *does not necessarily satisfy the condition* $\mu(0) = 0$*. For example, let* $(Y; \sim_2, \backsim_2, \wedge_2, 1_2)$ *be a pseudo equality algebra given by Example 3.3. Define a map* $\mu : Y \to Y$ *by* $\mu(0_2) = \mu(b_2) = b_2$, $\mu(a_2) = \mu(1_2) = 1_2$*. Then one can check that* (Y, μ) *is a strong state pseudo equality algebra.*

Proposition 5.7. *Let* (X, μ) *be a strong state bounded pseudo equality algebra with* $\mu(0) = 0$*. Then for all* $x, y \in X$*:*
(1) $\mu(x^-) = \mu(x)^-$ *and* $\mu(x^\sim) = \mu(x)^\sim$*;*
(2) $x \perp y$ *implies* $\mu(x) \perp \mu(y)$ *and* $\mu(x + y) = \mu(x) + \mu(y)$, $\mu(\mu(x) + \mu(y)) = \mu(x) + \mu(y)$*;*
(3) $\mu(x \to y) \leq \mu(x) \to \mu(y)$ *and* $\mu(x \rightsquigarrow y) \leq \mu(x) \rightsquigarrow \mu(y)$*. If* x, y *are comparable, we have* $\mu(x \to y) = \mu(x) \to \mu(y)$ *and* $\mu(x \rightsquigarrow y) = \mu(x) \rightsquigarrow \mu(y)$.

Proof. (1) By (SX0) and (SX5), $\mu(x^-) = \mu(x \to 0) = \mu(x) \to \mu(0) = \mu(x) \to 0 = \mu(x)^-$. In a similar way, we can get that $\mu(x^\sim) = \mu(x)^\sim$.

(2) Let $x \perp y$. Then $x^{-\sim} \leq y^\sim$. By (SX1) and (1), we have $\mu(x)^{-\sim} \leq \mu(y)^\sim$ and $\mu(x + y) = \mu(y^\sim \to x^{-\sim}) = \mu(y^\sim) \to \mu(x^{-\sim}) = \mu(y)^\sim \to \mu(x)^{-\sim} = \mu(x) + \mu(y)$. By Proposition 5.2 (2), $\mu(\mu(x) + \mu(y)) = \mu(\mu(x)^\sim \to \mu(y)^{-\sim}) = \mu(x)^\sim \to \mu(y)^{-\sim} = \mu(x) + \mu(y)$.

(3) Since $x \wedge y \leq y$, then $\mu(x \wedge y) \leq \mu(y)$. Hence by (SX2) and Proposition 2.4 (3), we have $\mu(x \to y) = \mu(x) \to \mu(x \wedge y) \leq \mu(x) \to \mu(y)$. Similarly, $\mu(x \rightsquigarrow y) \leq \mu(x) \rightsquigarrow \mu(y)$. If $x \leq y$, then $\mu(x \to y) = \mu(1) = 1 \leq \mu(x) \to \mu(y)$. Hence $\mu(x \rightsquigarrow y) = \mu(x) \rightsquigarrow \mu(y)$. If $y \leq x$, then by (SX2), we have $\mu(x \to y) = \mu(x) \to \mu(x \wedge y) = \mu(x) \to \mu(y)$. The other part is similar. □

Definition 5.8. *([21]) Let* $(X; \sim, \backsim, \wedge, 1)$ *be a pseudo equality algebra. A homomorphism* $\mu : X \to X$ *is called an internal state-morphism (or state-morphism operator) if* $\mu^2 = \mu$*, that is* $\mu(\mu(x)) = \mu(x)$ *for all* $x \in X$*, and the pair* (X, μ) *is called a state-morphism pseudo equality algebra.*

According to the definition of an internal state-morphism μ on a pseudo equality algebra, it follows that μ is isotone and μ preserves the operations $\to$ and $\rightsquigarrow$. Note that in any pseudo equality algebra $(X; \sim, \backsim, \wedge, 1)$, the identity map Id_X on X is an internal state-morphism.

By Lemma 2.5 we can get the following theorem.

Theorem 5.9. *Let* $(X; \sim, \backsim, \wedge, 1)$ *be a pseudo equality algebra, and* $\mu : X \to X$ *be an internal state-morphism on* X*. Then* μ *is a strong internal state on* X*. Of course,* μ *is also an internal state on* X.

Note that the converse of Theorem 5.9 is not true in general.

Example 5.10. *In Example 5.4, the map* μ *is a strong internal state on* X*, but it is not an internal state-morphism on* X*, because* $\mu(b \backsim a) = \mu(b) = b \neq 1 = b \backsim b = \mu(b) \backsim \mu(a)$.

In the following, we discuss the relations between generalized state maps, states and internal states on pseudo equality algebras. First, we give the relations between states and (strong) internal states on pseudo equality algebras.

Theorem 5.11. *Assume that $(X;\sim,\backsim,\wedge,0,1)$ is a good bounded pseudo equality algebra, and μ is a strong internal state with $\mu(0)=0$ on X. If s is a Riečan state on $\mu(X)$, then the function $s_\mu : X \to [0,1]$ defined by $s_\mu(x)=s(\mu(x))$ is a Riečan state on X.*

Proof. Clearly, $s_\mu(1)=s(\mu(1))=s(1)=1$. Let $x,y\in X$ such that $x\perp y$. Then by Proposition 5.7 (2), $\mu(x)\perp\mu(y)$ and $\mu(x+y)=\mu(x)+\mu(y)$. Hence $s_\mu(x+y)=s(\mu(x+y))=s(\mu(x)+\mu(y))=s(\mu(x))+s(\mu(y))=s_\mu(x)+s_\mu(y)$. This implies that s_μ is a Riečan state on X. □

Example 5.12. *Let $(X;\sim,\backsim,\wedge,0,1)$ be a good bounded pseudo equality algebra given in 3.7. Define a map $\mu : X\to X$ by $\mu(0)=\mu(a)=0,\mu(b)=\mu(1)=1$. Then one can check that μ is a strong internal state on X, where $\mu(X)=\{0,1\}$. Moreover, the function $s:\mu(X)\to[0,1]$ on $\mu(X)$ defined by $s(0)=0,s(1)=1$ is a Riečan state on $\mu(X)$. It can be calculated that the function $s_\mu : X\to[0,1]$ defined by*

$$s_\mu(x)=s(\mu(x))=\begin{cases}0 & \text{if } x=0,a\\ 1 & \text{if } x=b,1\end{cases}$$

is a Riečan state on X.

Theorem 5.13. *Assume (X,μ) is a state bounded pseudo equality algebra with $\mu(0)=0$. If s is a Bosbach state on $\mu(X)$ and μ preserves $\to$ and $\rightsquigarrow$, then the function $s_\mu : X\to[0,1]$ defined by $s_\mu(x)=s(\mu(x))$ is a Bosbach state on X.*

Proof. Clearly, $s_\mu(0)=s(\mu(0))=s(0)=0$ and $s_\mu(1)=s(\mu(1))=s(1)=1$. If μ preserves $\to$, then $s_\mu(x)+s_\mu(x\to y)=s(\mu(x))+s(\mu(x\to y))=s(\mu(x))+s(\mu(x)\to\mu(y))=s(\mu(y))+s(\mu(y)\to\mu(x))=s(\mu(y))+s(\mu(y\to x))=s_\mu(y)+s_\mu(y\to x)$. This shows that s_μ is a Bosbach state on X. In a similar way, since μ preserves $\rightsquigarrow$, we can show that $s_\mu(x)+s_\mu(x\rightsquigarrow y)=s_\mu(y)+s_\mu(y\rightsquigarrow x)$. It follows that $s_\mu(x)$ is a Bosbach state on X. □

Example 5.14. *Let $(X;\sim,\backsim,\wedge,0,1)$ be a bounded pseudo equality algebra given in Example 3.7. Define a map $\mu : X\to X$ by $\mu(0)=\mu(b)=0,\mu(a)=\mu(1)=1$. Then one can check that μ is an internal state on X and μ preserves $\to$ and $\rightsquigarrow$, where $\mu(X)=\{0,1\}$. Moreover, the function $s:\mu(X)\to[0,1]$ on $\mu(X)$ defined by $s(0)=0,s(1)=1$ is a Bosbach state on $\mu(X)$. It can be calculated that the function $s_\mu : X\to[0,1]$ defined by*

$$s_\mu(x)=s(\mu(x))=\begin{cases}0 & \text{if } x=0,b\\ 1 & \text{if } x=a,1\end{cases}$$

is a Bosbach state on X.

Corollary 5.15. *Let (X,μ) be a state-morphism bounded pseudo equality algebra and s be a Bosbach state on $\mu(X)$. Then the function $s_\mu : X\to[0,1]$ defined by $s_\mu(x)=s(\mu(x))$ is a Bosbach state on X.*

By Proposition 5.7 (3) and the above corollary, we can get the following corollary immediately.

Corollary 5.16. *Let (X,μ) be a strong state linearly ordered bounded pseudo equality algebra and s be a Bosbach state on $\mu(X)$. Then the function $s_\mu : X\to[0,1]$ defined by $s_\mu(x)=s(\mu(x))$ is a Bosbach state on X.*

The above results indicate that by using (strong) internal state (or internal state-morphism) μ, one can extend any state of the image space $\mu(X)$ into the state of the entire space X.

Next, we discuss the relationship between the generalized states (namely, G-states) and the states on pseudo equality algebras.

Theorem 5.17. *Assume that* $(X; \sim, \backsim, \wedge, 0, 1)$ *is a bounded pseudo equality algebra, and* $m : X \to [0, 1]$ *is a state-morphism on* X*. Then* m *is a G-state from* $(X; \sim, \backsim, \wedge, 0, 1)$ *to* $([0, 1]; \sim_R, \wedge_R, 1)$.

Proof. Suppose m is a state-morphism on X. (GSX1) holds as m is a Bosbach state by Proposition 4.12. Let $x, y \in X$. Then $m(x) \wedge_R m(y) = min\{m(x), m(y)\} \in m(X)$. Let $m(x) \leq m(y)$. Then $m(x) \sim_R m(y) = m(y) \to_R m(x) = m(y \to x) \in m(X)$. Hence (GSX3) and (GSX4) hold. Next, we prove (GSX2). It is obvious that Lemma 2.5 results in (GSX2) and therefore the proof is finished. □

The following example indicates that the converse of Theorem 5.17 is not true in general.

Example 5.18. *In Example 3.6, the map* $\mu : X \to [0, 1]$ *is a G-state, but* μ *is not a state-morphism on* X *since* $\mu(b \to a) = \mu(a) = 0.5 \neq 1 = \mu(b) \to_R \mu(a)$.

According to Theorem 5.17, Corollary 4.16 and Theorem 4.25, we can obtain the following corollary.

Corollary 5.19. *Assume* $(X; \sim, \backsim, \wedge, 0, 1)$ *is a bounded pseudo equality algebra and* $s : X \to [0, 1]$ *is a function from* X *to* $[0, 1]$.
(1) If X *is linearly ordered and* s *is a Bosbach state on* X*, then* s *is a G-state from* $(X; \sim, \backsim, \wedge, 0, 1)$ *to* $([0, 1]; \sim_R, \wedge_R, 1)$*;*
(2) If X *is involutive and* s *is a Riečan state, then* s *is a G-state from* $(X; \sim, \backsim, \wedge, 0, 1)$ *to* $([0, 1]; \sim_R, \wedge_R, 1)$.

Finally we discuss the relations between generalized internal states (namely, GI-states) and internal states, internal state-morphisms on pseudo equality algebras.

Theorem 5.20. *Let* $(X; \sim, \backsim, \wedge, 1)$ *be a pseudo equality algebra. Then*
(1) an internal state μ *on* X *is a GI-state from* X *to* X*;*
(2) a GI-state μ *from* X *to* X *is an internal state on* X *if and only if* $\mu^2 = \mu$.

Proof. (1) From Proposition 5.2 (2) $\mu(X)$ is a subalgebra of X, which implies (GSX3) and (GSX4) hold. Thus μ is a GI-state from X to X.

(2) Assume that μ is an internal state on X. Then $\mu^2 = \mu$ by Proposition 5.2 (1). Conversely, let μ be a G-state from X to X and $\mu^2 = \mu$. Then it follows from (GSX4) that there exists $a \in X$ such that $\mu(x) \wedge \mu(y) = \mu(a)$ for any $x, y \in X$. Hence we have $\mu(\mu(x) \wedge \mu(y)) = \mu(\mu(a)) = \mu(a) = \mu(x) \wedge \mu(y)$ and so (SX4) holds. Similarly, we can prove (SX3). □

Example 5.21. *In Example 3.7,* μ *is a GI-state from* X *to* X*, but* μ *is not an internal state on* X *since* $\mu(\mu(a)) = \mu(b) = 1 \neq b = \mu(a)$.

Theorem 5.22. *Let* $(X; \sim, \backsim, \wedge, 1)$ *be a pseudo equality algebra and* μ *be an internal state-morphism on* X*. Then* μ *is a GI-state from* X *to* X.

Proof. Similar to the proof of Theorem 5.17. □

Note that the converse of the above theorem is not true in general according to the following example.

Example 5.23. *In Example 3.7,* μ *is a GI-state from* X *to* X*, but* μ *is not an internal state-morphism on* X *since* $\mu(\mu(a)) = \mu(b) = 1 \neq b = \mu(a)$.

6 Conclusions

In this paper, we introduce a new notion of generalized state map (or simply, GS-map) by extending the domain X of a state operator to a more universal setting Y. Moreover, we define two types of special generalized state maps, namely, generalized states from X to $([0, 1]; \rightsquigarrow_R, \leftsquigarrow_R, \wedge_R, 1)$ (or simply, G-states), and generalized internal states from X to X (or simply, GI-states). Also we introduce and investigate Bosbach states and Riečan states. We give the relations between generalized state map, states and internal states. We come to the conclusions that one can extend any state of the image space $\mu(X)$ into the state of the entire space X by using an internal state μ (or an internal state-morphism μ). In addition, another important result is that, in a sense, generalized state maps can be viewed as a possible united framework of the states and the internal states, the state-morphisms and the internal state-morphisms on pseudo equality algebras.

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